Transactions
of the
American Philosophical Society
Held at Philadelphia
For Promoting Useful Knowledge
Volume 96 Parts 2 & 3

ALHACEN ON THE PRINCIPLES OF REFLECTION

A Critical Edition, with English Translation
and Commentary, of Books 4 and 5
of Alhacen's *De aspectibus*

VOLUME ONE
Introduction and Latin Text

VOLUME TWO
English Translation

A. Mark Smith

American Philosophical Society
Independence Square • Philadelphia
2006

COVER ILLUSTRATION: Paris, Bibliothèque Nationale MS Lat 7319, f 116v.
The author wishes to express his deep gratitude for permission to use the many figures from this manuscript that appear in this book.

Library of Congress Cataloging-in-Publication Data

Alhazen, 965–1039.

[Manazir. Book 4–5. English & Latin]

Alhacen on the priciples of reflection: a critical edition, with English translation and commentary, of books 4 and 5 of Alhacen's *De aspectibus*, the medieval Latin version of lbn al-Haytham's *Kitab al-Manazir*/[edited by] A. Mark Smith.

p. cm - (Transactions of the American Philosophical Society; v. 96, pts. 2 & 3)

Includes bibliographical references (p.) and indexes.

Contents: v. 1. Introduction and Latin text—v. 2. English translation.

1. Optics—Early works to 1800. I. Smith, A. Mark. II. Title. III. Transactions of the American Philosophical Society; v. 96, pt. 2–3.

ISBN-10: 0–87169–962–1 (pbk. © 2006)
ISBN-13: 978–0–87169–962–6
US ISSN 0065–9746

QC353
[.A32313 2001]
535'.09'021-dc21 2001041227

This digital edition is distributed online as volume 4 in *Interpretatio*: Sources and Studies in the History of Science (http://www.ircps.org/interpretatio) published by the Institute for Research in Classical Philosophy and Science.

ISBN-13: 978–1–934846–01–8 (digital edition)
ISBN-10: 1–934846–01–5 (digital edition)

To Lois, my sine qua non

TRANSACTIONS

of the

American Philosophical Society

Held at Philadelphia for Promoting Useful Knowledge

VOLUME 96, Part 2

Alhacen on the Principles of Reflection:

A Critical Edition, with English Translation and Commentary, of Books 4 and 5 of Alhacen's *De aspectibus*

VOLUME ONE

CONTENTS

VOLUME I

VOLUME II

PREFACE

The passage from books 1-3 to books 4-6 of the *De aspectibus* is marked by an abrupt shift in topical focus. In the first three books, Alhacen was at pains to show that, under the right circumstances, the visual system yields a true and accurate perception of external reality. If the light is adequate, the eye healthy, the intervening medium appropriately transparent, and so forth, we see things as they actually present themselves to us in physical space. But only when the radial links between object and eye are uninterrupted. Under no circumstances is sight veridical when those links are broken. No matter how good the light, how healthy the eye, or how clear the air, things are never perceived correctly in mirrors or refracting media because, at best, they appear displaced and, at worst, distorted in shape, size, and distance. Reflection and refraction, in short, are by their very nature sources of visual illusion.

Alhacen's purpose in books 4-6 is to explain precisely how and why reflection deceives the eye. To that end, he establishes the basic principles of reflection in books 4 and 5 (the focus of the current edition) and then goes on to show in book 6 how those principles can be applied to the problem of image-distortion in plane, convex, and concave mirrors. Yet, however straightforward this goal, its accomplishment is anything but. On the contrary, as becomes clear in the course of Alhacen's analysis, an adequate explanation of reflection, as well as of image-formation and distortion, requires a great deal of sophistication in terms of both empirical verification and mathematical demonstration.

As a result, Alhacen's study of reflection seems more "scientific" and modern in its approach than his study of immediate, or direct, vision. No longer is common experience sufficient for empirical verification; controlled experiments are increasingly necessary. Admittedly, most of these experiments are primitive and loosely organized, but, as we shall see, one in particular calls for extremely tight controls. Likewise, the focus on mathematics is far more intense in books 4-6 than in books 1-3. And much of that focus is on the surprisingly complex problem of finding the exact spot, or spots, on a convex or concave mirror where a given ray of light will reflect to a given point on the other side of that spot.

It is easy to be misled by these changes in topical and analytic focus into thinking that Alhacen's primary concern in books 4-6 has shifted from the physiology and psychology of sight to the physics of light. After all, the way Alhacen uses ray-geometry in these books makes his approach look much like that of modern physical optics. It is crucial to realize, however, that throughout his analysis of reflection Alhacen's ultimate goal is to explain how and why things appear as they do in mirrors, not how light or illuminated color interacts physically with reflecting surfaces. No matter how far removed from visual issues the problem may appear to be at first glance, those issues are always lurking in the wings.

How Alhacen deals with the problem of finding precisely where reflection occurs on convex or concave spherical mirrors exemplifies this point. The givens of his analysis are 1) the great circle on the mirror defined by the plane of reflection, 2) the centerpoint of that circle, 3) the equal-angles law of reflection, 4) a point-source of radiation, and 5) a point to which that radiation will reach after reflection. But—and here is where Alhacen's true intentions are revealed—this latter point is not just any random point in space. It is *always* a center of sight. This is not to say that Alhacen was unconcerned with how light acts in its own right, but of far greater concern to him was how it acts with respect to specific viewpoints. This holds true whether the view is uninterrupted or interrupted by reflecting or refracting surfaces: the ultimate reference-point for *all* optical analysis is the center of sight. The apparent similarity between Alhacenian and modern physical optics is therefore just that: apparent. For, unlike its modern counterpart, Alhacenian optics is subjective, not objective in its analytic orientation. To lose sight of this fact is to miss the real point of his analysis of reflection, a point that becomes eminently clear in book 6, when Alhacen turns his attention to image-distortion in convex and concave mirrors.

Thematically bound by their focus on reflection and mirror-images, books 4-6 thus form a discrete topical unit. Nevertheless, my decision to separate books 4 and 5 from book 6 for this edition makes sense for at least two reasons. The first is practical. Taken as a whole, books 4-6 constitute nearly 40% of the entire text of the *De aspectibus*, almost as much as is occupied by the first three books. Given the size of this segment of text, and given my own time-constraints, I wanted to reduce the task of editing it to manageable proportions. The second reason has to do with topical organization. The purpose of books 4 and 5 is to establish and validate the rules of image-location in plane, convex, and concave mirrors on the basis of point-objects and point-images. The purpose of book 6, is to apply these rules to the analysis of two-dimensional images in order to explain precisely how the formation of such images determines the ways in which they appear deformed to the onlooker. There is thus a clear distinction between the section consisting of books 4 and 5, where the theoretical foundations are laid, and the section consisting of book 6, where those foundations are built upon.

Reduced though it was, the task of editing books 4 and 5 has been a long and complicated one during which I have accumulated many significant debts that demand acknowledgement. To start with, I wish to thank the NSF for its generous support during 2001-2003 (SES 0080445). Thanks are also due to the MU Research Council, the UM Research Board, and the American Philosophical Society for funding.

At a more personal level, I want to thank the librarians and archivists in charge of the manuscript collections I consulted for allowing me access to the necessary manuscripts and for the various courtesies they extended to me. For permission to reproduce pages from manuscripts in their collections, I am grateful to: The Master and Fellows of Trinity College, Cambridge; the Bibliothèque de l'agglomération de Saint-Omer; the Biblioteca Nazionale Centrale of Florence, by concession of the Ministero per i Beni e le Attività Culturali della Repubblica Italiana; the President and Fellows of Corpus Christi College, Oxford; the Crawford Collection of the Royal Observatory, Edinburgh; the Royal College of Physicians, London; and the Bibliothèque Nationale de France.

At the most personal level, I need to acknowledge my debt to the following people: Matt Shaw, whose talents as an editorial jack-of-all-trades lightened my authorial burden considerably; Mark Singer, whose knowledge of computer technology was invaluable in the production of the figures; Lindsey O'Donnell, whose uncanny arithmetical abilities went into the formatting of this edition; and Melinda Lockwood, without whose help with PageMaker I would have been lost in cyberspace. I also wish to thank Sandy Kietzman and Mona Burkett for their services in proofreading, and to John Frymire and Sabetai Unguru I owe a debt of gratitude for serving as sounding boards. Sabetai's critique of my introductory account of mathematical development during the Middle Ages and Renaissance was not only helpful, but also extraordinarily generous, given that my interpretation of Witelo is somewhat at odds with his. Thanks also to Abdullahi Ibrahim for his help with Arabic terms and concepts. And, of course, I cannot adequately thank my wife, Lois Huneycutt, both for her encouragement and for her willingness to take on so many family responsibilities in order to provide me the free time to work on this project.

INTRODUCTION

1. *Alhacen's Analysis of Reflection: An Overview*

Since the relevant background to both Alhacen and the *De aspectibus* has already been given in the preface and introduction to the edition of books 1-3, there is no need to rehearse the details here. Let us therefore turn directly to an examination of the form and content of books 4 and 5.

Alhacen's overall intent in these books is to determine as precisely as possible where the image of any given object-point seen in a mirror will lie with respect to a particular viewpoint, as defined by the center of sight at the vertex of the cone of visibility.[1] That location, according to Alhacen, is specified by the so-called cathetus-rule, which states that, for any given center of sight, the image of any object-point will lie where the normal, or cathetus, dropped from that object-point to the reflecting surface meets the straight line extending from the center of sight through the point of reflection. This line coincides with the light-ray reflected from the mirror, or, as Alhacen denominates it, the "line of reflection" (*linea reflexionis*). Thus, as illustrated in figure 1, p. 521, if center of sight A and object-point B face reflecting surface EF—be it plane, convex spherical, or concave spherical—and if ray BD reflects to A along ray DA, then, from the perspective of viewpoint A, the image of B will appear at I, where cathetus BI intersects the extension of AD. In plane mirrors, the cathetus is perpendicular to the reflecting surface itself, whereas in the two types of curved mirrors, it is perpendicular to the plane tangent to the reflecting surface where it intersects that surface. In those two types of mirrors, therefore, the cathetus passes through center of curvature G.

With this rule firmly established by the end of book 5, Alhacen turns in book 6 to the analysis of image-formation and image-distortion for object-surfaces, which, unlike points, have physical dimensions and are thus actually visible.[2] This analysis is based on reducing such surfaces to a representative sample of constituent points, locating the images of those points, and then extrapolating the composite image from them. How this works in practice is most easily illustrated in the case of plane mirrors. Let A in the top diagram of figure 2, p. 522, be the center of sight, BS the cross-section of some object-surface, D the point where incident ray BD reflects to A, and EF the common section of the surface of a plane mirror and the plane containing A, BS, and D. If BS is resolved into constituent points B, P, Q, and S, the incident ray from each point will reflect to A from a specific point of reflection on the mirror's surface. Accordingly, image B′ of B will lie where cathetus BB′ intersects line of sight ADB′, image P′ of P where cathetus PP′ intersects line of sight AD_1P', and so forth for points Q and S. The resulting composite image B′S′ will thus be the same shape and size as BS and will lie the same distance below the reflecting surface as BS lies above it. So, apart from apparent left-right reversal, image B′S′ will be an exact replica of BS.

The same kind of analysis can be applied to the situation illustrated in the bottom diagram of figure 2, where center of sight A and cross-section BS face a convex spherical mirror centered on G and containing arc EF. Let incident ray BD reflect to A from D, and let EF be the common section of the mirror's surface and the plane containing A, BS, and D. When the image-locations of constituent object-points B, P, Q, and S on BS are determined, the resulting composite image B′S′ turns out to be smaller than BS and somewhat bowed according to the curvature of arc EF. Moreover, B′S′ does not lie the same distance below the reflecting surface as BS lies above it. Far from being an exact replica, then, the image of BS is distorted in several ways that depend upon the mirror's curvature.

The fundamental principles underlying these two point-analyses, as well as the cathetus-rule upon which they are based, are few and fairly obvious. The first such principle is that the center of sight, the object-point, the point of reflection, and the image all lie in a plane normal to the mirror at the point of reflection. This plane constitutes what Alhacen calls the "plane of reflection" (*superficies reflexionis*). The second principle is that the angle of incidence, as measured according to the normal at the point of reflection (i.e., angle BDN in figure 2), is equal to the angle of reflection, as measured according to the same normal (i.e., angle ADN). The third is that the image lies on the straight line extending through the point of reflection from the center of sight. And the fourth is that the image also lies on the cathetus dropped from the object-point to the mirror. A considerable portion of Alhacen's analysis from the beginning of book 4 through the first quarter or so of book 5 is devoted to establishing these four principles as clearly and incontrovertibly as possible.

Neither the cathetus-rule nor the principles underlying it were new to Alhacen. All of these principles are at least implicit in Euclid's analysis of image-formation in the *Catoptrics* (c. 300 B.C.). In fact, Euclid went so far as to offer a mathematical demonstration of the equal-angles principle in the first proposition of that work.[3] The same fundamental principles are also implicit in Hero of Alexandria's *Catoptrics* (c. 50 A.D.). Like Euclid, moreover, Hero essayed a mathematical demonstration of the equal-angles principle.[4] But the clear articulation of all four principles had to await Ptolemy, who laid them out explicitly in the third book of his *Optics* (c. 160 A.D.).[5] Furthermore, Ptolemy departed from both Euclid and Hero by basing his analysis of reflection not just on mathematical principles but on empirical observation as well. This is most evident in his notorious experimental confirmation of the equal-angles principle, which is based on careful, measured observation of the angles of incidence and reflection for plane, convex cylindrical, and concave cylindrical mirrors.[6]

Not surprisingly, Alhacen follows Ptolemy's lead in his analysis of reflection, balancing empirical observation with mathematical demonstration in much the same way as Ptolemy.[7] Yet, despite obvious similarities, there are crucial differences between Alhacen's and Ptolemy's approaches. The most obvious lies in the fact that Alhacen bases his analysis on light-rays rather than on visual rays. A second and more significant difference is that Alhacen's analysis is far more rigorous and comprehensive than Ptolemy's. Take, for instance, Alhacen's experimental confirmation of the equal-angles principle in chapter three of book 4 (see pp. 300-312 below). Clearly modeled on Ptolemy's, Alhacen's experiment is nonetheless light-years beyond Ptolemy's in its instrumental and conceptual sophistication. Furthermore, unlike Ptolemy, Alhacen goes to great lengths to detail the mathematical implications of his empirical observations. In this way, for instance, he draws on the experimental confirmation of the equal-angles principle to verify that the plane of reflection is always normal to the mirror at the point of reflection. In addition, the rigor and sophistication of Alhacen's approach extends beyond the empirical to the mathematical basis of his analysis. Alhacen's proofs are incomparably more rigorous, sophisticated, and elegant than Ptolemy's. Thus, although there are traces of Ptolemy's influence throughout Alhacen's account of reflection, those traces dwindle to insignificance in the light of Alhacen's many original contributions to that account.

The third and most significant difference between Alhacen's and Ptolemy's analyses of reflection is that Alhacen undertakes to determine precisely where on the surface of a convex or concave spherical mirror the radiation from a given object-point will reflect to a given center of sight. Ignored entirely by Ptolemy—and for good reason, given its complexity—this problem, or, rather, its definitive solution, constitutes the *pièce de résistance* of Alhacen's analysis of reflection in book 5 of the *De aspectibus*. It also caps Alhacen's effort to hone the cathetus-rule to perfect sharpness for subsequent application in book 6. With the object-point, the cathetus, the center of sight, and the point of reflection all given and perfectly determined, the image-location can at last be found with absolute and unvarying precision.

The following overview of Alhacen's reflection-analysis is organized according to the structure of books 4 and 5. I will therefore proceed through those books in step-by-step, chapter-by-chapter order. I have chosen this approach not simply because it is straightforward. I have also chosen it because, in following the actual flow of Alhacen's analysis as it builds from empirical foundations to the elaborate mathematical structure erected on them in book 5, this approach reveals the underlying comprehensiveness, elegance, and logical seamlessness of that analysis.

Book Four: Alhacen opens his study of reflection in chapter 2 of book 4 by describing a set of simple experiments designed to show that the reflection of light is regular, that this regularity extends to every kind of light or luminous color, whether primary or secondary,[8] and that reflection weakens the resulting light or luminous color. The experimental apparatus consists of a room with one small opening through which sunlight or daylight streams in a relatively narrow shaft. An iron or silver mirror can then be placed toward the incoming light so as to reflect it to various places in the room. In this way it can be shown that, when the mirror is rotated, the spot of reflected light will follow the arc of rotation (4, 2.2-2.5, p. 296). It can also be shown that reflected light or illuminated color is less intense than the light or illuminated color at the source (4, 2.7-2.13, pp. 297-298) and that, if the same light or illuminated color shining on the mirror also shines directly on a white background, the reflected light or illuminated color will be less intense than that shining directly on the background (4, 2.15-2.20, pp. 298-299). On the basis of such experiments, therefore, we can conclude with Alhacen that "the forms of lights and colors are reflected from polished bodies and [are thereby] weakened" and that a "reflected form is brighter than a secondary one when they share the same source" (4, 2.21, p. 299).

Having set the stage with these rough, preliminary observations, Alhacen turns in chapter 3 to an experimental confirmation of the equal-angles principle. Before setting up the actual experiment, he establishes a few guidlines. First, the reflectivity of any polished body is due to the physical smoothness of its surface (4, 3.1, p. 300). Second, no matter the shape of the reflecting surface, reflection can occur from every point on it. In such reflection, moreover, the lines of incidence and reflection, as well as the normal to the point of reflection, will all lie in a single plane normal to the reflecting surface at the point of reflection, and within that plane the lines of incidence and reflection "will maintain an equivalent situation with respect to [that] normal and will form equal angles [with it]" (*tenebunt . . . eundem situm* *respectu perpendicularis et equalitatem angulorum*—4, 3.2, p. 300). Third, if light reaches a mirror along the normal, it reflects back along the normal. In plane mirrors, this normal is perpendicular to the reflecting surface itself; in cylindrical and conical mirrors, whether convex or concave, it is perpendicular to a plane tangent to the reflecting surface along a line of longitude; and in spherical convex and concave mirrors, it is perpendicular to a plane tangent to the reflecting surface at a point only (4, 3.3, p. 300).

In the next 38 paragraphs (4, 3.4-3.41, pp. 300-308), Alhacen describes the apparatus to be used for the experiment. This apparatus consists of three main components, the first of which is the register described in 4, 3.3-3.8, pp. 300-301. It consists of a thin bronze plaque formed from a semicircular section six digits in radius.[9] When finished, it takes the form illustrated in figure 4.3.2, p. 193, with its outer arc OFV centered on E. From that same centerpoint an arc five digits in radius is scribed on its surface to pass through point G. A selection of lines, such as HE and AE on the right side of EF, is also scribed on its surface. These lines are matched on the left side of EF by lines forming the same angle with it. Thus, angle HEF = angle MEF, and so forth. The bottom edge of the register is cut along the outermost of these converging lines, as well as along line VO, so that what is left at the base is a small triangular section with centerpoint E of the register at its vertex.

The second component consists of a wooden ring fourteen digits in diameter and seven digits high, its wall being two digits thick. Its inner hollow is therefore ten digits in diameter. The construction of this ring is described in 4, 3.9-3.22, pp. 301-304. On its top surface, lines are scribed to correspond precisely with those scribed on the face of the register. Thus, as illustrated in figures 4.3.3 and 4.3.4, pp. 193-194, lines AE, BE, CE, and DE, which converge at centerpoint E of the top base of the ring, form angles BEA, CEA, and DEA equal, respectively, to angles HEF, AEF, and MEF on the face of the register. From the inner and outer endpoints of these lines, as represented in figure 4.3.4, perpendicular lines are drawn along the inner and outer surfaces of the ring to form lines of longitude on both surfaces. Hence, when the cylinder is placed upright with its top and bottom bases perfectly horizontal, those lines will be perfectly vertical, as illustrated from an inside view in figure 4.3.6, p. 195.

A circle is then drawn on the inner surface of the ring parallel to its base at a height of two digits minus half a grain of barley above that base.[10] A notch is scooped along this circle to a depth of one digit and no thicker than the bronze register so that the register can be inserted snugly into the notch with its upper face parallel to the base of the ring at a distance of two digits minus half a grain of barley above that base (see figure 4.3.6, p. 195, for a cutaway diagram of the ring with its notch). When the register is properly inserted into this notch, the ring's axis will pass through centerpoint E of the register, and the inner arc passing through point G on the register will coincide with the inner surface of the ring. Also, line EF on the register will pass through point B on the ring's inner surface, line EH through point C, and so forth. Each line scribed on the register's face will thus intersect a corresponding vertical line on the inner surface of the ring along the orthogonal.

At a height of precisely two digits above the base on the outer surface of the ring, a circle is drawn parallel to the ring's base. Where that circle intersects each of the lines drawn vertically along the ring's outer surface, a hole one grain of barley in diameter is drilled through the wall with its axis parallel to the corresponding line inscribed on the upper face of the register. Accordingly, the line of longitude on the ring's outer surface and the corresponding line of longitude on its inner surface will bisect the outer and inner bases of the hole. Likewise each line scribed on the face of the register will line up perfectly with its appropriate hole, as represented in figure 4.3.7, p. 195. According to that alignment, the line on the register's face will coincide with the line of longitude at the bottom of the hole at the point where the vertical line on the inner surface of the ring intersects it.

For the final step of this phase of the construction, a square block of wood fourteen digits on a side and over one digit thick is formed.[11] At the very center of the top surface of this block, a square four digits to a side is drawn so that its center coincides with the center of the block. That square is excavated to a depth of one digit so that its bottom is perfectly flat and therefore parallel to the top surface of the block, as represented in figures 4.3.8 and 4.3.9, p. 196. The base of the ring is attached to this block so that the center of the circle at the base of the ring's hollow coincides with the center of the square on the top surface of the block. The axis of the ring will therefore pass through the center of the square at the bottom of the cavity in the block, the center of the square at the top, and point E of the register inserted into the ring above it, as illustrated from a bird's-eye view in figure 4.3.10, p. 197.

The third and final component consists of seven panels, each with a specific type of mirror inserted into it. Distinguished by the shape of their reflecting surfaces, these seven mirrors range from plane, through convex spherical, cylindrical, and conical, to concave spherical, cylindrical, and conical. Alhacen's choice of these particular mirrors was dictated by the relative simplicity of their shape and his capacity to explain their relevant properties with the mathematical tools available to him.[12] Accordingly, within the limits of his ability to analyze them mathematically, the six types of curved mirror selected by Alhacen are as representative as possible of all curved mirrors, whatever the complexity of their curvature.

The construction of the seven selected mirrors, all of which are iron, is described in 4, 3.23-3.41, pp. 304-308. The plane mirror consists of a fairly thin, flat disk three digits in diameter, as represented in the upper left-hand diagram of figure 4.3.11, p. 198.[13] The convex and concave cylindrical mirrors are formed from a hollow cylinder three digits high and six digits in diameter at the bases. Two sets of parallel chords three digits in length are measured off on the top and bottom bases of the cylinder, and the resulting sections are cut off to leave two cylindrical segments three digits high and three digits wide along the chords. The construction of one of these mirrors is represented in the upper right-hand diagram of figure 4.3.11. Then follows the construction of the convex and concave conical mirrors from a hollow right cone whose base is six digits in diameter and whose lines of longitude are four-and-a-half digits from vertex to base. When each mirror is properly excised from the cone, it will be four-and-a-half digits high and three digits across at the base, as represented in the lower right-hand diagram of figure 4.3.11. Finally, the spherical convex and concave mirrors are formed from a hollow iron sphere six digits in diameter. When properly excised, both mirrors will be three digits in diameter at the base, as represented in the lower left-hand diagram of figure 4.3.11.

For each mirror a rectangular wooden panel six digits high, four digits wide, and at least thick enough to stand firmly upright on its own is formed.[14] The plane mirror is inserted into its panel so that the reflecting surface is perfectly flush with the panel's face and so that its centerpoint coincides with the centerpoint of the panel's face--i.e., where the midlines along the length and width of the panel's face intersect. Accordingly, as represented in the upper right-hand diagram of figure 4.3.13, p. 199, the centerpoint of the plane mirror lies on the midline of the panel's face at a distance of precisely three digits above the panel's base.

The insertion of the convex cylindrical mirror is represented in the lower left-hand diagram of figure 4.3.13. It fits into its panel so that the midline along its length is perfectly flush with the face of the panel and thus coincides with the midline along the length of the panel's face. Likewise, the midline along the width of the panel's face bisects the mirror so that the centerpoint of the mirror's surface lies precisely three digits above the panel's base.

As represented in the upper left-hand diagram of figure 4.3.14, p. 200, the convex conical mirror is set into its panel so that line of longitude VF bisecting its surface is perfectly flush with the panel's face, its vertex lying at the very top of the midline along the length of the panel's face. Thus, the midline along the width of the panel's face intersects the mirror at a distance of precisely three digits above the panel's base along its midline of longitude. In this case, therefore, the midpoint on the panel's face will not coincide with the midpoint on the line of longitude bisecting the mirror's surface. But all that really matters is that the mirror's base lies below the midpoint of the panel.

The insertion of the convex spherical mirror, finally, is illustrated in the lower left-hand diagram of figure 4.3.14. It must be set into its panel so that the centerpoint of its surface coincides with the centerpoint of the panel's face and thus lies precisely three digits above the base of the panel on the midline along the length of the panel's face. In this case, placing the centerpoint of the mirror's surface precisely at the panel's centerpoint is crucial.

The three concave mirrors, on the other hand, must be inserted into their panels so that the chord (or chords) at their bases is perfectly flush with the panel's face or, in the case of the concave spherical mirror, so that the entire base is perfectly flush with that surface. Accordingly, as represented in the lower right-hand diagram of figure 4.3.13, p. 199, the concave cylindrical mirror fits into its panel with the chords at both ends flush with the panel's face. Hence, the line of longitude bisecting the mirror's surface will lie directly below, and parallel to the midline along the length of the panel's face, and the midpoint on its surface will lie precisely three digits above the panel's base.

Like its convex counterpart, the concave conical mirror is set into its panel so that its vertex touches the panel's top. However, as represented in the upper right-hand diagram of figure 4.3.14, p. 200, it must be set into its panel so that line of longitude VF bisecting its surface lies directly below, and parallel to the midline along the length of the panel's face. As before, although the midpoint of the panel's face will not be in line with the midpoint on line of longitude VF that bisects the mirror's surface, all that matters is that the mirror's base lie below the midpoint of the panel.

Finally, the spherical concave mirror is set into its panel so that its base is perfectly flush with the panel's face and the centerpoint of its surface lies directly below the centerpoint of the panel's face, as represented in the lower right-hand diagram of figure 4.3.14. Hence, the mirror's centerpoint lies precisely three digits above the panel's base. Here, too, the placement must be absolutely precise.

With their mirrors properly inserted, the panels are ready to be placed one at a time into the ring and stood perfectly upright on the floor of the square cavity at its base so that line FGE of the inserted register is perpendicular to the face of the panel containing the inserted mirror as well as to the midline along its length. So oriented, the panel is then slid toward point E of the register until that point touches the mirror. In the case of the plane, cylindrical, and conical mirrors, point E will touch the line of longitude bisecting the mirror's surface. As we have seen, this line either coincides with, or is parallel to and directly below the midline along the length of the panel's face. Thus, the line connecting point E and the midpoint of the mirror will either coincide with, or be parallel to the midline along the length of the panel's face. Moreover, by construction, the top face of the register lies half a grain of barley below the midpoint of the panel, so, in the case of the plane, cylindrical, and conical mirrors, point E will touch the mirror's midline of longitude half a grain of barley below the level of the panel's midpoint.[15]

Such is not quite the case for the spherical mirrors. Because of their curvature, point E at the vertex of the register will touch their surfaces either in front of or behind the midline passing through their centerpoints, so E will not be properly aligned with respect to those centerpoints.[16] In order to insure proper alignment, the panel containing the convex mirror must be pulled back slightly from the point of contact, whereas a hole must be drilled into the concave mirror at the point of contact so that point E of the register can be pushed ever so slightly into the panel beyond the mirror's surface. In both cases the required accommodation entails an adjustment of around .36 mm.[17]

Given this description of the apparatus and its appropriate setup, the purpose and outcome of the experiment are virtually self-evident. No matter which of the mirrors in its panel is properly disposed against point E of the register, when all but one of the holes in the ring's wall is blocked, and when the open hole is exposed to light, the beam of light passing through that hole should reach the mirror along the line matched to the hole on the face of the register. Likewise, after reflecting, the light-beam should follow the corresponding line on the face of the register to the corresponding hole on the other side. To establish that this is in fact the case, Alhacen proposes a variety of tests in 4, 3.43-3.64, pp. 309-312, for all seven mirrors on the basis of both primary light (i.e., sunlight) and secondary light (i.e., daylight). The test for illuminated color he defers until the very end of the chapter (4, 3.107-3.108, pp. 323-324). Given the simplicity of both the tests and their results, nothing more need be said about them than that they all confirm the equal-angles principle in one way or another. They also confirm that the plane of reflection is indeed normal to the mirror at the point of reflection.

With the empirical confirmation of the equal-angles principle complete, Alhacen goes on in the next section (4, 3.65-3.87, pp. 313-317) to explain the implications of the results. Although verbal, this explanation amounts to a geometrical demonstration based on the structure of the apparatus and the disposition of the mirrors in it. For one thing, Alhacen notes, the ring is formed so that the plane of the floor of the square cavity upon which the panels stand, the plane of the ring's bottom base, the plane of the top surface of the register, and the plane of the axes passing through the holes in the ring's wall are all horizontal and therefore parallel. For another, the plane of the register's upper face cuts the face of the panel at a level three digits minus half a grain of barley above the floor of the square cavity upon which the panel stands. Since the axes of the holes lie precisely half a digit above that level, the plane of the axes cuts the face of the panel at the midline along its width, which lies precisely three digits above its base and passes through or right in line with the centerpoints of the spherical and cylindrical mirrors.

Likewise, the vertical lines passing through the centerpoints of their respective holes along the inner and outer surfaces of the ring are all parallel to one another, as well as to the midline along the length of the panel's face, which coincides with or is parallel to the midline passing through the seven mirrors. So all those lines are perpendicular to the aforementioned planes. Finally, midline EGF of the register is perpendicular to the face of the panel.[18] Given, therefore, that light propagates in straight lines, and given the geometry of the apparatus in its proper disposition, it follows that, no matter the angle at which the incoming light-beam strikes the mirror, that light-beam will reflect at precisely the same angle from the mirror, and throughout the process, it will follow the appropriate lines scribed on the register's face. The axial rays of the incident and reflected beams will thus be perfectly parallel to their corresponding lines on the register at a distance of half a grain of barley above them, and they will be precisely the same length as those lines. So it follows that in all cases the plane of reflection, which is defined by the axial rays coinciding with the axes of their respective holes, is normal to the mirror at its centerpoint, because it is perpendicular both to the face of the panel and to the midline along its breadth.

From this it should be clear why Alhacen insists on such painstaking exactitude in the construction of the experimental apparatus, even to the point of adjusting the placement of the spherical mirrors by the negligible amount of .36 mm. The slightest variation in the parallelism of planes and lines, or in the measurement of angles and distances, or in the placement of the mirrors within their panels can skew the experimental results badly enough to disconfirm—or at least fail to confirm—the principle it is designed to validate. After all, the equal-angles principle mandates a specific and exact relationship between the angles of incidence and reflection, so any results that fall short of revealing precisely that relationship represent either failure of the experiment or failure of the principle.

That Alhacen's apparatus can be constructed more or less according to the exacting standards needed to yield the appropriate results was demonstrated in 1977 by Saleh Omar.[19] But Omar had the benefit of modern technology and machining. Alhacen did not. So the question inevitably arises: could Alhacen have possibly produced the apparatus he describes within the extraordinarily narrow tolerances required? To ask this question is to raise the issue of whether he really did construct the apparatus, or whether he imagined it, knowing full well that, if it were ever built to his specifications, it would work precisely as planned.

There are good arguments for both alternatives. On the one hand, the punctiliousness with which Alhacen describes various procedures in the construction of the apparatus suggests that he actually followed them. In 4, 3.10, pp. 301-302, for instance, he gives fairly clear instructions about how to set up a lathe for drilling out the central hollow in the ring. And in 4, 3.12-3.13, p. 302, he details two methods for producing the vertical lines on the outer and inner walls of the ring.[20] Furthermore, the apparatus as described is small enough that very slight errors in its construction can be tolerated without skewing the results beyond at least close approximation.

On the other hand, given the technological limits of Alhacen's time, there is ample reason to doubt that he could have constructed the apparatus as described, much less gotten the perfect experimental results he details. It is difficult to believe, for instance, that Alhacen could have drilled the holes in the ring's wall to the degree of accuracy required by the experiment, since the axis of every hole must be perfectly parallel both to the appropriate line on the face of the register and to the plane of the ring's base, which itself must be perfectly parallel to the top face of the register. It is also difficult to believe that he could have produced circles precisely half a grain of barley in radius (i.e., c. .21 cm.) with a compass, as he instructs us to do in 4, 3.17, p. 303. The adjustment of .36 mm. for the two spherical mirrors is also problematic. Failure to take it into account would have had a nugatory effect on the experimental results, so why bother to make it when the effort involved so clearly outweighs the benefit? Worse yet, it is difficult to believe that Alhacen could have formed the iron mirrors used in the experiment. For a start, casting the hollow iron sphere, cylinder, and cone from which the curved mirrors are supposedly formed would have been virtually impossible with the smelting technology available in Alhacen's day, so those forms must have been wrought or ground.[21] But cold iron is extraordinarily difficult to work because of its hardness, so, even if those hollow mirrors were somehow produced from blanks, the process of grinding and polishing them would have been inordinately time-consuming and expensive without the aid of advanced power technology, A far more practical, and equally effective, alternative would have been to form the requisite mirrors from bronze, which is considerably more tractable than iron.[22] There is no doubt that Alhacen was aware of this fact, so his failure to choose that alternative is puzzling. Whatever the case, until we know more about the technology available to Alhacen, the question of whether he conducted his reflection-experiments in physical or mental space must remain open.

Done with the reflection-experiments and their geometrical explanation, Alhacen attempts in the concluding section of chapter 3 to justify the equal-angles principle on theoretical and physical grounds (4, 3.88-3.106, pp. 317-323). He begins in 4, 3.88-3.92, pp. 317-319, by establishing that each point on a luminous surface facing a mirror forms the vertex of a cone of radiation with its base on the reflecting surface. By the same token, each point on the reflecting surface forms the vertex of a cone of radiation with its base on the luminous surface, the light from every point on this surface converging at that spot on the mirror. Furthermore, every point between the luminous and reflecting surfaces constitutes the vertex of two intersecting cones of radiation, one with its base on the luminous surface, the other with its base on the reflecting surface. All these cones taken together form a shaft of light whose outer edges are defined by the peripheries of the luminous and reflecting surfaces.

To illustrate, let AC in the top diagram of figure 3, p. 523, represent the cross-section of a luminous surface facing cross-section DD_4 of a plane mirror within the plane of the page. BD_1D_3 constitutes one of an infinite number of cones of radiation emanating from point B. D_2AC constitutes one of an infinite number of cones of radiation whose light emanates from every point on the luminous surface to converge at a single point on the reflecting surface. Point X, where the edges of these two cones intersect, constitutes the vertex of two opposite cones AXB and XD_1D_2 with bases AB and D_1D_2 on the luminous and reflecting surfaces, respectively. All such cones taken together will form truncated conical shaft ADD_4C of radiation.

Now, as Alhacen has established earlier on empirical grounds in 4, 3.46-3.47, pp. 309-310, when that conical shaft of light is reflected from the mirror it continues to spread out commensurately as the light within it radiates away from the mirror. Hence, the shaft of light after reflection is continuous with the incident shaft in the same way it would be if the light in the incident shaft had proceeded the same distance without being interrupted by the mirror. Suppose, therefore, that opaque surface EF in the top diagram of figure 3 intercepts the light from AC after reflection from base DD_4 on the mirror. That light will be projected to base HK on EF to create truncated conical shaft HDD_4K, which is continuous with incident shaft ADD_4C.

Like that incident shaft, this one is also composed of an infinite number of intersecting cones based on the reflecting and illuminated surfaces.

From every point on luminous surface AC, therefore, an infinite number of cones of radiation can be formed with their bases on reflecting surface DD_4. For each such point on AC, one of the cones emanating from it will have the entire reflecting surface as its base; the rest will stand on smaller segments of that surface. For every point on luminous surface AC, moreover, there will be one particular cone whose continuation after reflection has its base on segment HK of opaque surface EF. Thus, as illustrated in the middle diagram of figure 3, the light within cones ADD_3, BD_1D_4, and CD_2D_5 is reflected to HK from bases DD_3, D_1D_4, and D_2D_5 on the mirror along truncated conical shafts $DHKD_3$, D_1HKD_4, and D_2HKD_5, respectively. For every ray within the incident cone there is within the reflected shaft a corresponding ray, which, had it not been interrupted by the reflecting surface, would have continued along the line of incidence.

By symmetry, then, if HK is taken to represent the luminous source and AC the opaque surface upon which HK's light is reflected, that light will reach points A, B, and C from the same points on the mirror along the same rays as the light from points A, B, and C reached HK when AC was the luminous source. For each ray within incident shafts HKD_3D, HKD_4D_1, and HKD_5D_2, there will be a corresponding ray within the reflected cones D_3DA, D_4D_1B, and D_5D_2C, according to the continuity of reflection. So, if a center of sight is placed at point B, as represented in the bottom diagram of figure 3, the light from HK will radiate to the mirror along an infinite number of rays, such as HD, LD_1, MD_2, and KD_3, emanating from an infinite number of points on HK that, after reflecting, will converge on B along corresponding rays DB, D_1B. D_2B. and D_3B. This correspondence, of course, lies in the symmetry with which the ray-pairs are inclined to the reflecting surface—i.e., at equal angles. The same holds for any point on AC; if a center of sight is placed there, the entire form of HK will converge on it. Accordingly, the form of HK can reflect to an infinite number of centers of sight from the mirror, and the form of each point on HK will reflect to that center of sight in such a way that the incident and reflected rays will form equal angles with the normal dropped to the point of reflection.

In addition to using this model to justify the equal-angles principle on theoretical grounds, Alhacen draws upon it to explain the effect of reflection on light-intensity. As he points out in 4, 3.93, p. 319, light can be weakened by reflection in three ways: through increasing distance as it draws away from its source, through the deadening effect of reflection itself, and through the dispersal of the light as it spreads out. Accordingly, in the top diagram of figure 3, the light at HK is less intense than the light at source

AC because all three of these factors come into play. On the other hand, in that same diagram, the light from AC converges on point D2 of the mirror, so it is strengthened by concentration at that point. This intensification also applies to points H and K in the middle diagram of figure 3, the appropriate rays from every point on AC being concentrated at those two points after reflection—and indeed at every other point on HK. Altogether, then, the intensity of light at HK depends on how the weakening due to distance and reflection (as well as to dispersal) balances out against the strengthening due to concentration. If the weakening factors outweigh the strengthening factor, the resulting light will be less intense than, the light at the source. If not, it will be just as intense as, or more intense than, the source-light. Here Alhacen presumably has in mind the case of concave spherical or parabolic mirrors at or near whose focal points incoming parallel light-rays are concentrated with extraordinary intensity.[23]

Moving from the geometry to the physics of reflection, Alhacen draws on an analogy between projectile motion and light-radiation in order to explain reflection in terms of physical rebound. He lays the theoretical groundwork for this analogy by resolving the surface of any given luminous source into quanta of "least light" (*lux minima*) that lie at the threshold of effective luminosity. Nothing below that threshold—i.e., no spot of light smaller than this quantum—constitutes actual light because it will have dwindled to invisibility (4, 3.97, p. 320). Ill-defined though it may be (see note 71, pp. 359-360, for elaboration), this notion of minimal light nonetheless serves to reduce the luminous surface to such tiny spots that the beams of light emanating from them have almost no breadth. Virtually equivalent to mathematical rays, therefore, they can be understood as trajectories followed by the light-quanta as they fly outward from their source (4, 3.98, pp. 320-321).

How such light-quanta interact with a given physical surface is determined by the structure of that surface. When it is rough or porous, the light-quanta striking it are dispersed haphazardly because of its unevenness, some of the impinging quanta being scattered outward in random directions, others being trapped in the pores and thereby absorbed.[24] The less rough the surface, the less haphazard the dispersion, until eventually the surface becomes smooth enough to cause some reflection, however imperfect. The more perfect the reflection, the less haphazard the dispersal, since reflection is regulated by the equal-angles principle. Perfect smoothness therefore yields perfect reflection, which is perfectly regulated by that principle (4, 3.99-3.100, p. 321).

Now, if we think of these light-quanta as tiny projectiles shot at great speed from luminous sources, the analogy between reflection and physical rebound becomes obvious. Accordingly, light-quanta striking a mirror along the normal and reflecting back along that normal act like hard spheres dropped to an unyielding, horizontal surface and bouncing straight back along the original line of descent (4, 3.100-102, pp. 321-322). The smoother and more reflective the mirror's surface, the stronger the rebound as measured by the intensity of the reflected light. By the same token, the more unyielding the surface struck by the sphere, the stronger the rebound. So the polish of the mirror is analogous to the physical hardness of the surface from which the sphere rebounds.[25] However perfect the reflection or rebound, though, the light or sphere loses some of its original impulse after impact with the resisting surface—hence, the deadening effect reflection has on light.

Like reflection or physical rebound along the normal, reflection or physical rebound along the oblique follows the same dynamic principles. We can thus draw an analogy between light striking a mirror at an angle and a hard sphere projected obliquely at great velocity against an unyielding surface (4, 3.102, pp. 321-321). If the light or sphere were allowed to penetrate the given surface unhampered, it would continue along its original path. But such penetration is balked by the surface, which poses the same resistance it did when the light or sphere struck it orthogonally. Being directly opposed to the vertical impulse of the light or sphere, this resistance forces both to reverse direction along the orthogonal component of the path they followed in their incidence. But the surface poses no resistance whatever along the horizontal, so, when the light or sphere strikes it obliquely, its motion in that direction remains unaffected. Consequently, the motion is fully reversed along the vertical, while it remains unchanged along the horizontal. Altogether, then, the incidence and rebound of the obliquely striking light or sphere are perfectly symmetrical with respect to the point of impact, which is to say that they occur at equal angles with respect to the normal erected at that point (4, 3.103-106, pp. 322-323).

The idea of likening reflection to physical rebound is hardly original with Alhacen. Both Hero of Alexandria and Ptolemy apply the same analogy to the analysis of reflection, and Ptolemy extends that analogy to refraction (as in fact Alhacen does later on in book 7 of the *De aspectibus*). However, Alhacen's dynamic account of reflection according to the composition of vertical and horizontal motions and forces does seem to be original to him, and it has significant ramifications for the development of optics in Europe from the thirteenth to the seventeenth century. Clear traces of that account can be seen in Descartes's vector-analysis of reflection and refraction in the *Dioptrique* of 1637 (see note 79, p. 361, for elaboration on this point).

Bringing chapter 3 to a close with his long-deferred experimental confirmation of the equal-angles principle for illuminated color (4, 3.107-3.108, pp. 323-324), Alhacen undertakes in the next chapter to refute both the visual-ray account of image-formation in reflection and the account of those who "claim that the form of the object is impressed upon a facing mirror [so as to be] seen in the mirror the same way that natural forms of objects are perceived in objects" (4, 4.1, p. 324). In fact, Alhacen does not address the visual-ray account until the fifth chapter (4, 5.3, p. 326), so the whole of chapter 4 is devoted to undermining the impression-theory of image-formation. The arguments Alhacen adduces against this theory in 4, 4.2-4.6, pp. 324-325, are obvious enough that they warrant neither retailing nor discussion here. Suffice to say, they center on the fact that, contrary to what would be expected if they were actually impressed on the mirror's surface, images seen in reflection shift their location and disposition on the mirror's surface as we look at them from changing perspectives.

Alhacen's purpose in the fifth and final chapter of book 4 is to establish that, for any given viewpoint, every point on the exposed portion of a mirror's surface constitutes a point of reflection. To that end, he deals in order with plane mirrors (4, 5.1-5.10, pp. 325-329), convex spherical mirrors (4, 5.11-5.14, pp. 329-331), convex cylindrical mirrors (4, 5.15-5.26, pp. 331-335), convex conical mirrors (4, 5.27-5.5.45, pp. 335-342), concave spherical mirrors (4, 5.46-5.50, pp. 342-343), concave cylindrical mirrors (4, 5.51-5.56, pp. 343-344), and concave conical mirrors (4, 5.57-5.60, pp. 344-345). For each mirror, Alhacen determines how much of its surface is exposed to a given viewpoint, how any given plane of reflection passing through this viewpoint will cut that surface, and how many possible reflection-points there are within each plane.

Under normal circumstances, Alhacen begins, the entire surface of a plane mirror is exposed to any facing viewpoint. Any plane of reflection passing through that viewpoint will cut the reflecting surface along a straight line, and every point on this line can serve as a point of reflection for that viewpoint (4, 5.9, pp. 328-329). Since an infinite number of such planes can be passed through the given viewpoint along the normal dropped from it to the mirror, reflection can occur to that viewpoint from every point on the mirror (4, 5.10, p. 329).

In convex spherical mirrors, on the other hand, less than half the surface is exposed to any facing viewpoint (4, 5.11, p. 329). Every plane of reflection passing through that viewpoint will form a great circle on the mirror's surface, and every point on every great circle within the exposed portion of the mirror can serve as a point of reflection for that viewpoint.

Since an infinite number of such planes can be passed through any given viewpoint along the normal dropped from it to the mirror's surface, every point on that surface can serve as a point of reflection for that viewpoint (4, 5.13, p. 330).

As with convex spherical mirrors, so with convex cylindrical mirrors, less than half the surface is exposed to any facing viewpoint (4, 5.14-5,16, pp. 330-331). Planes of reflection passing through that viewpoint can cut the mirror's surface in three ways (4, 5.21-5.22, p. 333). If the cut is parallel to the cylinder's axis, the plane will form a line of longitude on the exposed surface, and every point on that line can serve as a point of reflection for the given viewpoint (4, 5.23, pp. 333-334). If the cut is perpendicular to the axis, the plane will form a circular section on the mirror's surface, and every point on that circle within the exposed portion of the mirror can serve as a point of reflection (4, 5.24, p. 334). But if the cut is oblique to the axis, the plane will form an elliptical section (or, as Alhacen calls it, a "cylindric section" [*sectio columpnaris*]) on the mirror's surface. Only one point on that section within the exposed portion of the mirror can serve as a point of reflection, and it lies where the minor axis of the ellipse intersects the mirror's surface (4, 5.25-5.26, pp. 334-335). Since an infinite number of planes can be passed through the given viewpoint along the normal dropped from it to the mirror's surface, an infinite number of elliptical sections can be formed on the exposed portion of the mirror. For each elliptical section, there is a unique minor axis intersecting the mirror's surface at a unique pair of points. Such points cover the entire exposed surface, so each and every one of them can serve as a point of reflection for the given viewpoint.

In convex conical mirrors, the amount of surface exposed to any viewpoint depends on where that viewpoint lies with respect to the mirror's vertex. If the line of sight extending from it to the mirror's vertex is perpendicular to the mirror's axis, then precisely half the mirror's surface is exposed to view (4, 5.30, p. 336). If that line of sight forms an acute angle with the axis on the side of the viewpoint, less than half the mirror's surface is exposed to view (4, 5.29, p. 336). And if that line of sight forms an obtuse angle with the axis on the side of the viewpoint, more than half the mirror's surface is exposed to view (4, 5.31, pp. 336-337). All, or virtually all, of that surface is exposed to view when the line of sight coincides with the edge of the cone or penetrates the cone through the vertex (4, 5.32-5.34, pp. 337-338).

Again, depending on where the viewpoint lies with respect to the mirror's vertex, any given plane passing through that viewpoint will form a line of longitude, a circle, or a conic section on the exposed portion of the reflecting surface (4, 5.40-5.41, p. 340). Under no circumstances will a plane forming a circle on this surface constitute a plane of reflection, because in a right cone, the plane of any circular section is oblique to its edge (4, 5.42, pp. 340-341), so the diameters of that section will be oblique to its edge. Since none of those diameters is normal to the reflecting surface at the points where they intersect it, none of those intersection-points can serve as a point of reflection. Consequently, in conical mirrors reflection is limited to planes that cut lines of longitude or conic sections on the mirror's surface.

When the viewpoint lies directly above the mirror's vertex along the axis, every plane passing through it along that axis will form a line of longitude on the exposed surface, and every point on that line can serve as a point of reflection. Since there is an infinite number of such planes, every point on the mirror's surface can serve as a point of reflection for that viewpoint (4, 5.40, p. 340). On the other hand, if the viewpoint does not coincide with the mirror's axis, then, no matter where it is located with respect to the vertex, only one plane of reflection passing through it will form a line of longitude on the mirror's surface.[26] The rest will form conic sections.

When a given plane of reflection cuts a conic section on the mirror's surface, there can be one or two but no more than two points of reflection within that plane (4, 5.43, p. 341). If the plane is normal to the mirror's surface, there can only be one point of reflection, and it lies where the axis (or, in the case of an ellipse, the major axis) of the conic section intersects the mirror's surface (4, 5.44, p. 341). If the axis or major axis of the conic section is not normal to the mirror, there can be two points of reflection, because in that case two lines within the given plane can be drawn normal to the mirror's surface from the section's focus, which lies where the cutting plane intersects the cone's axis. For each such section, there is a unique pair of such lines intersecting the mirror's surface at a unique pair of points (4, 5.45, p. 341-342). Therefore, since an infinite number of cutting planes can be passed through the viewpoint along the normal dropped from it to the mirror, an infinite number of conic sections can be formed on the mirror's surface. Within each plane, the particular conic section will have a unique pair of lines extending from its focus to intersect the mirror along the normal. There will thus be an infinite number of such intersection-points on the exposed portion of the mirror's surface, each serving as a point of reflection for the given viewpoint.

In concave spherical mirrors, the entire surface is exposed to a viewpoint lying inside the mirror (4, 5.46, p. 342). Otherwise, no matter how far outside the mirror it lies, more than half the surface will be exposed to that viewpoint. Every plane passing through the given viewpoint along the normal dropped from it to the reflecting surface will cut a great circle on that surface, and every point on this circle within the exposed portion can serve as a point of reflection. Since an infinite number of planes can be passed through the viewpoint to form great circles on the exposed surface, every point on that surface can serve as a reflection-point (4, 5.48-5.50, p. 342-343). When the viewpoint lies at the very center of curvature, the lines extending from it will all be normal to the mirror. Since light radiating along a normal reflects back along it, and since there is an infinite number of normals reaching the mirror in all possible directions from the viewpoint, every point on the exposed surface constitutes a point of reflection for that particular viewpoint (4, 5.47, p. 342). In all cases, then, any point on the mirror's surface can serve as a point of reflection for any given viewpoint.

In concave cylindrical mirrors, when the viewpoint lies inside the mirror, its entire surface is exposed to view. Wherever else it lies outside the mirror, more than half the reflecting surface will be exposed to view (4, 5.51, p. 343). Like its convex counterpart, a concave cylindrical mirror can be cut in three ways by any given plane of reflection: i.e., along a line of longitude, along a circle, or along an elliptical section. When the cut occurs along a line of longitude or a circle, every point on that line or circle can serve as a point of reflection (4, 5.54-5.55, pp. 343-344). When the cut forms an ellipse, there will be only two points of reflection within the plane, those points lying where the minor axis of the ellipse intersects the mirror's surface (4, 5.56, p. 344). The same reasoning therefore applies to this mirror as to its convex counterpart. An infinite number of planes can be passed through the viewpoint along the normal dropped from it to the mirror so as to form elliptical sections on the exposed portion of the mirror. For each elliptical section, there is a unique minor axis intersecting the mirror's surface along the orthogonal at a unique pair of points. Such points cover the entire exposed surface of the mirror, so each of them can serve as a point of reflection for the given viewpoint.

In concave conical mirrors, finally, if the line of sight between the viewpoint and the mirror's vertex is perpendicular to the axis, precisely half the reflecting surface will be exposed to view. If, on the other hand, the line of sight forms an acute angle with the axis, more than half the reflecting surface will be exposed to view. The entire surface will be exposed to view if the viewpoint lies below the mirror's base and the line of sight coincides with the edge of the mirror or passes through the mirror's base to its vertex. By the same token, if the line of sight forms an obtuse angle with the axis, less than half the reflecting surface will be exposed to view, and none of it will be visible when the line of sight coincides with the edge of the cone or passes into the mirror through its vertex (4, 5.57, p. 344).

As for the selection of possible reflection-points, the same reasoning that applies to convex conical mirrors applies here. That is, if the plane passing through the viewpoint cuts a circle on the mirror's surface, there will be no point of reflection within that plane. If it cuts a line of longitude on the mirror's surface, every point on that line can serve as a point of reflection.[27] And if it cuts a conic section on the mirror's surface, there will be at least one and at most two points of reflection in that plane (4, 5.58, p. 344). Since an infinite number of planes of reflection can be passed through the viewpoint along the normal dropped from it to the mirror, and since each of them contains a unique set of reflection points, those points taken as a whole cover the entire reflecting surface. Every point on the exposed portion of that surface can therefore serve as a point of reflection for any given viewpoint.

Although descriptive rather than theorematic, Alhacen's analysis in chapter 5 involves complex geometrical reasoning based on mental visualization rather than on actual diagrams. At times, therefore his train of description is exceedingly difficult to follow. This is especially the case with his analysis of reflection-points in cylindrical and conical mirrors, because in those mirrors most of the planes of reflection form conic sections on the reflecting surface. The determination of reflection-points within such sections can be extraordinarily hard to visualize, particularly when it must be done in three dimensions. All of this effort seems wasted in view of Alhacen's ostensible purpose in chapter 5, which is to establish the intuitively and empirically obvious fact that reflection can occur to any facing viewpoint from every point on a reflecting surface. But within that context, Alhacen has a deeper purpose that betrays both the rigor and comprehensiveness of his approach. He means to show unequivocally that what holds for any given plane of reflection in any mirror necessarily holds for all such planes in that mirror. In other words, mathematical conclusions drawn on the basis of one, or one kind of plane of reflection will apply universally to every other plane of that kind. Accordingly, Alhacen has been at pains throughout chapter 5 to justify as fully as possible the detailed mathematical analysis of reflection to be developed in book 5 on the basis of single planes of reflection within each of the seven mirrors.

Book Five: By the end of book 4, Alhacen has established three of the four principles underlying the cathetus-rule: namely, that the plane containing the center of sight, the object-point, the reflection-point, and the image must be normal to the reflecting surface; that within this plane the angles of incidence and reflection are invariably equal; and that the image lies on the line of reflection. The first part of chapter 2 of book 5 is devoted to verifying the fourth and final principle, which puts the image on the cathetus dropped from the object-point to the reflecting surface. As with the other three principles, so with this one, the verification is empirical. Accordingly, for each mirror Alhacen proposes a variety of simple tests based on posing thin rods, cones, or needles normal to the reflecting surface, sighting along them in various ways, and observing that their images, or at least the image of their farther endpoints, lie in a direct line with the center of curvature. These tests, which pretty much speak for themselves, are described in 5, 2.1-2.5, pp. 385-387 for plane mirrors; in 5, 2.6-2.9, pp. 387-388, for convex spherical mirrors; in 5, 2.10-2.19, pp. 388-391, for convex cylindrical mirrors; in 5, 2.20, p. 391, for convex conical mirrors; in 5, 2.21-2.26, pp. 391-393, for concave spherical mirrors; in 5, 2.27-2.31, pp. 393-394, for concave cylindrical mirrors; and in 5, 2.32, p. 394, for concave conical mirrors.

Alhacen follows this empirical verification with a theoretical explanation of why the image appears on the cathetus rather than on some other line dropped from the object-point to the mirror. The gist of his argument is that, since images in plane mirrors are perfect replicas of their objects in size and distance from the reflecting surface, they must lie on the cathetus, because if they did not, they would appear smaller or larger than they should (5, 2.35-2.36, pp. 395-396). This argument is based on the assumption that we perceive images as if they were actual objects in front of us and judge their size and distance accordingly (5, 2.33-2.34, pp. 394-395). Thus, if the image-location were to lie on OI′ outside normal OAI in the top left-hand diagram of figure 5.5, p. 218, it would appear farther away and smaller than it should from viewpoint E, whereas if it were to lie inside that normal along OI″, it would appear closer and larger than it should. Alhacen then extends this argument to convex spherical mirrors in 5, 2.37-2.40, pp. 396-397, the point being to show that, in curved mirrors, the image, however distorted it may be, would not appear as it does were it to lie on a line other than the cathetus.

In the next section, Alhacen gives a brief description of the cathetus-rule and some of its implications for image-formation in plane mirrors (5, 2.42, p. 397), convex spherical mirrors (5, 2.43, pp. 397-398), convex cylindrical mirrors (5, 2.44, p. 398), concave spherical mirrors (5, 2.45, p. 398), and concave cylindrical and conical mirrors (5, 2.46, pp. 398-399). Alhacen's purpose here is to lay out various points to be dealt with later in his mathematical analysis of reflection. Most important among these is that in plane and convex mirrors there can only be one point of reflection for any given viewpoint and object-point, whereas in concave mirrors there can be as many as four.

The remainder, and by far the lion's share of book 5 consists of 54 propositions divided into seven unequal sets according to the type of mirror analyzed. Within each set Alhacen addresses three basic problems: how to determine image-location when the object-point, the center of sight, and the point of reflection are given; how to determine the reflection-point(s) when the center of sight and the object-point are given; and how to determine the number of reflection-points for any given center of sight and object-point. Propositions 1-4, pp. 399-403, deal with plane mirrors; propositions 5-25, pp. 403-432, with convex spherical mirrors; propositions 26-29, pp. 432-438 with convex cylindrical mirrors; propositions 30-31, pp. 438-446, with convex conical mirrors; propositions 32-49, pp. 446-475 with concave spherical mirrors; propositions 50-52, pp. 475-482, with concave cylindrical mirrors; and propositions 53-54, pp. 482-485, with concave conical mirrors.

The first 18 propositions of book 5, which deal with various aspects of reflection from plane and convex spherical mirrors, are fairly straightforward and unremarkable. Starting with proposition 19, pp. 415-419, however, there follows a set of six lemmas that are instrumental in one way or another to the subsequent determination of reflection-points in convex and concave mirrors. Of the six, however, only four—i.e., lemmas 3-6—are directly implicated in those determinations; the other two play ancillary yet critical roles in laying the requisite groundwork for them.

In the first lemma, which is dealt with in proposition 19, Alhacen shows how to generate a line extending from some randomly chosen point on a circle to the extension of its diameter such that the segment of this line between where it intersects the circle and where it intersects the diameter is equal to some randomly chosen line. Thus, as illustrated in figure 4, p. 524, a line must be extended from point A on the circle to meet the extension of diameter BG at point D in such a way that the segment between where it intersects the circle and point D is equal to randomly chosen line QE.

Alhacen subdivides this problem into four cases. In the first case, as illustrated in the upper left-hand diagram of figure 4, A is the point on the circle where lines AG and AB intersect so as to be equal. The objective here is to generate line AD from A to the extension of diameter BG such that HD = QE. The second case actually consists of three subcases according to the inequality of AG and AB. In the first subcase, which is illustrated in the upper right-hand diagram of figure 4, AG < BG in such a way that AD will be tangent to the circle. In that case, AD itself is to be produced equal to QE. The construction and proof in this particular subcase is noteworthy, because it depends on certain properties of hyperbolic sections (see esp. 2.147, p. 417). In the last two subcases, AG is either greater than or less than BG in such a way that AD intersects the circle. In the former case, illustrated in the lower left-hand diagram of figure 4, AD passes through point H on the circle such that HD = QE. In the latter case, illustrated in the lower right-hand diagram of figure 4, AD is extended to point H opposite point D such that DH = QE. This lemma comes into play only once in subsequent analysis (proposition 21, lemma 3, pp. 420-422—see esp. 2.169, p. 421).

Proposition 20, lemma 2, pp. 419-420, shows how to drop a line from some point on the circumference of a circle through the diameter such that the segment of this line from where it intersects the diameter to where it intersects the opposite arc on the circle is equal to some randomly chosen line. Thus, as illustrated in figure 5, p. 525, the objective is to drop line AD from randomly chosen point A through diameter GB of the circle so that ED equals randomly chosen line HZ. In all but one case, two such lines, i.e., AD and AD′ in the top diagram, will meet the specified condition, both DE and D′E′ being equal to HZ. The exception is illustrated in the lower diagram of figure 5, where AD passes orthogonal to diameter GB through the circle's center and therefore forms a diameter in that circle. Since no other line dropped from A through diameter GB can be as long as AD, no segment on that other line can be as long as ED. ED is therefore unique. Like the previous lemma, this one also depends on certain properties of hyperbolic sections. Unlike that lemma, this one is applied directy more than once to subsequent analysis (proposition 23, lemma 5, pp. 425-426—see esp. 2.187, p. 425—and proposition 24, lemma 6, pp. 426-427—see esp. 2.194 and 2.197, p. 427). It is also applied indirectly, but in a crucial way, in proposition 46, pp. 467-470.[28]

The point of proposition 21, lemma 3, pp. 420-422, is best explained in the context of figure 6, p. 526. Take right triangle ABG, and choose some point D on either of the legs forming right angle ABG. Let BG be the selected leg. As is evident from the figure, D can lie on the leg itself (as in the left-hand diagram at the top) or outside the triangle on the extension of the leg (as in the remaining two diagrams). From point D a line is to be drawn to or through point T on hypotenuse AG so as to pass to or through point Q on the opposite leg of right angle ABG in such a way that TQ is to TG as some randomly chosen line E is to some other randomly chosen line Z—i.e., TQ:TG = E:Z. Alhacen applies this lemma twice in subsequent analysis (proposition 22, lemma 4, pp. 422-425—see esp. 2.175, p. 422—and proposition 38, pp. 458-459—see esp. 2.375, p. 458).

In proposition 22, lemma 4, pp. 422-425, the problem is as follows: given two randomly chosen points, such as E and D in figure 7, p. 527, and given a circle, to find point A on that circle where the line tangent to that point bisects the angle formed by the lines extending from the given points to that point. Thus, as illustrated in the figure, the problem posed in this lemma is to find point A at which tangent AH bisects angle EAD. This lemma is used only once in subsequent analysis (proposition 31, pp. 441-446—see esp. 2.295, p. 445).

The penultimate lemma, proposition 23, lemma 5, pp. 425-426, entails dropping a line from outside a circle to a radius in that circle such that the segment of that line between where it intersects the circle and where it intersects the radius is equal to the segment of the radius between this latter point of intersection and the circle's center. Given the circle with radius GB in figure 8, p. 527, and given point E outside it, the construction calls for extending line EZ from E through point D on the circle to point Z on the radius such that DZ = GZ. Like the previous lemma, this one comes into play only once in subsequent analysis (proposition 31, pp. 441-446—see esp. 2.290, p. 444).

The sixth and final lemma, proposition 24, pp. 426-427, is really just a special case of proposition 21, lemma 3. As in that case, some point D in figure 9, p. 528, is taken on leg BG of the right angle in triangle ABG. From it a line is dropped to point T on the other leg of the right angle, and that line is extended in the opposite direction to point Q on hypotenuse AG so that TQ:QG = E:Z, E and Z being randomly chosen lines. As is shown in the figure, two such lines, TDQ and T'DQ', can be extended from point D to fulfill the requisite proportionality. This lemma is applied twice in subsequent analysis (proposition 25, pp. 427-432—see esp. 2.200 and 2.201, pp. 427-428—and proposition 47, pp. 470-471—see esp. 2.464, p. 471).

Immediately following this set of six lemmas, the problem of finding the point of reflection on a convex spherical mirror faced by a center of sight and an object-point that are chosen at random is taken up in proposition 25, pp. 427-432. This same problem is addressed in proposition 29, pp. 437-438, for convex cylindrical mirrors and in proposition 31, pp. 441-446 for convex conical mirrors. In the case of the three concave mirrors, the problem is considerably more complicated, because in those mirrors there can be as many as four points of reflection, depending on how and where the center of sight and the object-point are disposed with respect to one another, as well as to the reflecting surface. Accordingly, the determination of all possible reflection-points for concave spherical mirrors requires four separate solutions, which are given in propositions 36, 37, 38 and 47, pp. 452-459 and 470-471. On the basis of these solutions, the points of reflection are determined for concave cylindrical mirrors in proposition 52, pp. 478-481, and for concave conical mirrors in proposition 54, pp. 482-485. These propositions lie at the very heart of Alhacen's account of reflection, those devoted specifically to convex and concave spherical mirrors forming the core of what has come to be known as "Alhazen's Problem." Because of their complexity, I have remanded the detailed analysis and explanation of all of them to a separate section (pp. xlvi-lxvi below) in order not to disrupt the narrative flow of this section unduly.

Aside from 25, 29 and 31 (see pp. xlvii-lvi below for a detailed analysis), the remaining set of seven propositions devoted to convex mirrors—i.e., 25-31—is straightforward enough to warrant no discussion (see pp. xlvii-lii below for a detailed analysis of proposition 25). Starting with proposition 32, however, the set of 18 theorems devoted to concave spherical mirrors does merit discussion. As a whole, these theorems specify the conditions under which reflection will occur from one, two, three, or four points in a given plane of reflection within the mirror. As mentioned earlier, this specification will depend on how the center of sight and the object-point are disposed with respect both to one another and to the reflecting surface.

As far as their disposition with respect to one another is concerned, three factors come into play. First, the center of sight and object-point may or may not lie on the same normal. Second, if they do, they may or may not be equidistant from that centerpoint. And third, they may or may not lie on opposite sides of the mirror's centerpoint. As far as their disposition with respect to the reflecting surface is concerned, that depends on whether they lie on different normals. If they do, they face opposite arcs on the mirror according to the intersection of the normals at the mirror's centerpoint. The various dispositions just specified are illustrated in figure 10, p. 528, where B and B′ represent object-points, A a center of sight, and G the center of the mirror. In the left-hand diagram B, B′ and A lie on the same normal on opposite sides of G, and BG = AG, whereas B′G ≠ AG. In the right-hand diagram, the two object-points and the center of sight lie on different normals LF and KH, which necessarily intersect at G. As before, BG = AG, and B′G ≠ AG. In this case, reflection can occur from B to A, or from B′ to A from arc KDL facing angle KGL, or from arc LD′K facing angle LGK.

Alhacen deals with both these cases in proposition 34, pp. 450-451. For the situation in which B, B′, and A lie on the same normal, as represented in the left-hand diagram of figure 10, he demonstrates that within the plane of the circle (which constitutes a great circle on the sphere from which the mirror is formed) reflection can occur from B or B′ to A at two corresponding points D and D′ on the mirror within respective arcs of the circle. Overall, then, reflection can occur from every point in the circle generated on the mirror's surface by the rotation of D and D′ about normal AGB as axis.

Things are somewhat different when B and B′ do not lie on the same normal with A, as represented in the right-hand diagram of figure 10. Although in that situation there will be reflection from B or B′ to A from some points D and D′ on opposite arcs of the circle, points D and D′ for B′ and A do not lie at corresponding positions within their respective arcs. Hence, D and D′ for B′ and A will be specific to the plane of reflection, which is to say that reflection will not occur from a circle generated on the mirror's surface by the rotation of D and D′ about an axis. Furthemore, if B and A lie on the same normal and on the same side of G, as represented in the left-hand diagram of figure 11, p. 529, then no reflection can occur, because the normal dropped from centerpoint G to the supposed reflection-point D will not bisect angle BDA, leaving angle of incidence BDG ≠ angle of reflection ADG. By the same token, when A and B lie on different normals, as represented in the right-hand diagram of figure 11, reflection cannot occur from arcs LH or KF for the same reason. Hence, in that case, reflection is restricted to arcs KL and HF.

Once the method for determining points D and D′ for any center of sight A and any object-point B on the same normal is given in proposition 36, pp. 452-454 (see pp. lvi-lvii below for a detailed analysis), the analysis of reflection for the first case is complete. The remaining analysis of concave spherical mirrors centers on the case in which A and B lie on separate normals. Accordingly, in proposition 37, pp. 454-458, Alhacen shows, first, that if A and B lie outside the mirror on different normals, only one reflection can occur within the given plane. On the other hand, he continues, if those points lie inside the mirror, and if they are equidistant from the mirror's centerpoint, there can be as few as two or as many as four (but not three) reflections, depending on whether the circle passing through the center of sight, the object-point, and the center of the mirror intersects arc KL.

Thus, as illustrated in figure 12, p. 529, the circle passing through A, G, and B intersects arc KL at points D_1 and D_2, and these constitute legitimate points of reflection for A and B (see pp. lvii-lix below for a detailed analysis). So in this case there will be four points of reflection: D_1, D, and D_2 on arc KL, and D′ on arc HF. As Alhacen points out, circle AGB will intersect arc KL if and only if lines BD and AD drawn from the object-point and the center of sight to the midpoint of arc KL form acute angles with their respective normals on the side of G. Thus, when the circle is either tangent to arc KL, as represented by circle A′GB′ passing through viewpoint A′ and object-point B′, or when it falls short of that arc, reflection can occur from only two points, one of which is D on arc KL, the other D′ on arc HF. And in that case, as is clear from the diagram, lines A′D and B′D form obtuse angles (i.e., DA′G and DB′G) with their respective normals. Since, therefore, circle AGB necessarily touches or intersects arc KL at one or at most two points (or none if it falls short), and since in the one case (i.e., when circle AGB is either tangent to or falls short of arc KL) there will be one reflection only on arc KL, whereas in the other case there will be three on that arc, reflection cannot possibly occur from three points (including D′ on arc HF) in the entire circle. Implicit in this analysis, of course, is the method for finding the appropriate points of reflection according to the bisection of arcs KL and HF by line DGD′ and the intersection of arc KL by circle AGB (see pp. lviii-lix below for a detailed analysis).

Starting with proposition 38, pp. 458-459, the most complicated situation of all—that in which the center of sight and the object-point lie on different normals at different distances from centerpoint G of the mirror—becomes the focus of analysis. After showing in that proposition how to determine point of reflection D′ on arc HF when A and B lie at different distances from G, as illustrated in figure 13, p. 529 (see pp. lx-lxii below for the detailed analysis), Alhacen demonstrates in proposition 39, pp. 459-460, that any such point D′ on arc HF, except for point Z on line XZ bisecting angle KGL, can serve as a reflection-point for an infinite number of point-pairs A and B on normals KH and LF—provided, of course, that those points are not equidistant from G. He then establishes in proposition 40, p. 460, that, if D′ is the point of reflection for specific points A and B on those normals, there can be no reflection from B to A at any other point on arc HF.

So much for arc HF, now to arc KL. In proposition 41, p. 461, Alhacen shows that, if A in figure 14, p. 530, is given on normal KH, and if normal LF is also given, there will be some point of reflection such as D or D′ on arc KL for some object-point such as B or B′ on normal LF. Or, to put it another way, from every point on normal LF reflection will occur to point A at some point on arc KL. Alhacen goes on in proposition 42, pp. 461-462, to establish that, when both BG and B′G in figure 14 are unequal to AG and when reflection occurs from both B and B′ to A at points D and D′, respectively, angles BDA and B′D′A may be larger or smaller than angle LGH adjacent to angle KGL subtended by arc KL. Then, in proposition 43, pp. 462-463, he demonstrates that, while angles BDA and B′D′A can be either larger or smaller than angle LGH, they can never be equal to it under the specified conditions—i.e., with BG and B′G ≠ AG. On the basis, finally, of the points established in the preceding two propositions, Alhacen demonstrates in proposition 44, pp. 463-465, that, if reflection occurs from B to A at two points D and D′ on arc KL, as illustrated in figure 15, p. 530, angles BDA and B′D′A cannot both be smaller than angle LGH. As will become clear shortly, this does not mean that either of the angles need be smaller than LGH; they may both be larger.

The next two propositions, which are absolutely critical to the analysis of reflection from arc KL, are somewhat confusing, because their true intent is not clear from the way they are presented. In proposition 45, pp. 465-467, for instance, the ostensible point to be demonstrated is that, for any two points A and B lying unequal distances from G on normals KH and LF, reflection can occur from one to the other at two points on arc K. Implicit in the construction on which the proof is based, however, is that A and B in figure 16, p. 531, must be disposed on their normals in such a way that circle AGB passing through them will intersect arc KL. Implicit as well is that the lines from A and B to midpoint X of arc KL must form acute angles with their respective normals on the side of G. Accordingly, the points of intersection C and E must lie on opposite sides of X, where GX bisects angle KGL and arc KL along with it. If, however, circle AGB fails to intersect arc KL, or if it is merely tangent to it, as represented in figure 16a, only one reflection is possible, in which case angle BDA formed at the point of reflection will be smaller than angle LGH. This follows from the fact (demonstrated in Euclid, III.22) that angle BTA at tangent point T on circle AGBT and angle AGB at the opposite vertex of quadrilateral ATBG in the same circle sum up to two right angles, as do KGL and adjacent angle LGH. But we know from proposition 43 that reflection cannot occur from an angle, such as BTA, that is equal to LGH, so T cannot be a legitimate point of reflection. We also know that all angles, such as BDA, that intersect on arc KL outside tangent circle BGA will be more acute than BTA and thus more acute than LGH. Therefore, angle BDA must be smaller than LGH.

Likewise, if circle AGB in figure 16b, p. 532, intersects arc KL at bisection-point X, no reflection can occur within the segment of arc KL between X and point C of intersection to the left of it. This fact is not immediately apparent, because any angle, such as BD′A, within that segment of arc KL will be greater than angle LGH and will therefore fulfill the specification of proposition 43 that both angles BDA and BD′A not be smaller than LGH. So let us suppose that D′ is a legitimate point of reflection. The test of whether it is follows from certain points established earlier in proposition 44 and can be easily understood by recourse to point D in figure 16b. Let D in figure 16c, p. 532, represent that same point on the same circle KLHF with A and B posed on their respective normals as before. For a start, we know by previous conclusions that angle BDA < angle LGH, and we know by construction that angle of incidence BDG = angle of reflection ADG. Draw line AB connecting A and B, bisect it at M, and construct circle ADB passing through B, A, and point of reflection D. Draw PMN orthogonal to AB through midpoint M to form a diameter in circle ADB. PMN will therefore bisect arc AB at point P, leaving arc AP = arc BP. From Euclid, III.27 we know that equal arcs subtend equal angles, so we know that angle BDP subtended by arc BP is equal to angle ADP subtended by arc AP. Therefore, DP bisects angle BDA. But since normal DG also bisects that angle, DP must coincide with that normal. Therefore, point P will lie at the intersection of normal DG and circle ADB.

Applying the same test to D′ in figure 16b, p. 532, we locate that same point D′, in figure 16d, p. 533, on the same circle KLHF with A and B posed on their respective normals as before. We then construct circle AD'B, connect A and B with line AB, bisect it at point M, and produce diameter PMN through point M. Being orthogonal to AB at M, PMN bisects arc AB at P, leaving arc AP = arc BP. Therefore, D'P bisects angle BD'A. But D'P does not coincide with normal D'G, so D'G does not, bisect angle BD'A, leaving angle of incidence BD'G unequal to angle of reflection AD'G. D' has therefore failed the test. Moreover, under the specified conditions, every other point D' chosen to the left of X on arc KL will fail this test.[29] Unless circle AGB intersects arc KL on both sides of point X, then, reflection can occur from only one point D on arc KL, and the resulting angle BDA < angle LGH. However, as mentioned just above, having circle AGB intersect arc KL on both sides of X is not enough. Lines BX and AX dropped from the object-point and the center of sight to the midpoint of arc KL must also form acute angles with their respective normals. So the ulterior intent of proposition 45 is to prove that, as long as circle AGB does intersect arc KL on both sides of X, and as long as both BX and AX form acute angles with their normals, there can be *at least* two reflections from that arc.

Against this background, both the purport and significance of proposition 46, pp. 467-470, become crystal clear. For in that proposition Alhacen proves that, when there is one point of reflection D on arc KL such that angle BDA > angle LGH, there will necessarily be another point D' on that arc such that angle BD'A > angle LGH. The proof itself turns on the fact, demonstrated in proposition 20, lemma 2, that two lines, such as AD and AD' in figure 5, p. 525, can be drawn from a given point A such that segments ED and E'D' will be equal.[30] As was shown in the previous theorem, any such point of reflection must lie on the segment of arc KL between midpoint X and the point to the right of X where circle AGB intersects it (i.e., between X and E in figure 16, p. 531). Consequently, Alhacen actually proves two things in this proposition, one explicitly, the other implicitly. Explicitly, he demonstrates that, if there is one point D of reflection such that angle BDA > angle LGH, there must be another point D' such that angle BD'A > angle LGH. Implicitly, he demonstrates that, if circle AGB intersects arc KL to the right of X at some point E, there will necessarily be two points of reflection D and D' within segment EX of that arc such that both angles BDA and BD'A are greater than angle LGH.

As he sums them up in proposition 49, pp. 472-475, the points established in propositions 44-46 are as follows. First, if circle AGB fails to intersect arc KL to the right of midpoint X, there can only be one point of reflection D. It will lie on segment CK of arc KL in figure 16b, p. 532, and it will yield angle BDA < angle LGH. Second, if circle AGB does intersect arc KL to the right of midpoint X, there will be two points of reflection D and D' on segment EX of arc AK in figure 16, p. 531, and they will yield angles BDA and BD'A > angle LGH. Third, when circle AGB cuts arc KL to the left of midpoint X, as in figure 16, p. 531, there will be a third reflection from some point on segment KC of arc KL such that it yields an angle < FGD. If, however, circle AGB intersects arc KL at point K (which would thus coincide with A) or fails to cut arc KL to the left of midpoint X, then, as long as arc XE in figure 16 is exposed to point A, there will be two reflections from segment XE but none from any point on remainder KX or LE of arc KL. And, finally, if there is a third reflection, it must occur within the segment lying on the opposite side of X from the segment within which the other two reflections occur. Thus, as represented in figure 16, since points D and D' lie on segment EX to the right of X, the third reflection must occur from segment CK to the left of X, not from segment EL to the right. That this must be the case follows from the proof that, although angle BD'A in figure 16b, p. 532, is larger than angle LGH, no point D' within segment XC of arc KL can be a legitimate point of reflection if point D on arc CK is a legitimate point of reflection (see pp. xli-xlii above).[31]

Consequently, by the end of proposition 46, Alhacen has established definitively both that and under what conditions there can be one, two, or three reflections from arc KL when AG ≠ BG. If there is only one, it will yield an angle BDA < angle LGH, if there are two, they will yield angles BDA and BD'A > angle LGH, and if there are three they will yield one angle BDA < LGH and two angles > angle LGH. All that is left to do is show how to determine those points of reflection for any properly disposed O and E. This Alhacen does in proposition 47, pp. 470-471, for the two reflection-points that yield angles > LGH (see pp. lxiii-lxiv below for a detailed analysis), but in the text as it stands, no such determination is given for the point of reflection yielding an angle less than LGH. That determination can be easily reconstructed on the basis of the method provided in proposition 38 for determining the point of reflection on arc GD opposite arc KL (see note 138, pp. 513-514, for a detailed analysis)

Compared to the cluster of five propositions culminating in 47, the last six theorems of book 5—including propositions 52 and 54, where the points of reflection are determined for concave cylindrical and conical mirrors (see pp. lxiv-lxvi below for a detailed analysis)—are at best anticlimactic and therefore call for no discussion. What does remain to be discussed before this overview concludes is a set of three issues that crop up earlier in book 5. The first has to do with the location of images seen along the normal. As Alhacen points out in several places, these particular images can only be of the spot on the cornea through which the orthogonal line of sight passes to the mirror (see, e.g., propositions 2 and 8, pp. 399-401 and 404-405). According to the cathetus-rule, such images should lie where the cathetus dropped from that spot on the cornea intersects the line of reflection. But in this case the two lines coincide, so there is no point of intersection. How, then, do we judge the location of images along that line? Alhacen's response is that we do so within the context of proximate points, whose images are defined by appropriate intersections. Thus, when we look straight down into a mirror, we locate the image of the cornea's centerpoint perceptually by referring it to the images of surrounding points whose locations are definite. That way we perceive the image of the cornea's centerpoint to lie the same distance from the center of sight as the images of the rest of the points on the cornea's surface (see, 5.2.38, pp. 396-397). The ulterior point here is that, even when the image has no definite location according to the cathetus-rule, it will be perceptually located as if it did.

The second issue is why we see a single image with both eyes in curved mirrors when each center of sight receives the image of any given object-point from a different reflection-point on the mirror's surface and therefore perceives it along a different line of reflection. As a result, the object's image-location is different for each eye. Why, then, is it not perceived double? The answer, according to Alhacen, is that, under normal conditions, the separate image-locations are so close that the visual faculty naturally melds both images together to make a single composite from them (see, e.g., 5.2.221, p. 432). Therefore, as far as the cathetus-rule is concerned, what obtains in single-viewpoint vision essentially obtains in binocular vision.

The third and final issue involves image-formation in concave mirrors. The problem here is that, although there should be an image for every point of reflection, this does not always seem to be the case in concave mirrors, where some images are clearly apparent while others are not. As Alhacen shows in proposition 32, pp. 446-449, this problem can be resolved geometrically. Suppose that O_5R in the lower diagram of figure 5.5, p. 218, represents a line of incidence, RE a line of reflection, E a center of sight, and C the center of the mirror. When the form of object-point O_1 reflects from R to E, normal CO_1 will intersect line of reflection RE at I_1 behind the mirror. When, however, the form of O_2 reflects to E, normal CO_2 will be parallel to line of reflection RE, so there will be no intersection and thus no definite image for O_2. According to cathetus O_3C, meanwhile, the image of O_3 will lie at I_3 beyond the eye, whereas the image of O_4 will lie at the center of sight itself, because its cathetus O_4C passes through point E. The image of O_5 finally, will lie at I_5 between the reflecting surface and the center of sight. Consequently, of all five images, only two—I_1 and I_5—will be clearly perceived. The rest will either go unperceived, or they will be seen on the reflecting surface itself, in which case all that is seen on that surface is a blur the same color as the object whose image it is. Consequently, in concave mirrors, the apparent lack of images for certain reflection-points is only apparent because of the physical conditions under which they can or cannot be seen.

2. *Alhacen's Solutions to "Alhazen's Problem"*

In 1669, on the basis of his reaction to book 5 of the *De aspectibus*, Christiaan Huygens formulated the following problem, which quickly gained currency as "Alhazen's Problem": *Given a spherical convex or concave mirror, and given a point of sight and a point on a visible object, to find the point of reflection on the surface of the mirror*.[32] This problem exercised not only Christiaan Huygens but also several other seventeenth-century mathematicians, their interest piqued by impatience with Alhacen's solution of it. Or, rather, I should say "solutions," because Alhacen was forced to approach the problem from several different directions when it came to concave spherical mirrors. The root of his difficulty lay in the fact (already discussed at length in the previous section) that, depending on how the object-point and the center of sight are disposed with respect to the center of curvature in such mirrors, there can be as many as four points of reflection. Furthermore, both the number and determination of these points depend on how the object-point and the center of sight are disposed with respect to the reflecting surface itself.

Faced with an extraordinarily complex phenomenon, then, Alhacen felt compelled—and understandably so—to address it on a case-by-case basis. Moreover, he was concerned not only with convex and concave spherical mirrors, but with convex and concave cylindrical and conical mirrors as well. Alhacen's problem, in short, was considerably broader in scope and more demanding than "Alhazen's Problem." In this section, we will examine Alhacen's approach to this problem in its full scope, dealing in order with convex spherical, convex cylindrical, convex conical, concave spherical, concave cylindrical, and concave conical mirrors. Insofar as possible, we will follow Alhacen's actual train of analysis, although at times it will be necessary to disrupt that train for the sake of clarity.

Convex Spherical Mirrors: In proposition 18, p. 415, Alhacen addresses the problem of how to find the point of reflection on the surface of a convex spherical mirror when the point-source of radiation and the center of sight are equidistant from the mirror's center. The solution to this problem is so simple as to be trivial. Let B in figure 17, p. 534, be the object-point, A the center of sight, and G the center of the mirror. Pass a plane through all three points to form a great circle on the mirror. Draw normals AG and BG, and bisect angle AGB with line GDE. That D is the point of reflection is clear from the fact that triangles BDG and ADG are equal, as are their corresponding angles BDG and ADG. Hence, the adjacent angles BDE (the angle of incidence) and ADE (the angle of reflection) are equal.

Having set the stage with the series of six lemmas (propositions 19-24, pp. 415-427) discussed earlier on pp. xxxvi-xxxviii above, Alhacen addresses the same problem in proposition 25, pp. 427-432, but this time with points A and B lying different distances from the mirror's center. Thus, as illustrated in figure 17a, p. 534, normals AG and BG are not equal. Unlike the previous case, this one is far from trivial, and Alhacen's method for solving it is commensurately complex.

Take random line D′M′, and divide it at point Q′ so that M′Q′:Q′D′ = BG:AG. Bisect line D′M′ at point N′, and draw perpendicular B′N′T′ through it. From endpoint D′ of line D′M′ draw D′T′ to B′N′T′ to form angle D′T′N′ equal to half of angle BGA. As given in the figure, GE′ represents the actual bisector of angle BGA, so angle D′T′N′ = angle BGE′. Through point Q′ on D′M′ draw a line meeting B′T′ at B′ and D′T′ at G′ such that B′G′:G′D′ = BG:GD (i.e., the radius of the great circle on the mirror). This Alhacen has shown us how to do in the preceding theorem (proposition 24, lemma 6, pp. 426-427). Line B′G′ will form angle B′G′D′ with line D′T′. At the center of the circle form angle BGE equal to B′G′D′. Point D, where leg GE of that angle intersects the circle, will be the sought-after point of reflection.

Alhacen's proof that D is in fact the point of reflection is based on figure 17b, p. 535. The gist of the proof—and here I will take some liberties with his actual procedure—is as follows. For a start, it is clear by construction that triangles BGD and B′G′D′ are similar. Draw line BT to form angle BTD equal to half of angle AGB. Drop perpendicular DN to line BT, and continue DN to M so that DN = MN. DM will intersect BG at point Q. Accordingly, the entire figure consisting of triangle BDGT and line DQNM highlighted in bold will be similar to the entire figure consisting of triangle B′D′G′T′ and line D′Q′N′M′ to the left of the circle. Thus, angle BTD = angle B′T′D′ = half of angle BGA = angle E′GB (or E′GA) from the previous figure. From this it follows that angle GBT = angle E′GE from the previous figure—i.e., the difference between angle BGD and half of angle AGB.

Draw line BM to form triangle BMN. Since MN = DN, by construction, and since angles BNM and BND are right, and thus equal, by construction, triangle BMN = triangle BND. Since, moreover, angle DBN = angle DBG + angle E′GE from the previous figure (i.e., the difference between angle BGD and half of angle AGB), and since angle E′GE = angle GBT, angle DBM =

2DBG + 2GBT. From point D draw line DS parallel to BM, and extend BG to meet it at point S. Since vertical angle BQM = vertical angle DQS, and since alternate angles BMD and MDS are equal, triangles BQM and DQS are similar, leaving their respective sides proportional. Hence, BQ:QS = MQ:QD. But angle BMQ = angle BDQ, since triangle BMD is isosceles by construction. Therefore, given the equality of alternate angles BMQ and QDS, it follows that angle BDQ = angle QDS, so DQM bisects angle BDS. Therefore, by Euclid, VI.3, BD:DS = BQ:QS. But we have already established that BQ:QS = MQ:QD, and MQ:QD = BG:GA by construction, so BQ:QS = BG:GA.

Draw line DP such that angle PDS = angle BGA, and draw XDY tangent to the circle at point D. The remainder of the proof depends on showing that vertical angle XDP, which equals vertical angle ADY, also equals angle BDX. This follows from the fact that angle XDQ = angle NTD, because XDQ + QDG (i.e., NDT) sums up to a right angle, and so does NTD + QDG (i.e., NDT). Thus, since angle NTD = one-half angle BGA by construction, angle XDQ = one-half angle BGA. Now, angle PDS = angle BGA by construction, so angle QDS = angle BGA – angle PDQ. But angle QDS = angle BDQ, since QD bisects angle BDS, so angle BDQ + angle PDQ = angle BGA. Since angle BDQ = angle BDX + angle XDP + angle PDQ, then angle BDX + angle XDP + 2 angle PDQ = angle BGA. But we have already established that angle XDQ (which = angle XDP + angle PDQ) = one-half angle BGA. Thus, angle XDB + angle PDQ + one-half angle BGA (i.e., angle XDP + angle PDQ) = angle BGA. It follows, then, that angle XDB + angle PDQ = one half angle BGA, so angle XDB = angle XDP.

The next step is to demonstrate that, if PD is extended toward point A, it will intersect GA at that very point. That it does rests on showing that the resulting triangle PAG will be similar to triangle PDS, which follows from the fact that angle PDS = angle BGA by construction, while angle APG is common to both triangles. Therefore, PDA is a straight line, so vertical angles XDP and ADY are equal. But angle XDB = angle XDP, so angle XDB = angle ADY. Accordingly, given that angles XDE and YDE are both right by construction, and given that their parts XDB and ADY are equal, it follows that remainders BDE (the angle of incidence) and ADE (the angle of reflection) will be equal.

The key to Alhacen's method for finding D in this case is the construction of triangle B'G'D' with constituent angle B'G'D', and the construction of similar triangle BDG with constituent angle BGD at the center of the mirror. And the key to this construction lies in dividing line D'M' into segments M'Q' and Q'D' that are proportional to normals BG and AG. Both that and how this expedient works is clear enough. Far less obvious is how and why Alhacen battened on to it in the first place. Did some unfathomable flash of inspiration lead him to realize that forming triangle BGD according to the conditions just laid out would yield the appropriate angle B′G′D′? Perhaps so, but there is a more mundane explanation based on the particular construction he used for the proof and some fairly obvious patterns that emerge from that construction. In order to uncover these patterns, we need to backtrack from the construction and proof as completed to the initial grounds upon which they are laid.

Let us therefore take point D as given, and let us begin with the limiting case in which A and B are equidistant from the mirror's center, as represented in figure 17c, p. 535. Since normal GDE bisects angle AGB, there is no difference between angle BGD and half of angle AGB. Therefore, line BNT from the previous figure will coincide with line BQG, so N will coincide with Q, and G with T. Carry out the construction as before, drawing line DQ perpendicular to BT (which coincides with BG) and extending perpendicular DN (which coincides with DQ) to point M so that MN (i.e., MQ) = DN (i.e., DQ). On base MN form triangle BMN equal to triangle BND. Draw DS parallel to BM and extend BG to meet it at S. Triangles BMN and DNS will thus be similar (through equality) leaving their corresponding sides proportional. Hence, BN:NS = BD:DS, which is to say that BQ:QS = BD:DS, from which it follows that DQ bisects angle BDS (by Euclid, VI.3). Likewise, since GE bisects angle AGB, BE:EA = BG:GA. Thus, if we connect line AS and draw a line parallel to it from E, that second line will pass through point Q, cutting both lines BA and BS equiproportionally. In other words, BE:EA (which = BG:GA) = BQ:QS.

Triangles BGA and BDS therefore correspond in two crucial ways: the lines bisecting the angles at their respective vertices G and D cut equiproportional segments from them (i.e., BE:EA = BG:GA, and BQ:QS = BD:DS), and the segments cut off by those bisectors on respective bases BA and BS are equiproportional (i.e., BE:EA = BQ:QS). Therefore, BQ:QS = BG:GA. But triangles BMQ and DSQ are similar (through equality), so their corresponding sides are proportional, leaving BQ:QS = MQ:QD. Hence, MQ:QD = BG:GA.

If, therefore, we apply Alhacen's method for finding D in this case, we proceed as follows. Take random line D′M′, and divide it at point Q′ such that M′Q′:Q′D′ = BG:AG. Since BG = AG, M′Q′ = Q′D′, so line D′M′ is already bisected (i.e., points N′ and Q′ from figure 17b coincide). Through point Q′ pass a line perpendicular to M′D′, and from point D′ draw line D′T′ to that perpendicular so as to form angle B′T′D′ = angle BGD = one-half angle AGB. From point B′ on the perpendicular draw line B′G′ through point Q′, intersecting D′T′ at point G′ so that B′G′:G′D′ = BG:GD (i.e., the radius of the great circle on the mirror). When angle BGD is formed at the center of the mirror equal to the resulting angle B'G'D', point D where leg GD of that angle intersects the circle will be the point of reflection.

Now, using this simple limiting case as a guide, let us return to the original case in which AG and BG are unequal, and let us compare this case to the one just discussed. These two cases are illustrated in figure 17d, p. 536, with the limiting case backgrounded in light gray. Let A_1 be the new center of sight on line of reflection AD, and let B remain the object-point. Since A_1 is on the same line of reflection as in the previous case, point D is given. Bisect angle A_1GB with line GE_1, connect BA_1, and let bisector GE_1 intersect it at E_1. Again, from Euclid, VI.3 we know that bisector GE_1 cuts BA_1 in such a way that $BE_1:E_1A_1 = BG:GA_1$. At point B form angle GBN_1 equal to angle E_1GE (i.e., the difference between half of angle A_1GB and angle BGD), and extend BN_1 to meet DG at T_1. Hence, angle BT_1D = angle BGE_1 = half of angle BGA_1. From point D draw DN_1 perpendicular to BT_1, and extend it to M_1 so that $M_1N_1 = N_1D$.

Carry out the construction as before, drawing DS_1 parallel to BM_1 and extending BG to meet it at point S_1. Extend BT_1 to meet the extension of DS_1 at V_1. Accordingly, because of the equality of corresponding angles and sides, triangle DN_1V_1 = triangle BN_1M_1, which equals triangle BDN_1 by construction, so triangle BDN_1 = triangle V_1DN_1. Consequently, DN_1 bisects angle BDV_1. Earlier we established that $BE_1:E_1A_1 = BG:GA_1$, so, if we draw a line from point E_1 parallel to A_1S_1, it will pass through point Q_1, cutting lines A_1B and S_1B equiproportionally. Hence, $BE_1:E_1A_1 = BQ_1:Q_1S_1$, from which it follows that $BQ_1:Q_1S_1 = BG:GA_1$. But Q_1 is where DM_1, which bisects angle BDS_1, cuts line BS_1. Therefore, given the similarity of triangles BM_1Q_1 and Q_1DS_1, it follows that corresponding sides are proportional. Hence., $BQ_1:Q_1S_1 = M_1Q_1:Q_1D$. But $BQ_1:Q_1S_1 = BE_1:E_1A_1 = BG:GA_1$, so $M_1Q_1:Q_1D = BG:GA_1$. Triangles BDS_1 and BGA_1 thus correspond in the two crucial ways mentioned before: the lines bisecting the angles at respective vertices G and D cut equiproportional segments from them (i.e., $BE_1:E_1A_1 = BG:GA_1$, and $BQ_1:Q_1S_1 = BD:DS_1$), and the segments cut off by those bisectors on respective bases BA_1 and BS_1 are equiproportional (i.e., $BE_1:E_1A_1 = BQ_1:Q_1S_1 = BG:GA_1$). Finally, since $BQ_1:Q_1S_1 = M_1Q_1:Q_1D$, it follows that $M_1Q_1:Q_1D = BG:GA_1$. And the same pattern holds for any other center of sight chosen on line of reflection AD. No matter where the new point A lies on that line, it will always be the case that BE:EA = BG:GA = BQ:QS = MQ:QD. The ratio of normals BG and GA, in short, is absolutely fundamental and systemic to the construction.

So far we have taken point D of reflection, object-point B, center of curvature G, and line of reflection AD as given and, using D as the anchor-point, constructed triangle $BDGT_1$ and line $DQ_1N_1M_1$ highlighted in bold lines in the lower diagram of figure 17d. To find point D when only A, B, and G are given is simply a matter of recapitulating that construction outside the mirror and then importing it back into the mirror. This we do by creating an exact replica in triangle $B'D'G'T'$ and line $D'Q_1'N_1'M_1'$ in the upper diagram of figure 17d. To that end we need to know, first, the ratio of BG to AG, and second, the measure of angle BGA. These, of course, are given in the problem as it is set up, so points Q_1' and N_1' on $M_1'D'$ are given, as is angle $D'T_1'N_1'$ (= half of angle BGA) on perpendicular $T_1'N_1'$ passing through the midpoint of $M_1'D_1'$. The sticking-point lies in knowing how to generate line $B'G'$ through point Q_1' to $D'T_1'$ such that $B'G':G'D'$ = BG:GD. The real test of Alhacen's ingenuity in determining the location of D was therefore to figure out how to generate this line, and he passed it with flying colors in proposition 24, lemma 6. At bottom, then, that lemma forms the crux of Alhacen's solution to the reflection-problem for convex spherical mirrors.

It is worth noting, finally, that the ratio of normals is also fundamental to reflection from plane mirrors. In this case, of course, there is no center of curvature, so the normals will never intersect. Thus, in figure 17e, p. 537, if D is the point of reflection, DE the normal to that point, and AD the line of reflection, and if A, A_1, and A_2 represent various viewpoints on that line, then normals AG, A_1G_1, and A_2G_2 dropped from them to the mirror will be parallel to one another as well as to normals DE and BG_3. Since the resulting triangles AGD, A_1G_1D, A_2G_2D, and BG_3D are all similar, their corresponding sides will be proportional. Thus, for example, sides AD, AG, and GD in triangle AGD will be proportional to corresponding sides A_1D, A_1G_1, and G_1D in triangle A_1G_1D, as well as to corresponding sides BD, BG_3, and G_3D in triangle BG_3D.

Now, according to the conditions specified, angle BDE = angle ADE, since DE bisects angle BDA. If we connect line BA, then, according to Euclid, VI.3, DE will intersect that line at point E such that BE:EA = BD:AD. The same holds for the remaining points A_1 and A_2: $BE_1:E_1A_1 = BD:A_1D$, and $BE_2:E_2A_2 = BD:A_2D$. So the location of D is ultimately determined by the ratio of normals BG_3 and AG, BG_3 and A_1G_1, or BG_3 and A_2G_2. Accordingly, given points B and A, A_1, or A_2 facing reflecting surface G_1G_3, we can find point D by dropping the normals from B and A, or from B and A_1, or from B and A_2, and cutting line G_3G, G_3G_1, or G_3G_2 according to the ratio of those normals: i.e., $G_3D:DG = BG_3:AG$, $G_3D:DG_1 = BG_3:A_1G_1$, or $G_3D:DG_2 = BG_3:A_2G_2$. And the same holds for any other point A chosen on line of reflection AD. There is thus a close analogy between this analysis of reflection from plane mirrors and Alhacen's analysis of reflection from convex spherical mirrors. The key thing in both cases is the ratio of normals and the recapitulation of that ratio in lines BA, BA_1, and BA_2 through bisection, although in the case of reflection from plane mirrors it is the angle of reflection rather than the angle formed by the intersection of normals that is bisected.

Whether Alhacen saw the patterns and connections just discussed is an open question, because he never mentions them explicitly. But they are clearly implicit in the construction upon which he based his proof, and that construction is clearly implicit in the construction for the limiting case. Granted, this latter construction is not self-evident, but it is fairly obvious—obvious enough, I suggest, that Alhacen could have seen in it the model for his solution to the problem of finding D when AG and BG are unequal. To suppose that he did takes some of the mystery out of that solution and its derivation, but it takes nothing whatever from its ingenuity or originality.

Convex Cylindrical Mirrors: Alhacen implicitly addresses the problem of finding the point of reflection in convex cylindrical mirrors in proposition 28, pp. 435-437, where he subdivides the problem according to three cases, depending upon how the plane of reflection cuts the mirror and its axis. On the one hand, if that plane cuts the cylinder along a line of longitude, it will include the entire axis. Thus, as illustrated in figure 18, p. 538, plane of reflection AXYB passing through object-point B and center of sight A contains line of longitude ST and axis XY. Point D of reflection must therefore lie on line ST. Furthermore, since ST is the common section of the plane of reflection and the plane tangent to the mirror along line ST, finding point D reduces to finding the point of reflection on a plane mirror with ST as the common section of the reflecting surface and the plane of reflection. Alhacen has already provided the solution to this problem in proposition 1, p. 399. Accordingly, once point of reflection D is found and normal FDE is erected at that point, angle BDE of incidence will equal angle ADE of reflection.

If, on the other hand, the plane of reflection intersects the axis orthogonally, it will cut a circle on the cylinder's surface. This case is represented in figure 18a, p. 538, where plane ABG of reflection intersects axis XY orthogonally at point G and cuts the circle in light gray on the cylinder's surface. Point D will therefore lie on that circle, and to find it we need only apply the method in proposition 25 based on the proportionality of normals BG and AG. Consequently, with normal GDE produced, angle BDE of incidence will equal angle ADE of reflection.

In the third and most complex case, which is represented in figure 18b, p. 538, the plane of reflection ABHK intersects the cylinder's axis obliquely at point F and therefore cuts the cylinder's surface along an ellipse. Here the solution requires passing a plane through point A so as to form the circle with centerpoint G represented in light gray on the cylinder's surface. Then line BB′ is dropped from point B to that plane parallel to the edge of the cylinder and thus perpendicular to its base-planes. Points A and B′ will therefore face the circle centered on G within the same plane. To find point D′ on that circle where the form of point B′ would reflect to point A is, as before, a matter of applying the method outlined in proposition 25 according to the proportionality of normals B′G and AG. Having determined that point, we then pass line of longitude D′D through it. Point D, where the plane of reflection intersects this line of longitude, will be the point of reflection. Hence, if we extend normal FDE through that point, angle BDE of incidence will equal angle ADE of reflection.

Convex Conical Mirrors: As with convex cylindrical mirrors, so with convex conical mirrors, points A and B can be disposed to the mirror in various ways. The problem of finding the point of reflection can thus be subdivided into a variety of cases. The first and most trivial of these has the plane of reflection cutting a line of longitude on the cone's surface. As with the first case in cylindrical mirrors, so with this one, the problem of finding point D reduces to finding the point of reflection on a plane mirror. The remaining cases, which number six according to Alhacen's analysis, depend upon where A and B are situated with respect to the plane passing perpendicular to the cone's axis through its vertex. Let us deal with these cases in order as they occur in proposition 31, pp. 441-446.

In the first case (pp. 441-442) A and B are assumed to lie below that plane, as represented in figure 19, p. 539, where lines SXT and UXV define the plane passing perpendicular to axis XY through vertex X of the cone. Let B be the object-point and A the center of sight, and let ABHK be the plane of reflection. Through point A pass a plane orthogonal to axis XY of the cone, and let it meet the axis at point Y. That plane will thus cut a circle with centerpoint Y on the cone's surface. Drop a line from vertex X of the cone to point B, extend it to the plane of the circle, and let it intersect that plane at point B′. B′ and A will therefore face the circle centered on Y within the same plane, so, by applying the method in proposition 25 according to the ratio of normals B′Y and AY, we can find point D′ where the form of B′ would reflect to A. In this case, of course, the circle must be treated abstractly, as if it lay on the surface of a spherical or cylindrical mirror, because in its actual situation normal YE′ strikes the cone's surface obliquely, which means that plane B′AY cannot be a true plane of reflection. Now, draw line of longitude D′X. Point D, where plane of reflection ABHK intersects line of longitude D′X, will be the point of reflection. Hence, if we erect normal FDE at that point, angle of incidence BDE will equal angle of reflection ADE. Needless to say (which is undoubtedly why Alhacen does not), the construction just outlined holds if A and B lie in a plane, such as AYB', that cuts the axis orthogonally rather than obliquely and, therefore, that already forms a circle on the cone.

In the second case (pp. 442-443), points A and B in figure 19a, p. 539, are both assumed to lie in the plane that passes perpendicular to axis XY through vertex X of the cone. In this case, X can be treated abstractly as an already-defined point of reflection, so the angles of incidence and reflection are determined by bisecting angle AXB with line XE. Pass a plane through line XE along axis XY to produce line of longitude XK on the cone's surface. From point E drop a normal to line of longitude XK, and extend it to point F on the axis. BEAF will thus be the plane of reflection, D the point of reflection, and FDE the normal to that point. Accordingly, angle of incidence BDE will equal angle of reflecton ADE.

In the third case (pp. 443-444), points A and B are both situated above the plane passing perpendicular to axis XY through vertex X, as represented in figure 19b, p. 540. Produce the opposite section of the original cone. If line AB is disposed orthogonally with respect to axis XY, then pass a plane through it parallel to the base of the lower cone to form the circle centered on T in the opposite, upper cone. If line AB is disposed obliquely with respect to axis XY, then let A lie lower than B. From X draw a line through A and extend it until it meets line AB at new point A such that line AB is disposed orthogonally with respect to axis XY. Either way, A and B will lie in the plane of the circle. Find point D' on the concave section of that circle where the form of B would reflect to A, that circle being taken abstractly as if it were in a concave spherical or cylindrical mirror. Alhacen's method for determining this point will come later, when he analyzes reflection in concave mirrors, so for now we must take D' as properly determined.[33] Accordingly, with normal D'E produced, angle BD'E of incidence will equal angle AD'E of reflection. From point D', as appropriately determined, draw line of longitude D'X in the upper, opposite cone, and continue it to point L in the lower cone. From point E on line BA drop normal ED to line of longitude XL, and extend it to point F on axis XY. D will therefore be the point of reflection so that, with normal FDE produced, angle BDE of incidence will equal angle ADE of reflection.

In the fourth case (pp. 444-445), one of the points is assumed to lie in the plane passing perpendicular to axis XY through vertex-point X, the other below that plane. Let B, in figure 19c, p. 540, lie in that plane and A below it. Pass a plane through point A to intersect axis XY orthogonally, thus forming the circle centered on Y on the cone's surface. Draw line BX, and drop BL perpendicular to the plane of the circle just formed. Connect LY, which is thus parallel to BX, and from A drop a line to LY, cutting it at Q such that D′Q = QY. To do this we follow the procedure Alhacen has outlined previously in proposition 23, lemma 5. Through point D′, where line AQ intersects the circle, extend line YE′. From point L draw line LB′ parallel and equal to YD′, and then draw lines BB′ and B′D′. LB′D′Y will therefore be a parallelogram, so angle LB′D′ = opposite angle LYD′, and angle B′LY = opposite angle B′D′Y. Likewise, angle B′D′E′ = alternate angle LYD′. But angle LYD′ = angle QD′Y, since, by construction, triangle D′QY is isosceles on base D′Y. Therefore, angle B′D′E′ = angle QD′Y. But angle AD′E′ is also equal to angle QD′Y, since they are vertical angles. Hence, angle B′D′E′ = angle AD′E′, so D′ is the point from which the form of B′ would reflect to A. Draw line of longitude XD′. Point D, where plane of reflection ABHK intersects it will be the point of reflection for B and A. Hence, with normal FDE produced, angle BDE of incidence will equal angle ADE of reflection.

The fifth case (p. 445) is illustrated in figure 19d, p. 541, where B lies in the plane passing perpendicular to axis XY through vertex-point X, and A lies above it. Form the complementary cone above the base cone, and pass a plane through A perpendicular to the axis so as to produce the circle centered on T. Drop perpendicular BB′ to the plane of that circle. Hence, A and B′ face the concave surface of the circle. Find point D′ on that surface where the form of B would reflect to A. As before, Alhacen's method for determining this point has yet to be revealed, so we must take D′ as given. With normal D′E′ drawn, angle of incidence B′D′E′ = angle of reflection AD′E′. From point D′, as appropriately determined, draw line of longitude D′X on the upper cone and extend it to L on the lower one. Point of reflection D will therefore lie where plane of reflection ABHK intersects line of longitude XL. Thus, with normal FDE produced, angle of incidence BDE will equal angle of reflection ADE.

In the sixth and final case (pp. 445-446), the two points lie on either side of the plane passing perpendicular to axis XY through vertex X. Hence, as represented in figure 19e, p. 541, A lies below that plane, B above it. Form the opposite cone above the base cone, and pass a plane through B to intersect the axis orthogonally at point T. Point T will therefore be the center of the circle cut by that plane on the cone's surface. From point A draw line AK through vertex X and continue it until it meets the plane of that circle at point K. Then, on the basis of proposition 22, lemma 4, find point D′ through which tangent SD′C passes in such a way as to form angle SD′K equal to angle SD′B. Through centerpoint T of the circle draw diameter D′TE′, which is therefore perpendicular to tangent SD′C. From point D′ draw line of longitude D′X on the upper cone and extend it to point L on the lower cone. Produce line AA′ parallel to that line of longitude, and let it meet the plane of the upper circle at point A′. Since line AK and line of longitude D′L intersect at X, they are in the same plane, and since line AA′ is parallel to line of longitude D′L, all three lines AK, D′L, and AA′ lie in the same plane, so points K, D′, and A′ will lie in the same plane. Connect them with line KD′A′.

Now, angle KD′S = angle BD′S by construction, and angle KD′S = vertical angle A′D′C, so angle BD′S = angle A′D′C. But, since D′TE′ is perpendicular to SD′C by construction, angle SD′E′ = angle CD′E′, both being right. Hence, because their corresponding parts BD′S and A′D′C are equal, remainders BD′E′ and A′D′E′ will be equal. Accordingly, D′ is the point from which the form of B would reflect to A′. The point of reflection D for points A and B will therefore lie at the intersection of the plane of reflection and line of longitude D′XL. Hence, with normal FDE produced, angle of incidence BDE will equal angle of reflection ADE.

In all, therefore, Alhacen's method for finding the point of reflection in convex cylindrical and conical mirrors is based on defining the appropriate line of longitude and determining the point at which the plane of reflection intersects it. To define the appropriate line of longitude, in its turn, requires passing a plane through either A or B to cut the cone or its complement along a circle and then projecting the other point onto that plane.[34] With both points thus facing the circle, the final step is to find the point on that circle at which the form of the one would reflect to the other and produce the line of longitude from that point. Granted, this method is sometimes complex at the operational level, but it is fairly simple at the conceptual level.

Concave Spherical Mirrors: As mentioned earlier, the analysis of concave spherical mirrors is considerably more complex than that of convex spherical mirrors because of the multiplicity of possible reflection-points, depending on how the source-point of radiation and the center of sight are disposed with respect to one another as well as to the center of curvature and the reflecting surface itself. As will become clear in the course of our analysis, however, there are fundamental linkages between Alhacen's approach to convex mirrors and his approach to concave ones.

Let us start with the case in which both A and B are equidistant from the center of curvature. The most trivial form of this case, which Alhacen addresses in proposition 36, case 1, p. 452, occurs when A and B lie on the same normal at equal distances from centerpoint G of the mirror, as in figure 20, p. 542. Given that AG = BG, finding the normal to the point of reflection is a matter simply of dropping DG perpendicular to AB at centerpoint G of the mirror. From the equality of triangles AGD and BGD it is obvious that angle of incidence BDG = angle of reflection ADG. So D will be a point of reflection for A and B, and, by symmetry, so will D_1. In fact, if line DD_1 is rotated on axis AGB, the form of B will reflect to A from every point on the resulting circle formed by D on the mirror's surface. That it will reflect from no other point D_2 on the mirror is obvious enough from the diagram to require no proof. Furthermore, it makes no difference whether A and B lie inside or outside the circle of the mirror as long as point D on the reflecting surface is exposed to them (i.e., as long as BD can reach the mirror and DA can reach A).

But what if A and B do not lie on the same normal? Alhacen addresses this case in proposition 37, pp. 454-458. The first and most trivial instance is represented in figure 20a, p. 542, where A and B lie far enough outside the great circle of the mirror that the line passing through them does not intersect that circle. To find the point of reflection under these conditions, we need only bisect angle AGB with line GE and extend the bisector to point D. Like its corollary in case one for convex mirrors, the problem in this case is so simple in its solution and proof as to need no elaboration. From the equality of triangles AGD and BGD it is obvious that angle BDE of incidence is equal to angle ADE of reflection. That the form of B cannot reflect to A from any point, such as D_1, on arc FD follows from the fact that, no matter where D_1 lies on that arc, normal GD_1 dropped to it will not form equal angles with AD_1 and BD_1. Nor can the form of B reflect to A from any point D_2 on arc FK, because normal GD_2 will lie outside angle AD_2B and will thus not bisect it. The same holds by symmetry for arcs DH and HL. Consequently, since reflection cannot occur from any point other than D on the entire circle, D is the only possible point of reflection in this instance. It should be noted that in this case, as well as in every other case in which A and B lie on different normals, reflection will occur only in the great circle formed on the mirror by the plane of reflection. In other words, there is no circle of reflection as in the previous case.

Now, if we shift A and B toward G along the same normals so that the line passing through them intersects the circle, and if A and B remain equidistant from G, as represented in figure 20b, p. 543, it is obvious that the previous analysis holds for arc FH: i.e., that the form of B will still reflect to A from point D but from no other point on arc FH. However, as long as A and B lie below XY, which is tangent to the circle at point D_1, where bisector GE intersects the circle opposite D, points A and B also face some segment of arc KL bounded by the extensions of normals AG and BG. Accordingly, if we assume that the reflecting surface extends through arc KL, then it is obvious from the diagram that the form of B will reflect to A from point D_1 and that the same holds for any other points—such as A' and B'—that lie below tangent XY. It is also evident that the form of B will not reflect to A from any point D_2 on arcs FK or HL, for in that case normal D_2G would lie outside angle AD_2B. In this instance, therefore, the form of B will reflect to A from at least two points, D and D_1, the former on arc FH and the latter on arc KL.

Whether the form of B can reflect to A from any point other than D_1 on arc KL depends upon whether the circle passing through points A, B and G intersects the circle of the mirror within that arc, as discussed in the previous section (pp. xl-xli). Moreover, as we pointed out in that discussion, circle ABG will intersect arc KL if and only if the lines dropped to points A and B from the midpoint of arc KL form acute angles with their respective normals. Thus, as illustrated in figure 20c, p. 543, when A and B are disposed as they are, circle ABG never touches the circle of the mirror. Assume that D_2 is a legitimate point of reflection. Draw normal GD_2, extend it to P on the opposite side of the mirror, and let it pass through the inner circle at point T. Connect AT and TB. Angle ATG is subtended by arc AG, and angle BTG is subtended by arc BG, and both arcs are equal by construction (i.e., chord AG = chord BG). Thus, by Euclid, III.21, angle ATG = angle BTG.

If D_2 is a legitimate point of reflection, then angle of incidence AD_2G should equal angle BD_2G. Extend AD_2 to M and BD_2 to N. Within the circle of the mirror, then, arc MP subtended by angle AD_2G should equal arc NP subtended by angle BD_2G. Extend line MO perpendicular to normal D_2P, and let R be where the two lines intersect. Accordingly, since angle MRP = angle ORP (both being right by construction), arcs PM and PO subtending those angles are equal. But arc PM subtends angle PD_2M, whereas arc PN subtends angle PD_2N, and arc PN > arc PO. It therefore follows that angle PD_2N, which coincides with angle of incidence BD_2G, is greater than angle PD_2M, which coincides with angle of reflection AD_2G, so the form of B cannot reflect to A from point D_2 or, by extension, from any other point on arc KL except D_1.

If, however, circle ABG does intersect the arc, it will necessarily do so at two points. Those two points will be points of reflection, as claimed in the previous section. Accordingly, as represented in figure 20d, p. 544, points D_2 and D_3, where circle ABG intersects arc KL, will be points of reflection, because the arcs on circle ABG subtending angles AD_2G and BD_2G within circle ABG—i.e., arcs AG and BG—are equal, by construction, and those same arcs subtend angles AD_3G and BD_3G within that same circle. In the previous case we excluded reflection from arcs KD_2 and LD_3, where circle ABG does not touch the circle of the mirror. We can also exclude reflection from any point D_4 on arc D_2D_1 according to the inequality of arcs MP and PN, since arcs MP and MO are equal by construction, and we can extend that exclusion to arc D_3D_1 by symmetry. In this instance, then, the form of B can reflect to A from points D_1, D_2, and D_3 on arc KL and from point D on arc FH. Four is therefore the maximum number of reflections when A and B lie inside the circle at equal distances from G.

From this sketchy analysis it should be evident that, even in the relatively trivial case of equality between AG and BG, the determination of D is complicated by the need to take into account not only where A and B lie with respect to G, but also where they lie with respect to the surface of the mirror. It should be clear as well that the problem of finding D is intimately connected with the problem of determining precisely how many legitimate points of reflection there can be under the various conditions specified. Yet, despite these complicating factors, once the conditions are properly specified, the two methods for finding the appropriate point or points of reflection—i.e., through the bisection of angle AGB and the intersection of circle ABG—are almost intuitively obvious in their simplicity.

Things become more complicated when we turn to the case in which normals AB and AG are unequal, as represented in figure 20e, p. 544. There are, however, certain parallels between this case and the preceding one. For one thing, as we showed in the previous section (p. xli), the form of B can reflect to A from one, and only one, point on arc FH, and it cannot reflect from any point on arcs FK or HL. For another, if the line passing through A and B does not touch the circle of the mirror, the only reflection that can occur will be from the given point on arc FH. For yet another, when A and B lie inside the circle of the mirror, reflection can occur from as many as, but no more than, four points: one from arc FH and the rest from arc KL. And for yet another, the circle passing through points A, B, and G is instrumental in determining how and where reflection can occur from arc KL.

Despite these parallels, there are certain limitations that are specific to the case at hand. For instance, if the form of B reflects to A from some point D on arc KL in figure 20e, then angle ADB (to which I will henceforth refer as the "reflected angle") cannot be equal to angle LGH adjacent to KGL. Alhacen proves this point in proposition 43, pp. 462-463, which was discussed earlier on p. xli. Furthermore, as was shown at the same point in the previous section, if the form of B reflects to A from two points on arc KL, it is impossible for both reflected angles to be less than angle LGH. In addition, there can be two reflections from arc KL such that both reflected angles are greater than adjacent angle LGH. Thus, when the circle passing through points A, G, and B intersects arc KL, two reflections may occur in arc CE cut by that circle (but only on segment XE), and in both of them the reflected angle will be greater than adjacent angle LGH. An additional reflection will occur from arc CK, and the reflected angle in that case will be less than adjacent angle LGH. Altogether, then, there can be three reflections from arc KL and one from arc FH. Four, in short, is the maximum number of reflections that can occur when AG ≠ BG. As was also pointed out earlier (pp. xl-xli), if circle AGB intersects arc KL at midpoint X, as represented in figure 16b, p. 532, or if it is tangent to or falls short of that arc, as represented in figure 16a, p. 531, the form of B will reflect to A from only one point on arc KL, and the resulting reflected angle BDA will be less than adjacent angle LGH.

With these specifications in mind, let us turn to the actual problem of determining the point or points of reflection in concave sperical mirrors when points A and B lie inside the circle of the mirror and normals AG and BG are unequal. The first and simplest case, which Alhacen addresses in proposition 36, case 2, pp. 453-454, has A and B lying on the same normal, as represented in figure 20f, p. 544. Extend line AG through and beyond point F on the circle of the mirror. Find point E on that extension such that BE:GE = BG:AG. In other words, BE is to GE in the same ratio as the normals dropped to the center of the mirror from the object-point and the center of sight. From point E draw a circle passing through points D, G, and D_1 within the mirror. The form of A will reflect to B from the two points of intersection D and D_1. The proof rests on showing that within the circle of the mirror, normal DG bisects angle BDA. This point, in turn, rests on the fact that DB:DA = BG:AG.[35]

Furthermore, as happens when A and B lie equidistant from G on the same normal, reflection in this case occurs not just from D and D_1 but from the entire circle produced by D as line DAD_1 rotates about axis AGB. That the form of A cannot reflect to B from any other point D_2 on the circle of the mirror follows from the fact that, when normal D_2G is drawn, $BD_2:AD_2 \neq BG:AG$. At bottom then, the determination of D and D_1 comes to ground in the proportionality between normals AG and BG, just as it does for convex spherical mirrors.

Now, let A and B lie on different normals at unequal distances from G, as represented in figure 20g, p. 545, where B lies on normal KGH and A on normal LGF. Let line XGY bisect angle FGH. As Alhacen formulates it in proposition 38, pp. 458-459, the method for finding point D of reflection on arc FH is as follows. Take some line D'M', and divide it at point Q' such that M'Q':Q'D' = BG:AG. Bisect D'M' at point N', and through that point draw line N'B' perpendicular to D'M'. At point D' form angle N'D'T' equal to half of angle LGH adjacent to FGH.[36] Then, following the procedure Alhacen has already outlined in proposition 21, lemma 3, draw a line from point Q' that meets perpendicular N'B' at point B' and passes through D'T' at point G' such that B'G':D'G' = BG:DG (i.e., the radius of the mirror). At centerpoint G of the mirror form angle BGD equal to angle B′G′D′. Point D, where leg DG of that angle intersects the mirror, will be the point of reflection.

The construction according to which Alhacen proves that D is in fact the point of reflection begins with the formation of angle GDQ equal to angle G′D′Q′, as represented in figure 20h, p. 545. Thus, as is obvious from the diagram, triangles QDG and Q′D′G′ are similar. Likewise, if we draw BN perpendicular to the extension of DQ at point N, it is obvious from the diagram that triangles QBN and Q′B′N′ are similar. Equally obvious from the diagram is that triangles BGT and B′G′T′ are similar. Extend DN to M so that DN = NM. Accordingly, the entire figure constructed on line D′M′ to the left is recapitulated in the figure highlighted by bold lines in the circle of the mirror to the right.

Connect MB, draw line DS parallel to it, and continue BH until it meets DS at S. Since vertical angles SQD and MQB are equal, and since alternate angles DSQ and MBQ are equal, triangles DSQ and MQB are similar, so their corresponding sides will be proportional. Hence, SQ:QB = DQ:QM. Moreover, since DM is bisected by perpendicular BN, triangles BDN and BMN are equal, so their corresponding angles BDN and BMN are equal. But angle BMD = alternate angle MDS, so angle BDQ = angle SDQ, from which it follows that DQ bisects angle SDB. The rest of the proof is based on constructing angle GDP equal to angle GDB and then showing that line AP is a rectilinear continuation of line DP, from which it follows that angle of incidence BDG = angle of reflection ADG.

The similarity between Alhacen's method for determining D in this case and his method for determining D in convex spherical mirrors becomes clear if we extend line BN in figure 20k, p. 546, to meet the extension of line DS at point V and repeat the process with lines B′N′ and D′S′. Then, with line D′B′ produced, the entire figure B′D′S′V′ formed on line D′M′ to the left will be recapitulated in figure BDSV formed on line DM in the circle of the mirror.

Now, if we join AB, it is clear that XY, which bisects angle KGL—and thus angle FGH—will intersect AB at point E. But, by Euclid, VI.3 we know that AE:EB = AG:BG. We also know that MQ:QD = BG:AG by construction. But we have established earlier that MQ:QD = BQ:QS, and, since DQ bisects angle SDB, DB:DS = MQ:QD = BQ:QS. Hence, BQ:QS = BD:DS = BG:AG. Triangles BDS and BGA thus correspond in the two crucial ways discussed earlier in the context of convex spherical mirrors: the lines bisecting the angles at respective vertices G and D cut equiproportional segments from them (i.e., BE:EA = BG:GA, and BQ:QS = BD:DS), and the segments cut off by those bisectors on respective bases BA and BS are equiproportional (i.e., BE:EA = BQ:QS = BG:GA). Furthermore, it should be obvious that dropping line Q′G′B′ from point Q′ on D′M′ through point G′ on D′T′ so that D′G′:G′B′ = DG:BG is tantamount to producing line B′Q′S′ so that B′S′:S′D′ = BS:SD. In other words, implicit in Alhacen's construction for determining D in this case is his construction for determining D in convex spherical mirrors.[37]

So much for arc FH, in which there is only one possible point of reflection. Arc KL is another matter altogether, for, as we have already observed, the form of B may reflect to A from as many as three points on that arc if the circle passing through A, G, and B intersects the arc. Let A and B in figure 20n, p. 547, lie on normals FGL and KGH at unequal distances from G, and let them be disposed in such a way that circle ABG does intersect arc KL. Let us start with the case in which the point of reflection lies outside circle ABG so that, according to previous discussion, the resulting reflected angle ADB is less than angle LGH. Oddly enough, Alhacen never addresses this case explicitly, but the method to be applied to it is essentially the obverse of the one applied to reflection from arc KH. Accordingly, take some line D′M′, and cut it at point Q′ so that M′Q′:Q′D′ = AG:BG. Bisect D′M′ at N′, draw perpendicular N′T′ through that point, and form angle N′D′T′ equal to half of angle LGH. From point Q′ drop line Q′A′G′ through N′T′ so that it meets D′T′ at point G′ in such a way that Q′G′:D′G′ = BG:GD (i.e., the radius of the circle of the mirror). Alhacen has already demonstrated how to do this in proposition 24, lemma 6. At centerpoint G of the mirror, form angle AGD equal to angle A′G′D′ (i.e., Q′G′D′). D will be the point of reflection.

The proof that D actually is the point of reflection is essentially the same as that in the previous case of reflection from arc FH. The construction is illustrated in figure 20p, p. 547. From that diagram it is obvious that the entire figure DGTANQM highlighted in bold in the mirror of the circle is similar to figure D′G′T′A′N′Q′M′, so all the corresponding triangles are similar. Draw DA and AM. Since AN is perpendicular to DN and bisects it according to construction, triangles DAN and MAN are equal. Draw DS parallel to AM, and continue GAQ to meet it at point S. Therefore, angle SDM = alternate angle DMA, which equals angle MDA by construction, so DQM bisects angle SDA. Furthermore, since angle SDM = alternate angle DMA, while angle DSQ = alternate angle QAM, triangles DSQ and QAM are similar, so their corresponding sides will be proportional. It follows, then, that SQ:QA = DQ:QM. But DQ:QM = BG:AG by construction, so SQ:QA = BG:AG. The rest of the proof is based on constructing angle GDP equal to angle GDA and then showing that line DB is a rectilinear continuation of BP, from which it follows that angle of incidence BDG equals angle of reflection ADG.

The patterns that underly the two previous methods underly this one as well, the two associated triangles being ADS and BAG in figure 20q, p. 548. In the former, bisector DQ cuts base AS at point Q such that DS:DA = QS:AQ. But, since it has been established that triangles DQS and AQM are similar, then QS:AQ = DQ:QM = BG:AG. Likewise, XY bisects angle BGA to intersect line BA at point E so that BE:EA = BG:GA. Therefore, corresponding bases AS and BE of the two triangles are cut according to the proportionality of normals BG and AG. That proportionality, in short, is systemic throughout the construction.[38]

We are now left with the problem of finding the two remaining points of reflection on the arc inside the intersecting circle. However, as mentioned earlier, just because circle ABG intersects arc KL does not necessarily mean that reflection will occur from the arc within the intersecting circle. There is one additional restriction. If reflection is to occur from more than one point on arc KL (in which case it will necessarily occur from three) the lines drawn from the point of bisection on arc KL to A and B must form acute angles with their respective normals on the side of G. In other words, angles XBG and XAG in figure 20t, p. 550, must be acute.[39] Let A and B be properly disposed according to this final restriction. As given in proposition 47, pp. 470-471, the method for finding the two points of reflection in this case is as follows.

Take some line D′M′, in figure 20t, p. 550, and divide it at Q′ such that M′Q′:Q′D′ = BG:AG. Bisect D′M′ at N′ and pass perpendicular line B′N′T′ through that bisection-point. Form angle M′D′T′ equal to half of angle LGH. Draw line B′Q′G′ through point Q′ so that it intersects B′T′ at point B′ and meets D′T′ at G′ in such a way that B′G′:G′D′ = BG:GD. This we can do by following the procedure laid out by Alhacen in proposition 24, lemma 6. Form angle BGD in the circle equal to angle B′G′D′. D will be a point of reflection.

Now, as we established earlier in the discussion of proposition 24, lemma 6, it is possible to project two lines through point Q′ to meet the specified conditions. Accordingly, in the figure below and to the right of the circle, line D_1′M″ is cut proportionate to line D′M′ in the figure above and to the left of the circle, so that M″Q″:Q″D_1′ = M′Q′:Q′D′ = BG:AG, and M″D_1′ is bisected at point N″, with line B″N″T″ passing through it perpendicular to M″D_1′. Under these conditions, it is possible to pass a line B″Q″G″ through point Q″ such that B″G″:G″D_1′ = BG:GD in the circle. Thus, B″G″:G″D_1′ = B′G′:G′D′. Nevertheless, the two resulting angles B″G″D_1′ and B′G′D′ will be unequal.[40] Accordingly, form angles BGD and BGD_1 in the circle equal to angles B′G′D′ and B″G″D_1′, and D and D_1 will both be points of reflection.

Not only does Alhacen not mention the construction of this second angle in proposition 47, but he does not demonstrate that D is in fact a legitimate point of reflection. He does suggest the direction the reader might take in order to prove that fact, but the approach he suggests, which is based on conclusions drawn in proposition 46, is different from the one we have been taking to this point in the analysis. The construction for the proof according to our approach is given in figure 20x, p. 551. As is clear from the diagram, the entire figure D'G'T'B'Q'' formed on line D'M' is recapitulated in figure DGTBQ formed on line DM in the circle of the mirror. Accordingly, DS is parallel to BM by construction, so angle BDQ = angle QDS, from which it follows that DQM bisects angle BDS. Hence BQ:QS = BD:DS = MQ:QD = BG:AG. By the same token, BE:EA = BG:GA, so triangles BDS and BAG correspond in the two ways discussed earlier: i.e., the lines bisecting the angles at respective vertices G and D cut equiproportional segments from them (i.e., BE:EA = BG:GA, and BQ:QS = BD:DS), and the segments cut off by those bisectors on respective bases BA and BS are equiproportional (i.e., BE:EA = BQ:QS = BG:GA). The same construction applies *mutatis mutandis* to reflection-point D_1, and it also applies to the limiting case in which normals BG and AG are equal and the reflection-point lies at X, where arc KL is bisected.[41]

Concave Cylindrical Mirrors: Alhacen's method for finding the point, or points, of reflection in concave cylindrical mirrors is analogous to his method for finding the point of reflection in convex cylindrical mirrors. Accordingly, the simplest case involves A and B lying in a plane of reflection that cuts the mirror along a line of longitude and includes the axis. In this case, of course, the problem reduces to finding the point of reflection on a plane mirror in which the line of longitude represents the common section of the reflecting surface and the plane of reflection. In the second case, the plane of reflection containing A and B is orthogonal to the axis so that it cuts the cylinder's surface along a circle. Since that circle lies in the plane of reflection, finding the point or points of reflection on it is a matter of applying the methods for determining such points in concave spherical mirrors. Thus, the form of B can reflect to A from as few as one point and as many as four, depending on where A and B lie in relation to the center of the circle and the mirror's surface.

The third and final case, which Alhacen addresses in proposition 52, pp. 478-482, involves A and B lying in a plane of reflection that intersects the axis obliquely and therefore cuts the cylinder's surface along an ellipse. Let the cylinder with axis XG in figure 21, p. 553, represent the mirror, and let B lie above A inside the cylinder. Through A pass a plane orthogonal to axis XG so as to cut the cylinder's surface along circle FKLH centered on G. From point B drop a perpendicular to the plane of the circle, and let it meet that plane at B′. Connect lines AB′ and AB.

To find the point or points on circle FKLH from which the form of A would reflect to B′, we need only apply the methods for finding such points on concave spherical mirrors. Let D′ be one of those points. Accordingly, normal E′D′G will bisect angle AD′B′ to render the angles of incidence and reflection equal. From point D′ produce line of longitude D′T on the cylinder. Then, within plane D′GXT formed by the axis and the line of longitude draw line YDE through AB parallel to E′D′G. Since line E′D′G is normal to D′T, line YDE will also be normal to D′T, and it will lie in the same plane as AB, which it intersects. Point D, will therefore be the point of reflection within plane of reflection ADBY, from which it follows that angle BDY of incidence will equal angle ADY of reflection. Quadrangle ADBY, with its associated diagonals AB and YD, is thus the projection of quadrangle AD′B′G, with its associated diagonals AB′ and GD′, onto the plane of reflection.

The same procedure can be applied to every other point from which the form of A would reflect to B′ within the base-circle centered on G. For each such point, there will be an appropriate point D on the cylinder, and for each such point D the plane of reflection will cut an ellipse on the cylinder's surface, each ellipse being particular to the given plane of reflection. Suffice it to say, the number of actual reflections will depend upon how much of the reflecting surface is exposed to A and B.

Concave Conical Mirrors: As is the case with all the other curved mirrors, if A and B lie in a plane that includes the axis, that plane will cut a line of longitude on the surface of the cone. Finding the point of reflection thus reduces to finding the point of reflection on a plane mirror in which the line of longitude forms the common section of the reflecting surface and the plane of reflection.

On the other hand, if A and B lie on a line through which a plane can be passed orthogonal to the cone's axis so as to cut a circle on the cone's surface, then the first step is to find the point or points of reflection within that circle. Let A and B in figure 22, p. 554, be two such points, and let them lie in the plane of circle FKLH centered on point G. That plane, of course, will be orthogonal to axis XGY of the cone. Applying the methods for determining the point, or points, of reflection in concave spherical mirrors, find the point, or points, within circle FKLH from which the form of B would reflect to A. Let D′ be one of them. From that point produce line of longitude D′X on the mirror's surface, and join line AB. Then, within the plane formed by line of longitude D'X and axis XG of the cone, draw line YDE through AB normal to line of longitude D'X. That line will intersect axis XG at point Y, and point D, where it intersects D'T, will be the point of reflection within the plane of reflection ADBY. Accordingly, angle of incidence ADY will equal angle of reflection BDY. Quadrangle ADBY, with its associated diagonals AB and DY, is therefore just a projection of quadrangle AD'BG, with its associated diagonals AB and GD', onto the plane of reflection, and AB is the common section of the planes containing the two quadrangles. The same procedure can be applied to every other point from which the form of A would reflect to B within the base-circle centered on G.

In the third and final case, A and B are disposed so that the plane passing through them cuts the axis obliquely. For instance, let B in figure 22a, p. 555, lie above A so that the plane passing through line AB strikes axis XG at a slant. Pass a plane through A orthogonal to axis XG to form circle FKLH with centerpoint G on the cone's surface. From vertex X draw XB and extend it until it meets the plane of circle FKLH at point B'. Applying the methods for finding the point, or points, of reflection in concave spherical mirrors, determine the point, or points, on circle FKLH from which the form of point B' would reflect to A. Let D' be one of them. From point D' produce line of longitude D'X on the cone's surface. Within the plane formed by D'X and axis GX of the cone, produce line YD through AB to intersect D'X orthogonally. Point D, where that line meets D'X, will be the point of reflection within the plane formed by ADBY, so angle of incidence BDY will equal angle of reflection ADY. Thus, quadrangle ADBY, with its associated diagonals AB and YD, is simply a projection of quadrangle AD'B'G, with its associated diagonals AB' and GD', onto the plane of reflection, and line AM passing through the intersection of DY and D'G is the common section of those two planes. The same procedure can be followed for any other point of reflection on circle FKLH, so there can be as few as one and as many as four such points on the cone, and each point will lie on a particular conic section.

3. *The Sources of Alhacen's Reflection-Analysis*

The clearly identifiable sources for Alhacen's analysis of reflection in books 4 and 5 of the *De aspectibus* are extremely limited in both type and number, consisting of Ptolemy's *Optics*, Euclid's *Elements*, Apollonius' *Conics*, and Serenus' *On the Cone and Cylinder*. Since these sources fall into two categories by type—optical and mathematical—the following examination of how Alhacen used them will be organized according to that division.

Optical Sources: As we noted in the discussion of sources in the previous edition of books 1-3, Alhacen drew on a variety of Greek and Arabic optical sources in formulating his overall theory of visual perception. We also noted that, rather than cite those sources specifically, Alhacen referred to them according to such generalities as "mathematicians" or "natural philosophers."[42] Still, it is possible to isolate probable sources on the basis of internal and external evidence. Such evidence, for instance, points to Ptolemy as the crucial authority among the "mathematicans" drawn upon by Alhacen in constructing the theory of visual perception and illusions laid out in books 1-3.[43]

The same problem applies to identifying the optical sources for Alhacen's analysis of reflection in books 4 and 5 of the *De aspectibus*. The possibilities are legion, but internal evidence suggests strongly that Ptolemy's account of reflection in books 3 and 4 of the *Optics* was the major, if not the sole, optical source for Alhacen. In fact, as will be clear in fairly short order, Ptolemy's account of reflection served as nothing less than a model for Alhacen's, determining not only the general structure, but also specific elements of analysis. This is not to say that Alhacen followed Ptolemy slavishly in every particular, but it would, I think, be fair to say that Alhacen's account of reflection represents a critical elaboration on Ptolemy's in terms of both rigor and comprehensiveness.

A case in point is Alhacen's experimental confirmation of the equal-angles law of reflection in book 4 of the *De aspectibus*. Alhacen sets the stage in chapter 2 with a series of simple experiments to show that light and illuminated color reflect in a consistent way, no matter the source or type of illumination (i.e., primary or secondary light). While these experiments have no counterpart in Ptolemy's *Optics*, the actual set of confirmatory experiments in chapter 3 of book 4 does. Described in *Optics* III, 8-11 (Smith, *Ptolemy's Theory*, 134-135), Ptolemy's version of these experiments is based on the following apparatus.

Let bronze disk BEDG in figure 23, p. 556, be divided into quadrants by lines BD and GE, each quadrant being subdivided into 90 degrees. From three thin strips of iron form three mirrors, one plane, one convex cylindrical, and one concave cylindrical.[44] Place these mirrors upright on the disk at point A such that line BAD is perpendicular to their bottom edges. Line GE will thus coincide with the reflecting surface of the plane mirror and will be tangent to the reflecting surfaces of the two curved mirrors TK and ZH at point A. Let line BA be drawn in white, and in arc BG draw some line LA in another color. Fix a sighting device at point L, and establish a line of sight along LA to point A, which is marked with a pin. Finally, while sighting along line LA, move a tiny colored marker along arc BE until its image I appears to lie perfectly in line with LA, as represented for all three mirrors in figures 23a-23c, p. 556, where cathetus MI intersects line of reflection AL or its extension in the case of the plane and convex mirrors. Let point M be where the colored marker lies on arc BE. Under those conditions, arc MB = arc LB, which is to say that angle of reflection MAB = angle of incidence LAB. No matter what point L is chosen on arc BG, point M, where the colored marker ends up, will be such that the equality of angles is preserved for all three mirrors.

There are, of course, significant differences in detail between this experiment and the one described by Alhacen in the third chapter of book 4, the most obvious being that Ptolemy's experiment measures the reflection of visual rays, while Alhacen's measures the reflection of light-rays. By choosing different things to measure, however, Ptolemy and Alhacen had to measure them in fundamentally different ways. Since visual rays are impossible to detect by direct observation, Ptolemy necessarily fell back upon an indirect method for confirming the equal-angles law based on image-location. In measuring the reflection of light-rays, on the other hand, Alhacen could approach the phenomenon directly by restructuring the experiment as he did so as to highlight the passage of light-beams through the wooden ring to the mirror and thence to the ring's inner wall. Even so, in the case of secondary color-radiation, Alhacen had no choice but to confirm the equal-angles law by indirect means because of the weakness of that radiation.[45]

A second major difference between the two experiments is that Ptolemy's involves only plane, convex cylindrical, and concave cylindrical mirrors, while Alhacen added convex and concave spherical and conical mirrors to bring the total of tested mirrors from three to seven. In this case, of course, the addition of spherical and conical mirrors represents an effort to make the experiment more comprehensive and general. To some extent, moreover, the need to accommodate the four new mirrors—particularly the convex and concave spherical ones—to the experiment explains the greater sophistication and complexity of Alhacen's apparatus over that of Ptolemy's.[46] By the same token, the fact that Alhacen, unlike Ptolemy, was measuring the passage of light-rays through his apparatus helps explain the extensiveness of his tests for both primary and secondary light- and color-radiation. It also helps explain his inclusion of the preliminary experiments in chapter 2. These experiments, of course, would have made no sense within the context of Ptolemy's analysis of reflection because that analysis was not based on light-rays.

By no means exhaustive, this brief discussion should nonetheless be adequate to show not only that Alhacen's core experiment for validating the equal-angles law of reflection was based on Ptolemy's, but that, where the two experiments differ, they do so in degree rather than in kind. In other words, those differences stem from differences both in what the two experiments were intended to measure and in the universality of the resulting conclusions. It is therefore at the level of implementation rather than conception that Alhacen actually broke from Ptolemy in his approach to optical experiment.

This point is clearly exemplified in the series of experiments Alhacen proposes at the beginning of book 5, paragraphs 2.1-2.20, to confirm that, no matter which of the seven mirrors is tested, all point-images seen in those mirrors will appear on the normal, or cathetus, dropped from the object-point to the reflecting surface. These experiments are suggested by Ptolemy in *Optics*, III, 4, where he observes that, "if we stand long, straight objects at right angles to the surface of mirrors, and if the distance is moderate, the images will appear [to lie] perfectly in line with those objects [as they] are properly viewed outside the mirror."[47] Hence, in establishing this point empirically for each of the seven types of mirror, Alhacen was simply following up on Ptolemy's suggestion and, in the process, validating it as comprehensively and rigorously as possible.

Even clearer than the link between Alhacen's and Ptolemy's empirical justification of the principles of reflection are the links between their respective applications of those principles to the mathematical analysis of reflection. These links are especially clear in the way Ptolemy and Alhacen approach the analysis of concave spherical mirrors in order to determine how many points of reflection there can be for a given center of sight and object-point.

As we saw on pp. lvi-lvii above, Alhacen begins this analysis in proposition 36, pp. 452-454, with the center of sight and the object-point lying on the same diameter within the mirror and flanking the center of curvature. Accordingly, he turns first in proposition 36, case 1, to the situation in which the two points are equidistant from the center of curvature, as illustrated in figure 5.2.36, p. 255, where E is the center of sight, H the object-point, and D the center of curvature, and where ED = DH. On that basis, he demonstrates that within the circle containing diameter ZEDHA, points G and B will be points of reflection for E and H and, furthermore, that if circle GABZ containing those points is rotated about diameter ZEDHA as axis, G and H will form a circle of reflection within the sphere of the mirror. Alhacen then concludes that no other point, such as C, within circle AZBG can be a point of reflection.

Ptolemy demonstrates the first two points in essentially the same way as Alhacen by showing in theorem IV.2 (Smith, *Ptolemy's Theory*, 176-177) that, if points H, Z, and E in figure 24, p. 557, represent the center of sight, the object-point, and the center of curvature on diameter BHEZD, and if HE = EZ, point A will be a point of reflection. Like Alhacen, he goes on to explain that, if triangle HAZ is rotated about axis BD, point A will form a circle of reflection centered on E. To prove the third point—i.e., that no point outside of the circle of reflection formed by A can serve as a point of reflection for Z and H—Ptolemy takes a slightly different tack from Alhacen's. Thus, in theorem IV.3 (Smith, *Ptolemy's Theory*, 177), he shows that, if ZE > HE, as represented in figure 24a, p. 557, A cannot be a point of reflection. He then shows in theorem IV.4 (Smith, *Ptolemy's Theory*, 177) that, if ZE > EH, as represented in figure 24b, p. 557, there will be no point of reflection, such as K, in arc AD of circle ADGB. On that basis, he concludes in theorem IV.9 (Smith, *Ptolemy's Theory*, 181-182) that, if there is some point of reflection, such as L, for H and Z in semicircle BAD in figure 24c, p. 557, there can be no other point of reflection, such as S, within that semicircle—which means by extension that, if A is the point of reflection for H and Z when HE = HZ, as in figure 24, p. 557, there can be no other point of reflection in semicircle BAD. He closes that theorem by observing that, if triangle LHZ in figure 24c, p. 557, is rotated about axis BZ, point L will produce a circle of reflection for H and Z.

Having demonstrated the previous three points for the case in which the center of sight and the object-point lie equidistant from the center of curvature on the same diameter within the spherical mirror, Alhacen then turns in proposition 36, case 2, to the situation in which the two points lie different distances from the center of curvature on the same diameter. Given these conditions, Alhacen wishes to show, first, how the point of reflection within the circle containing the diameter can be determined, and second, that there can be only one such point within the circle. Accordingly, if E and H in figure 5.2.36a, p. 256, represent the center of sight and object-point on diameter ADZ of circle AGZB within a concave spherical mirror, and if ED > HD, then we first locate point Q on the extension of AZ such that EQ:DQ = ED:DH. With point Q as center, we draw a circle of radius QD. Points G and B, where that circle intersects circle AGZB of the mirror, will be points of reflection. Those points, moreover, are the points at which the line drawn from point Q' on diameter Q'D of the cutting circle are tangent to circle AGZB of the mirror. Furthermore, no points other than G and B (e.g., C) can serve as points of reflection for E and H within circle AGZB. It therefore follows, according to Alhacen, that, if triangle EHG is rotated about axis

QE, point G will describe a circle of reflection for points E and H, which is the same point established by Ptolemy in theorem IV.9 discussed above.

At first glance, Ptolemy's version of this determination appears markedly different from Alhacen's, but under closer inspection it turns out to be essentially the same. For instance, in theorem IV.5 (Smith, *Ptolemy's Theory*, 177-179), Ptolemy locates the point of reflection as follows, according to figure 25, p. 558. Let Z and H be the center of sight and object-point on diameter BED, let E be the center of the mirror, let ZE > HE, and let point T be taken on ZE such that ET = EH. Find point K on the extension of diameter BD according to which ZT:EH = ZH:KH. It therefore follows that ZT:ZH = EH:KH, from which it follows that KZ (i.e., KH + ZH):KH = ZE (i.e., ZT + ET):ET = ZE:HE, since ET = HE. When line KL is then drawn tangent to circle ABGD, point L, where it touches arc AB of the circle, will be a point of reflection, and by implication there will be a corresponding point of reflection where the tangent from point K touches arc BG of the circle.

Now, if we reletter the diagram for Alhacen's demonstration to correspond with the diagram for Ptolemy's, as in figure 25a, p. 558, Alhacen's original proportion for determining the point of reflection becomes QZ:QE = ZE:HE. But QZ:QE = KZ:KH, so we end up with KZ:KH = ZE:HE, which is the proportion upon which Ptolemy's determination in theorem IV.5 is based. Accordingly, in the next theorem (IV. 6, in Smith, *Ptolemy's Theory*, 179), Ptolemy shows that, if KZ:KH ≠ ZE:EH, there will be no reflection within circle ABGD, which is tantamount to saying that, if some point other than L is chosen on arc AB of the circle, and if the tangent is dropped from that point to the extension of diameter BD, point K, where it intersects that diameter, will not yield the appropriate ratio.

Having fully analyzed the case in which the center of sight and object-point lie on the same diameter at equal or different distances from the center of sight, Alhacen turns to the case in which they lie on different diameters at equal or unequal distances from the center of curvature. As in the previous account, the simpler version of this case, which Alhacen addresses in proposition 37, cases 2 and 3, pp. 455-458. has the center of sight and object-point equidistant from the center of curvature. Under these circumstances, Alhacen shows first that, if H and T in figure 5.2.37b, p. 258, are the two points, and if they are situated in such a way that the circle passing through them and center of curvature D does not cut facing arc GB, there can be only one point of reflection—i.e., E. The proof centers on showing that if some other point O is chosen on arc GB, angles TOD and HOD will be unequal. On the other hand, if H and T are situated such that the circle passing through them and center of curvature D does cut the facing arc, as represented in figure 5.2.37c, p. 259, there will be three points of reflection on arc BG—i.e., E and points M and L where circle HDT intersects arc BG.

The Ptolemaic counterparts of these two propositions are to be found (in reverse order) in theorems IV.10 and IV.17 (Smith, *Ptolemy's Theory*, 182-183 and 187-188). In theorem IV.17, Ptolemy demonstrates that, if Z and H in figure 26, p. 559, represent the center of sight and object-point, if ZE = EH, and if circle ZDH does not cut facing arc LM of the mirror, there can be only one point of reflection, B, from that arc.[48] Like Alhacen, Ptolemy proves this by showing that, if some other point T is chosen on LM subtending angle LDM, the resulting angles HTD and ZTD will be unequal. When circle ZDH does cut arc LM, Ptolemy demonstrates in theorem IV.10, there will be three points of reflection for Z and H, those points being B, T, and K in figure 26a, p. 559. Ptolemy then goes on in theorems IV.11 and 12 (Smith, *Ptolemy's Theory*, 183-184) to show that no more than three such reflections are possible on arc LM.

When we move to the more complex case in which the center of sight and object-point lie on different diameters at unequal distances from the center of curvature, the links between Alhacen and Ptolemy become more tenuous, in great part because Alhacen's ultimate goal in analyzing this case is to set up the conditions for determining precisely where the points of reflection are located and thus to solve part of his eponymous problem. As a result, Alhacen's analysis is far richer and more sophisticated than Ptolemy's because, unlike Ptolemy, Alhacen is concerned not only with how many points of reflection there can be for a particular configuration of center of sight and object-point on their respective diameters, but also with the size of the reflected angles vis-à-vis the angle adjacent to the angle subtended by the arc facing the two points.[49] Nevertheless, there are some obvious points of convergence between the two analyses, and the most effective way to illustrate those convergences is by recourse to Alhacen's summation of the results of his analysis in proposition 49.

For instance, in proposition 49, case 1, p. 472, Alhacen establishes that, if A and B on diameters CG and HG in figure 5.2.49, p. 277, represent the center of sight and object-point facing arc CH of the mirror, if BG ≠ AG, and if circle AGB passing through the mirror's center of curvature does not cut that arc, there can be only one point of reflection. Moreover, in proposition 49, case 2, pp. 472-473, where circle AGB in figure 5.2.49a, p. 277, touches arc CH at point T (and thus represents a special case of the previous one), and given that AG > BG, Alhacen establishes that the single point of reflection—i.e., T'—will fall on arc CT.

Ptolemy establishes these two points in theorem IV.18 (Smith, *Ptolemy's Theory*, 188-189), as follows. Let Z and H in figure 27, p. 560, represent the center of sight and object-point facing arc ML of the mirror, let ZE > EH, which means that ZD > HD, and let circle ZDH passing through those two points and center of curvature D not cut arc ML. In addition, let KTD bisect line ZH at point T. Under those conditions, Ptolemy goes on to demonstrate, there can be only one reflection, and it will necessarily take place at some point N within arc KL.

In proposition 49, subcase 3b, pp. 473-474, Alhacen addresses the situation in which circle AGB cuts arc CH at points E and F in figure 5.2.49c, p. 277. According to this configuration, he concludes, it is possible for there to be as many as three reflections from arc CH, two of them within arc EF and one within arc EC or arc FH. However, as we noted in our discussion of propositions 45 and 46 on pp. xli-lxiv above, the fact that circle AGB cuts arc CH in two places is not enough by itself to ensure that there will be three reflection-points. Point F, where it intersects the arc on the right-hand side, must lie to the right of the midpoint of that arc. Thus, as represented in figures 16a and 16b, pp. 531-532, if that intersection-point falls on or to the left of midpoint X, there can be one, and only one reflection-point, and it will lie on the arc between K and the point where circle AGB intersects arc CH on the left-hand side (note that in figure 16a the intersection-point T represents the tangent).

Ptolemy establishes this latter point in theorem IV.16 (Smith, *Ptolemy's Theory*, 186-187) according to figure 27a, p. 560, where Z and H represent the center of sight and object-point, B the midpoint of arc LM, and ZDH the circle passing through Z, H, and center of curvature D. In that case, Ptolemy goes on to demonstrate, there can be no reflection from either arc BM or TB, so the only possible reflection will occur from arc TL. On the other hand, as Ptolemy proves in theorems IV.12 and IV.13 (Smith, *Ptolemy's Theory*, 183-185), if circle ZDH in figure 27b, p. 560, cuts arc LM at points T and K on either side of midpoint B, there will be three reflections altogether, two from arc BK and one from arc LT. This, of course, is the same conclusion that would follow from an Alhacenian analysis of the same configuration for center of sight Z and object-point H.

This relatively brief comparative account of Alhacen's and Ptolemy's analyses of reflection from concave spherical mirrors should suffice to show how deeply indebted Alhacen was to Ptolemy not only for fundamental points of analysis (e.g., the number of possible reflections for a particular configuration of the center of sight and object-point), but also for analytic techniques (e.g., the cutting circle that passes through the center of sight, the object-point, and the center of curvature). Alhacen's genius and originality thus lay not in reconstructing the science of optics from the ground up but in building upon the foundations already laid by Ptolemy and restructuring the subsequent analysis to tie up various loose ends left by Ptolemy. Foremost among these, of course, was determining exactly where the point or points of reflection will fall in the six curved mirrors Alhacen chose for analysis when the center of sight and object-point face those mirrors at random locations.

Mathematical Sources: Alhacen mentions only two mathematical source-authorities by name in books 4 and 5 of the *De aspectibus*, and then only sparingly. The first and more frequently cited of these, Euclid, is named only seven times, and only once is a locus in the *Elements* provided.[50] The second, Apollonius (or "Ablonius"), is named three times, and in only one instance is a locus in the *Conics* given this time by both book and proposition.[51]

Among unmentioned sources, there are numerous possibilities. To explain various details of Alhacen's mathematical reasoning in books 4 and 5, for example, Friedrich Risner, the late-sixteenth-century editor of the *De aspectibus*, refers to Theon's recension of Euclid's *Elements* and commentary on the *Almagest*, Proclus' commentary on the *Elements*, Theodosius' *Sphaerics*, and Serenus' *On the Section of a Cylinder*. In citing these authorities, however, Risner was simply appealing to sources that would have helped contemporary readers follow his explanation of Alhacen's logic. In no way was he attempting to isolate Alhacen's actual sources—a point borne out by his citation of Campanus of Novara's thirteenth-century edition of the *Elements*, a work to which Alhacen could not possibly have had access.[52]

It is, of course, highly likely that Alhacen at least knew of, and was even familiar with, the ancient sources cited by Risner. It is equally likely that these and many other ancient Greek and contemporary Arabic sources contributed in various ways to the mathematical knowledge Alhacen brought to bear on his analysis of reflection. The problem is whether and how he might have drawn on specific points within these sources in the course of that analysis. Was he actually thinking of Proclus' commentary on *Elements*, I.29 when he concluded in proposition 25 (paragraph 2.212, p. 430) that, if line BI in figure 5.2.25, p. 239, is drawn parallel to line DL, then line DQ will intersect BI? Perhaps so. But if we ask whether he needed to refer to Proclus' commentary in order to reach this conclusion, the answer is clearly no. The most rudimentary understanding of Euclidean geometry is sufficient warrant for that conclusion.

If, therefore, we restrict our examination to mathematical sources that seem directly and necessarily pertinent to Alhacen's analysis, the list reduces to three: Euclid's *Elements*, Apollonius' *Conics*, and Serenus' *On the Section of a Cylinder*. The choice of the first two is of course dictated by

Alhacen's own citation of them. The choice of the third follows from Alhacen's awareness that a plane cutting a cylinder obliquely will form a true ellipse, not simply an ellipse-like section, on the cylinder's surface. Since this fact is by no means self-evident, it stands to reason that Alhacen knew it either directly or indirectly from Serenus' demonstration in proposition 20 of *On the Section of a Cylinder*.[53]

Alhacen's explicit use of Apollonius in the fifth book of *De aspectibus* is limited to a handful of propositions in book 2 of the *Conics*, where Apollonius deals with hyperbolic sections. Twice, in proposition 19, lemma 1, and proposition 20, lemma 2, Alhacen cites proposition II.4 of the *Conics*. He also cites proposition II.8 twice, both times in proposition 19, lemma 1, and once he has recourse to proposition II.16, in proposition 20, lemma 2. Other than these particular references, there are no other instances in books 4 and 5 of the *De aspectibus* where specific loci in the *Conics* can be pinpointed with certainty.[54] There is no question, however, that Alhacen's narrative account of conical and cylindrical mirrors in book 4, chapter 5 of the *De aspectibus* is based on a thorough understanding of conic sections and their focal properties. This is hardly surprising, given that Alhacen had adequate mastery of the *Conics* to essay a reconstruction of the missing eighth book of that treatise.[55]

Euclid's *Elements* is by far the most widely and consistently applied source for Alhacen's analysis of reflection in book 4 and, more to the point, book 5. This claim is supported by the fact that, despite Alhacen's almost unvarying failure to cite specific loci, it is easy to tie particular steps in his constructions and proofs to particular theorems in the *Elements*, a process made even easier by Risner's explanatory interpolations.[56] Practically speaking, however, it would be pointless to tie every step in Alhacen's constructions and proofs to its relevant propositional source in the *Elements*. It makes virtually no sense, for instance, to assume that every time Alhacen mandated erecting a line perpendicular to another at a given point on it he was consciously drawing on *Elements*, I.11. Nor is it likely that he expected his readers to have explicit recourse to that proposition in order to grasp the instruction and its full meaning. After all, the required construction is conceptual, not practical.

Barring such relatively insignificant links, Alhacen's mathematical reasoning at various points throughout book 5 of the *De aspectibus* is based on—or at least implicitly supported by—some 42 Euclidean propositions, virtually all of them distributed among the first six books of the *Elements*. Or, to put it in slightly different terms, without tying various logical steps in Alhacen's reflection-analysis to these particular propositions, it is extremely difficult, if not impossible, to fully comprehend the mathematical reasoning upon which that analysis is based. Take as an example Alhacen's construction for proposition 25, as illustrated by figure 5.2.25, p. 239. In the course of this construction, Alhacen concludes that, since angle BDG of triangle BDG is bisected by DQ, it follows that BQ:QG = BD:DG. Far from evident in its own right, this conclusion become evident only in the light of *Elements*, VI.3, where Euclid demonstrates that, if an angle of a triangle is bisected by a line that intersects its base, the ratio of the segments cut from the base by that bisector is the same as the ratio of the other two sides of the triangle. Like this particular proposition, each of the remaining 41 just mentioned clarifies specific steps taken by Alhacen when the logic underlying those steps is not immediately apparent.

Although the 42 Euclidean source-propositions isolated according to this criterion all fall within the first six books of the *Elements*, their distribution among those books is highly irregular. Only one proposition in book IV (IV.5) figures in Alhacen's analysis, and it occurs only once. Likewise, propositions from book II of the *Elements* rarely crop up in the course of book 5, and the ones that do are limited to II.1-II.3. Book I of the *Elements* is represented by nine propositions, four of which (I.4, I.19, I.26, and I.32) recur with moderate frequency in Alhacen's analysis—I.26 and I.32 six times, I.4 five times, and I.19 four times. The lion's share of propositions—29 in all—is thus distributed among books III (12), V (8), and VI (9). Within book III, the most frequently recurring propositions are III.22 (nine times) and III.31 (six times), and within book VI, propositions VI.3 and VI.4 recur most frequently by far, the first showing up sixteen times and the second twenty-one times. Within book V, proposition V.22 recurs six times, V.16 and V.18 four times, and IV.17 three times. None of the rest crops up more than twice.[57]

As far as total use is concerned, the propositions in book V, which deal generally with ratios and proportionality, and book VI, which deal with the proportionality of areas (as applied to triangles in particular), figure somewhat more prominently than those from the preceding four books. The reason for this prominence is the extensive use to which Alhacen puts proportionality theory throughout his analysis. Indeed, proportionality theory lies at the very heart of Alhacen's most intricate and technically demanding constructions and demonstrations. From a purely Euclidean standpoint, Alhacen's reliance on compound ratios at certain points in his analysis is somewhat unusual, if not idiosyncratic, but there is ample precedent for that in Apollonius' *Conics*.[58] Otherwise, Alhacen's treatment of proportions and ratios is Euclidean to the core.

All told, then, it is possible to follow Alhacen's analysis in book 5 to the letter on the basis of a remarkably small fund of mathematical sources, Euclid's *Elements* foremost among them. And the same holds for his optical sources, which reduce essentially to Ptolemy's *Optics*. One might therefore expect that, given the paucity of sources upon which he relied more or less explicitly, Alhacen's application of those few sources to his analysis of reflection would be commensurately limited. But, as should be eminently clear from previous discussion, the very opposite is true. Alhacen's account of reflection is extraordinarily complex and sophisticated in terms not only of structure and content but also of conception. In those regards, in fact, it far outstrips the sources upon which it was based. Rather than detracting from Alhacen's achievement in book 5, the fact that he drew on so few sources and yet managed to make so much of them is a testament to his logical acuity and rigor as well as to his powers of invention.

4. *The Reception of Alhacen's Reflection-Analysis in the Latin West*

The account of reflection in the fifth book of the *De aspectibus* represents a critical turning point for Alhacenian optics. Everything, or virtually everything, up to proposition 1 of that book (with the possible exception of chapter 5 of book 4) is readily understandable to an intelligent but mathematically untutored reader. Such is clearly not the case with the theorematic portion of book 5. Many of the conclusions drawn in that portion are difficult to grasp because they are so deeply buried in dense thickets of mathematical reasoning. And the same holds for the ulterior intent of several of the theorems, propositions 45 and 46 being prime examples.[59] In order to understand the optical content of book 5 in its full array of implications and ramifications, therefore, it is necessary to understand the underlying analysis as fully as possible. This simply cannot be done without a fairly solid grasp of the first six books of Euclid's *Elements* and a reasonably good understanding of the basic geometry of conic sections.

More to the point, without a fairly sure grounding in the Euclidean theory of proportions laid out in book 5 (and to some extent in book 6) of the *Elements*, even the most assiduous reader is likely to flounder quite early in the course of Alhacen's reflection-analysis. This point has a crucial bearing on the reception and use of the *De aspectibus* during the thirteenth and succeeding centuries. Because of its conceptual difficulty, the fifth book of the *Elements* was regarded by medieval students as the *pons asinorum* of Euclidean geometry. Its crossing was therefore considered to be an achievement. But to cross it was not necessarily to command it. Those who had truly mastered Euclidean proportionality theory thus represented a limited segment of the university-educated elite of medieval Europe. In addition, there is no firm evidence that Apollonius' *Conics* was available in Latin translation during the Middle Ages.[60] If it was, it had almost no impact on mathematical learning in the Latin West before the Renaissance. Likewise, there is little or nothing to suggest that a Latin version of Serenus' *On the Section of a Cylinder* was in circulation during the Middle Ages.

By the early thirteenth century, then, lack of appropriate texts and a somewhat superficial training in geometry left most scholastic readers unprepared to make real sense of Alhacen's mathematical analysis in book 5 of the *De aspectibus*. There was at least one exception, though. Thanks to Marshall Clagett we know that, in proposition IV.20 of his *De triangulis*, Jordanus Nemorarius appealed to proposition 20, lemma 2, of book 5 of the *De aspectibus* to support a step in his method for trisecting an angle.[61] Although the dating of this work is problematic, it was certainly written before 1260 and perhaps as early as the 1230s.[62] Whichever the case, Jordanus' use of the Alhacenian lemma indicates that the *De aspectibus* was actually being read by the mid-thirteenth century, not only for its optical content, but also for its mathematical content. It also suggests that by this time there was an adequate fund of mathematical knowledge—and therefore, presumably, of appropriate texts—to make that reading meaningful.[63]

A truly gifted mathematician, Jordanus was exceptional for his time, so his ability to mine the *De aspectibus* for its mathematical treasures tells us little about the general level of mathematical learning in thirteenth-century Europe. A better indicator is Witelo, whose *Perspectiva* of c. 1275 was so closely modeled on Alhacen's *De aspectibus* that the late-sixteenth-century scientific impressario Giambattista della Porta dubbed him "Alhazen's ape."[64] Born in Silesia, probably in Wroclaw, in the early-to-mid-1230s, Witelo apparently received his formative education in Poland before entering the University of Paris as an arts student in 1253. Master's degree in hand, Witelo seems to have returned home for a short while before enrolling for higher study, perhaps in canon law, at the University of Padua. There he remained from the very early to the very late 1260's, at which time he moved to the papal court at Viterbo. It was during his sojourn there that Witelo composed the *Perspectiva* under the mentorship of William of Moerbeke, who provided him both encouragement and practical help in the form of textual translations.[65]

Porta's flippant dismissal of Witelo as "Alhazen's ape" was both unfair and inaccurate. In fact, Witelo wrote the *Perspectiva* not simply to recapitulate Alhacen's optical analysis, but to make it readily accessible to contemporary readers. To that end he subdivided Alhacen's analysis into relatively short, digestible theorematic chunks, restructured it accordingly, and added elaboration or explanation whenever he saw fit. As part of this effort to render the *De aspectibus* accessible to contemporary scholars, Witelo devoted the first book of the *Perspectiva* to a set of 137 theorems (plus sixteen definitions and five postulates) that were clearly intended to provide the requisite mathematical foundations for the subsequent account of reflection and image-formation in books 5-9 of the *Perspectiva*.[66]

Some of the theorems among the set provided by Witelo in the opening book were undoubtedly included for the sake of rigor rather than for actual instructional purposes. The fact that two spheres can touch at only one point is virtually self-evident to anyone with a rudimentary understanding of spheres and their properties. But self-evidence is not proof, so Witelo offers a demonstration in I.76 (Unguru, 102). The same holds for the fact that two planes parallel to the same plane are parallel to each other, which Witelo demonstrates in I.24 (Unguru, 61).[67] Such demonstrations of "self-evident" points are in the minority though. Most of the theorems in book 1 show things that are not immediately obvious in order to lay the ground for their later application in the body of the *Perspectiva*. For instance, in propositions I.3-13 (Unguru, 49-55), Witelo demonstrates a number of points about proportions and the manipulation of their terms in order to explain such operations as alternation, composition, separation, and compounding of ratios, all of which play a crucial role in the subsequent analysis of reflection.[68] But propositions I.3-13 add little or nothing to what Euclid had already established in the fifth book of the *Elements*.[69] Why, then, did Witelo bother to include them? High among the reasons, I suggest, was a concern on Witelo's part that his readers might not be adequately schooled in proportionality theory to grasp the point of these propositions without help.

It is surely with this consideration in mind that Witelo undertook the analysis of cones and cylinders presented in propositions I.89-118 (Unguru, 110-136). The fundamental purpose of that analysis is twofold: first, to provide a framework for understanding how planes of reflection passing at various angles through cones and cylinders create particular sections within which reflection occurs in determinate ways, and second, to introduce the appropriate conditions for solving the six lemmas provided in propositions 19-24 of the *De aspectibus*. Forming the capstone of book 1 of the *Perspectiva*, those six lemmas are recapitulated in the final ten propositions: I.128 and I.130 (Unguru, 145-146 and 147-152) corresponding to proposition 19, lemma 1; I.133 (Unguru, 153-155) corresponding to proposition 20, lemma 2; I.134 (Unguru, 155-158) corresponding to proposition 21, lemma 3; I.135 (Unguru, 158-162) corresponding to proposition 22, lemma 4; I.136 (Unguru, 163-165) corresponding to proposition 23, lemma 5; and I.137 (Unguru, 165-167) corresponding to proposition 24, lemma 6.

Witelo's analysis of cones and cylinders in the second half of book 1 is especially interesting for the insights it provides into the level of Witelo's mathematical expertise and thus the state of mathematical development in his day. After all, the study of conic sections represented a major advance in post-Euclidean geometry and, as such, was regarded as a higher-order subdiscipline within mathematics well up into the Renaissance and beyond. The fact that Witelo was able to deal with cones (and their extension to cylinders) would seem to indicate a surprisingly high level of mathematical sophistication on his part. It also suggests that he was at least aware of, if not familiar with, Apollonius' *Conics* and Serenus' *On the Section of a Cylinder*. But an examination of the actual theorems included by Witelo in his analysis of cones and cylinders belies the first of these suppositions and therefore brings the second into question. Almost all of the proofs offered in the course of that analysis can be formulated on the basis of Euclid's *Elements* and a fairly rudimentary understanding of the generation of right cones from the rotation of isosceles triangles about the axis formed by the line bisecting the angle at the vertex and the generation of cylinders from the rotation of rectangles about one of their sides as axis.[70]

There is, however, at least one intriguing exception. In I.98 (Unguru, 115-116), Witelo explains the generation of the three principal conic sections descriptively according to the obliquity of the plane cutting the cone. He then denominates each of the resulting sections in three ways: according to whether they are generated by a right-angled, obtuse-angled, or acute-angled planar cut; according to whether that cut produces a parabola, hyperbola, or ellipse on the cone's surface; or according to whether the generated section conforms to "what the Arabs call" *mukefi*, increased *mukefi*, or diminished *mukefi*.[71] Three things are worth noting about these denominations. First, the characterization of conic sections according to the type of angle is pre-Apollonian, so its use by Witelo suggests a pre-Apollonian source (perhaps Archimedes) for at least some of his knowledge of conic sections. Second, because the terms parabola, hyperbola, and ellipse were coined by Apollonius in the *Conics*, Witelo's use of them suggests some familiarity either with the *Conics* itself or with a post-Apollonian source. And third, Witelo's adverting to the Arabic terminology (which reflects the Apollonian terminology) indicates that at least some of what he knew about conic sections came through Arab channels other than Alhacen, who refers to all conic sections generically as "sectiones piramidis."[72]

All of this suggests that Witelo actually knew Apollonius' *Conics*, or at least the section of book 1 dealing with the generation of the three principal conic sections (i.e., I.11-13 at a minimum) and the relevant analysis of hyperbolic sections in book 2 (i.e., I.4, I.8, and I.16 at a minimum). The problem is that, in terms of its actual application in book 1 of the *Perspectiva*, most of this knowledge, particularly as regards book 2 of the *Conics*, could have come from Alhacen rather than from the *Conics*.[73] In fact, the propositions in book 1 of the *Perspectiva* where the traces of that knowledge are especially prominent (i.e., I.128-133—Unguru, 145-155) are based entirely upon propositions 19-20 (lemmas 1 and 2) of the *De aspectibus*, and it is precisely here that Alhacen brings book 2 of the *Conics* to bear in his analysis.

That this may in fact have been the route, or at least the main route, by which Witelo became acquainted with Apollonius' analysis of hyperbolic sections finds support in certain peculiarities of Witelo's analysis. For one thing, *Conics*, II.8 is absolutely central to the construction of the figure (5.2.19c, p. 231) that accompanies proposition 19, lemma 2 (see esp. case 2, pp. 416-417). The key point in this construction is to establish that MO = LC, which follows from Apollonius' proof in II.8 that, if a line cuts a hyperbolic section at two points, and if it is extended on both sides to the asymptotes, the segments cut off on each side by the hyperbola and the respective asymptote will be equal. This proof is relatively straightforward, yet Witelo fails to include it among the propositions given in book 1 of the *Perspectiva*. Instead, in proposition I.129 he offers an extraordinarily maladroit description of what it demonstrates, concluding that the proof itself is not worth giving, "since it depends on many preliminary principles of [the *Conics*]" (Unguru, 147).

Witelo is correct. Proposition II.8 does depend on several preliminary principles, but for all practical purposes these can be reduced to the four points demonstrated in *Conics*, I.32, II.3, II.4, and II.7.[74] Hence, given his apparent familiarity with the *Conics*, Witelo's failure even to take a stab at recapitulating the proof in II.8 is puzzling. On the other hand, if proposition 19, lemma 2, of the *De aspectibus* was the sole, or even the main, conduit through which Witelo had access to *Conics*, II.8, and if, therefore, he learned *what* it demonstrated without learning *how* it was demonstrated, then that failure becomes far less puzzling.[75] None of this precludes the possibility that Witelo had a version of the *Conics* at hand in some form or another, but it does, I think, raise serious doubts about how well he understood the contents of that version.[76]

That Witelo was not particularly adept in the geometry of conic sections should not blind us to the fact that he knew enough to make basic sense of the two Alhacenian lemmas in which it figures prominently. Nor should it blind us to the likelihood that Witelo availed himself of textual resources beyond Alhacen's *De aspectibus* to acquaint himself at least to some extent with that geometry. Indeed, Witelo seems to have drawn on a surprising variety of mathematical sources in the course of book 1. Included among these, according to Unguru's count, may be Pappus' *Mathematical Collection*, Campanus' recension of the *Elements*, Theon's commentary on the *Elements*, Eutocius' commentary on Archimedes' *On the Sphere and the Cylinder*, Jordanus Nemorarius' *Geometry*, and Theodosius' *Sphaerics*.[77]

As Unguru rightly observes, Witelo was a mathematician of no great originality or genius. Many of his interpolations and elaborations (e.g., the "proof" of the parallel postulate) are at best misconceived and at worst logically insupportable.[78] But what Witelo lacked in innate talent he made up for in perseverance, and it was this trait combined with a more-or-less adequate fund of mathematical resources that enabled him to cope with Alhacen's reflection-analysis as well as he did. That, in a nutshell, is what makes Witelo so significant as a gauge of the intellectual temper of his times. The very fact that a mathematician of such limited talent could nonetheless make good sense of Alhacen's reflection-analysis indicates that, by the last quarter of the thirteenth century, mathematics had reached a high enough level of development in the Latin West to permit mathematicians of far lower caliber than Jordanus a generation or so earlier to read the *De aspectibus* from cover to cover with almost perfect comprehension. Such readers were doubtless thin on the ground, but there were enough to ensure the entrance of Alhacen's treatise into the scholastic mainstream by the close of the thirteenth century.

Like Newton's *Principia*, however, Alhacen's *De aspectibus* had more than one audience. On the one hand, there were those, like Witelo, who absorbed the work in all its technical, mathematical detail. The traces of this group can be seen in the copious annotations and corrections to be found in the surviving manuscripts of the *De aspectibus*. On the other hand, there were those (undoubtedly a significantly larger group) who were more interested in the gist than in the minute details of Alhacen's analysis. The two foremost examples of this latter group are Witelo's contemporaries, Roger Bacon and John Pecham.[79] Of the roughly 168 pages comprising the modern Latin edition of Bacon's *Perspectiva*, for example, only about seventeen are devoted to reflection. This amounts to around ten percent of the entire treatise, as opposed to something over fory-five percent of Alhacen's and Witelo's counterparts. Furthermore, in the course of his brief account of reflection, Bacon barely mentions the key points in book 5 of the *De aspectibus* concerning the number of possible reflections in the six curved mirrors, and he makes no mention whatever of the method for locating them.[80] But Bacon was less interested in providing a comprehensive analysis of reflection than in explaining image-distortion as clearly and briefly as possible on the basis of the cathetus-rule. Moreover, his interest in optics was spurred less by scientific than by theological concerns.[81] So it was not necessarily out of mathematical incompetence that Bacon avoided the technical issues of Alhacen's reflection-analysis. It was out of a desire to follow particular analytic imperatives that included those issues only peripherally.[82]

Pecham's case is somewhat different from Bacon's. For one thing, he devoted far more of his optical compendium (roughly thirty percent) to reflection than did Bacon. For another, as he put it himself, Pecham's primary goal in writing the *Perspectiva communis* was "to compress into concise summaries the teachings of perspective, which [in existing treatises] are presented with great obscurity. . . ."[83] Unlike Bacon, in short, Pecham focused on the scientific rather than the theological implications of optics. As a result, he confronted some of the technical issues that Bacon had sidestepped.

For instance, in his discussion of concave spherical mirrors in *Perspectiva communis*, II.45, II.46, and II.48 (Lindberg, *Pecham*, 197-205), Pecham establishes the conditions under which one, two, or four images can be seen in such mirrors when the center of sight and object-point lie on different diameters at equal distances from the center of curvature. These three theorems correspond to the three cases presented in proposition 5.37 of the *De aspectibus*. Accordingly, in proposition II.45 (cf. 5.37, case 1, pp. 454-455), Pecham shows that, when the center of sight and object-point lie outside the sphere of the mirror, only one reflection is possible. Not only is his proof essentially the same as Alhacen's, but the figure on which he bases it is virtually identical in form and lettering to the one used by Alhacen (cf. figure 5.2.37, p. 396). Pecham then goes on in proposition II.46 (cf. 5.37, case 2, pp. 455-456) to show that, if the center of sight and object-point both lie inside the sphere of the mirror, there can be as many as two reflections. In this case, Pecham's demonstration is such a rough distillation of Alhacen's as to be no demonstration at all. Finally, in proposition II.48 (cf. 5.37, case 3, pp. 456-458), Pecham shows that there can be as many as four reflections, the proof depending on a figure that is clearly abstracted from the one used by Alhacen (cf. figure 5.2.37c, p. 259).

All three of these propositions are predicated on having the center of sight and object-point equidistant from the center of curvature. But what about the more complex situation in which the two points lie different distances from that center? Pecham addresses this case obliquely in proposition II.47, where the ostensible purpose, as stated in the enunciation, is to show that three images are possible in concave spherical mirrors. Entirely descriptive, the body of this brief proposition appears to be based somewhat loosely on proposition 5.49 of the *De aspectibus*, where Alhacen offers a summary of results from propositions 43-48. Pecham's exposition is worth quoting in full:

> Take two points on different diameters, one inside the circle and the other on the circumference of the circle or outside. And if a [second] circle is drawn including these two points along with the center of the mirror, then if the [second] circle intersects the circle of the mirror in one place, reflection will take place from one arc only; if intersection occurs in two places, reflection can take place from one point of the arc interposed between the diameters or from two or from three or sometimes from four.[84]

What Pecham seems to have in mind for the first situation is illustrated in figure 5.2.49d, p. 278. In this case, if A and B′ represent the relevant points, with B′ lying on the mirror's circumference and A inside the mirror so that at point E circle AGB intersects arc CB′ (i.e., "the circle of the mirror"), then reflection can only occur from segment EB′ of arc CB′. Thus, to use Pecham's own words, "reflection will take place from one arc only" (i.e., EB′) and not from both CE and EB′. In addition, as we established earlier on pp. xlii-xliv, there will be two reflections in this case, both from arc EB′. One reflection will also occur from arc KD, bringing the total to three.

On the other hand, if "intersection occurs in two places" on arc CH, then both A and B must lie inside the circle of the mirror, as represented in figure 5.2.49c, p. 277.[85] Under these conditions, as we established on p. xliv above, reflection can occur from segment FE as well as from segment CE or HF of arc CH, depending on whether A or B is closer to centerpoint G. Moreover, within segment FE there can be two reflections, so if we add the single reflection from segment CE or HF, there can be as many as three reflections from arc CH. Add to those the reflection from arc KD, and the number rises to four.

However, as we noted earlier on pp. xli-xlii above, unless point F, where circle AGB cuts arc CH, lies to the right of the midpoint of CH, there can be no reflection from segment FE, in which case reflection is only possible from segment CE or FH. With the single reflection from KD added, there will thus be a total of two reflections. Altogether, then, counting the case in which both A and B lie outside the mirror (as in II.46), "reflection can take place from one point . . . or from two or from three or sometimes from four."

Suffice to say, this interpretation of Pecham's intent in proposition II.47 is forced to the point of torture, and its forcing depends upon considerable knowledge of what Alhacen establishes in propositions 43-46 of the *De aspectibus*. But, as far as the actual mathematical demonstrations are concerned, those propositions exemplify "the great obscurity" of optical analysis that Pecham was at such pains to avoid, which is doubtless why he ignored them.[86] Hence, in his effort to streamline the account of reflection in concave spherical mirrors, Pecham chose as his paradigm the simpler case, in which A and B are equidistant from center of curvature G. In the interest of comprehensiveness, however, he was compelled to give at least lip service to the more complex case, in which A and B lie different distances from G. Both simplistic and confused, the resulting account misses the real import of that case, and, as a result, Pecham failed to grasp the central point of Alhacen's reflection-analysis, which was to formulate a general method for locating the point or points of reflection in the six curved mirrors chosen for analysis in books 4-6 of the *De aspectibus*.

Hence, while Pecham's account of reflection is somewhat less superficial than Bacon's, it is only marginally so in comparison to the one provided in Alhacen's *De aspectibus* and Witelo's *Perspectiva*. Yet it was precisely because of their superficiality (which passed for simplicity) that Bacon's and Pecham's optical treatises enjoyed considerably wider diffusion in scholastic circles than those of Alhacen or Witelo. Indeed, as remarked earlier, Pecham's *Perspectiva communis* became a standard text for the teaching of optics in the Middle Age precisely because of its apparent simplicity and comprehensiveness. Accordingly, many medieval scholars were introduced to Alhacenian optics through what amounted to popularizations in the works of Pecham and Bacon, who focused on the basic conclusions of Alhacen's reflection-analysis with virtually no regard for the deep mathematical structure of analysis. There were thus two distinct avenues for the dissemination of Alhacenian, or Perspectivist, optics in the Middle Ages. The one mapped out by Alhacen and Witelo takes the reader through the science of optics in all its analytic detail and complexity. The other mapped out by Bacon and Pecham offers a short cut that intentionally bypasses such analytic detail and complexity.

David Lindberg has done yeoman service in showing how these four treatises shaped the Perspectivist optical tradition that lasted well into the seventeenth century.[87] It is largely through his efforts, for instance, that we have a fairly complete census of the manuscripts and subsequent printed editions of all four treatises.[88] This census, of course, tells us a great deal about the reception and circulation of those works throughout the Middle Ages and Renaissance. We also know a fair amount about their use during that period, either from direct citations or from the context within which they crop up. Hence, we have a number of ways to trace the influence of these works, some of them direct, some indirect.

The problem is that, although we have ample evidence *that* these works were read and used, and are even able to pinpoint their actual use, we know very little about *how* they were read and used. This problem is especially acute in the case of Alhacen's *De aspectibus* and its proxy, Witelo's *Perspectiva*. Both were cited copiously during the Middle Ages and Renaissance, and both were drawn upon in a variety of ways without explicit reference.[89] But citing a work or drawing on it selectively does not require mastery of it. Nor for that matter does it require more than passing familiarity based on a knowledge of certain key points in the text. It may well be the case, then, that, as they were picked over during the decades after their initial appearance, Alhacen's *De aspectibus* and Witelo's *Perspectiva* dwindled in scope to a relatively small set of canonical loci in the same way as happened to so many other authoritative texts over the period.[90]

This being so, we would expect to find few if any traces of the technical details of Alhacen's reflection-analysis in later works inspired by the *De aspectibus* or Witelo's *Perspectiva*. This expectation is, in fact, met. There is no evidence, apart from the manuscripts themselves of the two treatises, that the hard, mathematical core of Alhacen's analysis was studied in earnest before the early fifteenth century. As far as we know, the earliest witness to such study is an Oxford statute of 1431 that allowed Alhacen's or Witelo's treatises to be substituted for Euclid's *Elements* in the teaching of geometry at the undergraduate level.[91] Otherwise, although there are vague indications that the two works were used for university instruction during the fourteenth and fifteenth centuries, we have no clear idea of how they were employed for that purpose. In view of the optical commentaries and epitomes that survive from this period, though, it seems clear that the two works were overwhelmingly favored for their optical rather than their mathematical content.[92]

That this latter content is all but missing in derivative studies of the period suggests that very few scholastic thinkers appreciated the complexity of Alhacen's reflection-analysis, particularly as it pertains to the most original and ingenious part of that analysis: the method for locating points of reflection. This suggestion is reinforced by the fact that, despite their relative lack of theoretical and analytic sophistication, Euclid's *Catoptrics* and Ptolemy's *Optics* were even more widely studied than Alhacen's *De aspectibus* and Witelo's *Perspectiva* during the Middle Ages and Renaissance. Indeed, interest in these two works, Ptolemy's *Optics* in particular, appears to have burgeoned during the Renaissance.[93] Why? For one thing, the visual-ray theory of Euclid and Ptolemy is mathematically equivalent to the light-ray theory of Perspectivist optics, so, as far as the basic geometry of reflection is concerned, there is nothing to choose between the two. For another, at a practical level, Euclid's and Ptolemy's works provide a perfectly adequate account of reflection. To understand the essential rules of reflection and image-location does not require the sort of intricate mathematical analysis provided in the Alhacenian account.[94]

That the most mathematically sophisticated and interesting portion of Alhacen's reflection-analysis failed on the whole to capture the interest of Renaissance scholars was not due to a corresponding lack of interest in mathematics during that period. On the contrary, from the mid-fifteenth century on, the study of mathematics underwent explosive development through the recovery of Greek texts and their eventual translation and publication. For instance, the Latin version of Euclid's *Elements*, based on Campanus' translation from the Arabic, appeared in print for the first time in 1482 and was followed in 1505 by a translation from the Greek. The first Greek edition was published not long after, in 1533.[95] Four years later, in 1537, books 1-4 of Apollonius' *Conics* appeared in print for the first time in Latin translation, and in 1566 Federico Commandino's magisterial Latin version was published along with Eutocius' commentary and Pappus' *lemmata*. Serenus' *On the Section of a Cylinder* was also included in that edition.[96] Along with these key mathematical sources, moreover, Witelo's *Perspectiva* saw the first light of publication in 1535, the resulting edition having been popular enough to warrant reprinting sixteen years later.[97]

Within the context of these newly published sources, Friedrich Risner's tandem edition of Alhacen's *De aspectibus* and Witelo's *Perspectiva*, which appeared in 1572 under the title *Opticae Thesaurus*, is doubly significant. Friend and protegé of Petrus Ramus, who was one of the foremost mathematicians of the sixteenth century, Risner was inspired by his mentor to undertake the edition in the first place. Ramus, in fact, provided him with the necessary manuscripts upon which to base the edition.[98] A notable mathematician in his own right, Risner was considered eminent enough in that discipline to have been earmarked by Ramus for a chair in mathematics that he had endowed at the University of Paris.

Risner's quality as a mathematician and his familiarity with the sources just discussed come clear throughout the tandem edition of Alhacen's and Witelo's treatises. In short, the *Opticae Thesaurus* represents the cutting edge in mathematics for its day. On the other hand, Risner was no less an optician than a mathematician and, as such, collaborated with Ramus in the production of an optical tract that was eventually published well after both men had died.[99] The *Opticae Thesaurus* thus represents the cutting edge in optics for its day. In its double capacity as a mathematical and an optical text, therefore, Risner's edition of Alhacen and Witelo offered the best of both disciplines for several generations to come.

The list of those who were introduced to Alhacen and Witelo through Risner's edition includes a remarkable array of scientific luminaries of the later sixteenth and seventeenth centuries. Among these, of course, Johan

Kepler, René Descartes, and Christiaan Huygens loom large, but many equal or lesser lights, such as Willibrord Snel and Thomas Hariot, carry that list to the very end of the seventeenth century.[100] But Alhacen's model of visual imaging had been pretty thoroughly discredited by the mid-seventeenth century on the basis of Kepler's searching critique in the *Ad Vitellionem paralipomena* of 1604. Popularized by the likes of René Descartes, Kepler's alternative model of retinal imaging, which struck at the very heart of Alhacen's account of visual imaging, become more or less canonical by mid-century.[101] Moreover, as part of his critique, Kepler raised serious doubts about the applicability of the cathetus-rule to the determination of image-location in convex and concave mirrors.[102] These doubts, which called into question Alhacen's entire account of specular and refractive image-formation, were taken seriously by subsequent opticians as they attempted to make better sense of mirrors and lenses. Consequently, the analysis of reflection and refraction was approached in new, more sophisticated (and complex) ways, as witness the series of optical lectures Isaac Barrow gave as Lucasian Professor of mathematics at Cambridge in 1667 and published two years later under the title *Lectiones Opticae*.[103]

The last effective traces of Alhacen's reflection-analysis are to be found in the discussion of "Alhazen's Problem," which centered on a general solution conveyed by Christiaan Huygens to Henry Oldenbourg, secretary of the Royal Society, in June of 1669.[104] Huygens' solution involved the generation of opposite branches of a hyperbola within particular asymptotes constructed according to the placement of the center of sight and object-point with respect to the given spherical mirror. The four points where the two branches cut the circle of the mirror constitute the points of reflection.[105] A little over a year later, in July, 1670, René François de Sluse of Liège wrote a letter to Oldenbourg in which he claimed to have been inspired to solve the same problem in the course of reading Isaac Barrow's *Lectiones Opticae* of 1669.[106] Indefatigable correspondent that he was, Oldenbourg shared Hugyens' solution with a number of scholars, including Sluse, to whom he sent a copy in September, 1670.[107] Sluse then responded in November of that year, informing Oldenbourg that Huygens had "followed just the same line of analysis" as he had, although, "as two means of effecting [the solution] present themselves as a result, each by a hyperbola upon asymptotes, he [i.e., Huygens] chose one as the easier to use and I the other."[108] Thus began a spate of correspondence between the two through Oldenbourg's mediation that lasted until 1673, when Oldenbourg saw fit to publish excerpts from it in the *Philosophical Transactions*.[109] The resulting controversy, a mild one at that, focused on which of the two solutions was the easier and more "natural," but this very controversy was predicated on the understanding that, being difficult and unnatural because of its inelegance, Alhacen's method was fatally flawed.[110] In the end, therefore, Alhacen's last significant bequest to the Latin West was a problem to which his solution was rendered, if not wrong, at least superfluous. Henceforth, Alhacen would be little more than a footnote in the history of optics and mathematics, a history in which he had nonetheless played a crucial formative role. After that (to rephrase Sarton on Apollonius' fate at the end of the seventeenth century), the Alhacenian tradition was lost in the new optics of the time, like a river in the ocean.[111]

5. *Conclusion*

Viewed from a comfortable post-Keplerian perch, books 4 and 5 of Alhacen's *De aspectibus* appear to represent little more than wasted effort. Most of the basic principles established in these two books are so obvious now as to be axiomatic. That light reflects at equal angles within a plane normal to the surface of reflection needs no proof today, nor does the fact that the equal-angles law applies to all light-radiation, regardless of source or type. Light, after all, is light—or, rather, a composite of discrete colors. Worse, as Kepler showed in his critique of Alhacen's reflection-analysis, Alhacen generalized the cathetus-rule of image-location beyond its legitimate scope to include convex and concave mirrors and, by extension, refracting media. Even Alhacen's solution to his eponymous problem, though mathematically correct, is so inelegant by today's standards as to be all but worthless except as a quaint historical artifact.

Within the context of his time, however, Alhacen's analysis of reflection in books 4 and 5 was a *tour de force* in terms not only of empirical and mathematical rigor, but also of originality. This point becomes eminently clear when we compare Alhacen's analysis to that of its primary source in Ptolemy's *Optics*. Having substituted light-rays for visual rays as the basis for analysis, Alhacen was faced with certain issues at the empirical level that Ptolemy never had to confront. Consequently, Alhacen was forced to take into account all forms of visible radiation, from direct sunlight to the shining of brightly illuminated color. He was also forced to re-establish the principles of reflection for such radiation. Much of what today strikes us as wasted effort in his account thus derives from the need to demonstrate those principles for all possible forms of radiation from reflecting surfaces of all possible shapes.

The substitution of light-rays for visual rays also forced Alhacen to recast Ptolemy's reflection-experiment. Part of that recasting involved an effort at generalizing the experiment to include convex spherical and conical mirrors, as well as concave spherical and conical mirrors. But the brunt of that recasting involved the creation of an experimental apparatus within which the actual passage of light to and from the selected mirrors could be rendered visible—hence the hollow ring with its inserted register and selection of holes through which the light could be directed to the mirrors. That this apparatus was probably never constructed as described (and, therefore, that the various tests based on that apparatus were probably never conducted as described) does not detract from Alhacen's ingenuity in devising the experiment as he did, knowing full well that such an instrumental setup *would* have yielded the appropriate results.

As with his attempts to validate the principles of reflection empirically, so with his application of those principles to the mathematical analysis of reflection, Alhacen far outstripped Ptolemy in terms of rigor, scope, and ingenuity. As we have seen, Alhacen bent Ptolemy's analysis to the specific end of locating the point or points of reflection in the seven mirrors chosen for analysis. To that end, particularly as it applies to concave spherical mirrors, Alhacen went much further than Ptolemy in specifying the conditions under which the number of possible points varies according to the placement of eye and object within the mirror. He also went further in restricting those possibilities according to the reflected angle and, on that basis, demonstrated conclusively that there can be no more than four reflections from a concave spherical mirror for any given center of sight and object-point. This restriction is implicit in Ptolemy's analysis, but it is never rigorously demonstrated.

The most salient difference between Alhacen's and Ptolemy's reflection-analysis, of course, lies in Alhacen's having tied a major loose end left dangling by Ptolemy: defining with absolute mathematical precision the relevant point or points of reflection on convex and concave spherical mirrors. As we have seen, this is a hideously complex problem, requiring four separate solutions, although in fact those solutions break down into two closely related pairs. But it is crucial to understand that, having inherited the problem from Ptolemy, Alhacen also inherited the analytic *structure* of that problem from him. Within the confines of that structure, Alhacen's method for defining the points of reflection is truly remarkable for its ingenuity, originality, and rigor. It is grossly anachronistic, therefore, to tax him with not having resolved the problem according to some other analytic structure, such as that within which Huygens and Sluse operated toward the end of the seventeenth century. It is also grossly anachronistic to tax Alhacen with failing to fully grasp the implications of his solution, which eventually led to its subsequent streamlining. Judged, therefore, according to the conceptual and analytic tools that were effectively available to Alhacen, his reflection-analysis stands as a landmark not only in the history of mathematical optics but also in the history of science overall.

NOTES

[1]For a brief discussion of this cone and its implications for visual perception, see A. Mark Smith, *Alhacen's Theory of Visual Perception*, Transactions of the American Philosophical Society, 91.4 and 5 (Philadelphia: American Philosophical Society, 2001).

[2]As Alhacen observes later on in book 4, pp. 320-321 below, points have no dimensions and are therefore invisible. In order to be seen, light and color must radiate from physical spots on the surfaces of luminous or illuminated bodies. Those surfaces, in turn, are seen according to the light and color radiated from all such spots on them, and it is through the perception of their surfaces that we perceive the bodies themselves. Nonetheless, for the sake of analytic convenience, we can treat those spots as if they were points.

[3]I. L. Heiberg, ed., *Euclidis Opera Omnia*, vol. 7 (Leipzig: Teubner, 1985), 286-289; for an English translation, see A. Mark Smith, *Ptolemy and the Foundations of Ancient Mathematical Optics*, Transactions of the American Philosophical Society, 89.3 (Philadelphia: American Philosophical Society, 2001), 80. This "proof," which is based on visual rays rather than light-rays, depends on presupposing that normal EA dropped from center of sight A to the mirror (as represented in the top diagram of figure 2, p. 522) is to ED as normal BG dropped from object-point B to the mirror is to GD (i.e., AE:ED = BG:GD).

[4]For this demonstration, see Smith, *Ptolemy and the Foundations*, 80-81. Far more ingenious than Euclid's, Hero's proof, which is also based on visual rays, assumes that the distance the visual ray travels from the center of sight to the object-point via the point of reflection will be the shortest possible. Accordingly, R_3 will be the point at which the lines of incidence and reflection will add up to the least amount among all possible combinations according to other points on the mirror. From this it necessarily follows that the angles of incidence and reflection will be equal. A major shortcoming of this proof is that it does not work for concave mirrors.

[5]See *Optics* III, 3-5, in A. Mark Smith, *Ptolemy's Theory of Visual Perception*, Transactions of the American Philosophical Society, 86.2 (Philadelphia: American Philosophical Society, 1996), 131-132. Ptolemy also applies the point-analysis described above to image-formation for object-surfaces in plane and curved mirrors.

[6]See ibid., 134-136; see also the account on pp. lxvii-lxviii above. Although Ptolemy does not specify that the two curved mirrors be cylindrical, it is clear from the structure of the experiment that the curved mirrors used were in fact cylindrical rather than spherical or conical.

[7]For some discussion of Alhacen's reliance on Ptolemy's *Optics* as a source, see A. Mark Smith, "Alhazen's Debt to Ptolemy's *Optics*," in T. H. Levere and W. R. Shea, eds., *Nature, Experiment, and the Sciences* (Dordrecht: Kluwer, 1990), 147-164. See also the discussion on pp. lxvii-lxxiv above.

[8]According to Alhacen's analysis, primary light is the light inherent in luminous sources, whereas secondary, or accidental, light is the light cast on opaque objects from those sources. However, the light in objects illuminated by luminous sources becomes primary when that light shines on other objects to create secondary light in them. See Smith, *Alhacen's Theory*, liii-liv.

[9]A digit is approximately 1.9 cm or 3/4 in. This measure is confirmed by implication in book 3 of the *De aspectibus*, where the distance between the pupils of the eyes is assumed to be four digits; see Smith, *Alhacen's Theory*, 573-574.

[10]As a measure, a full grain of barley is around .42 cm. or 1/6 in, so half a grain of barley is about .21 cm. Accordingly, the circle formed here on the ring's inner wall is somewhat less than 4 cm. above the ring's base.

[11]Alhacen gives no specification for the thickness of this block, but later construction makes it clear that it must be thicker than one digit.

[12]As will become clear later on, the virtue of these particular shapes is that planar cuts through them will yield points (if through the vertex), straight lines, circles, or the three conic sections: i.e., ellipses, parabolas, or hyperbolas. These sections can be mathematically analyzed on the basis of Euclid's *Elements* or Apollonius' *Conics*, both of which Alhacen was thoroughly familiar with (see pp. lxxiv-lxxvii above).

[13]Again, Alhacen specifies no particular thickness, but it makes sense that the mirror be as thin as possible for convenience's sake.

[14]Yet again, Alhacen specifies no particular thickness, but it is clear from subsequent use that these panels must be less than two digits thick yet thick enough to stand perfectly upright without wobbling.

[15]This, of course, follows from the fact that the top of the register has been inserted into its notch so that its face lies two digits minus half a grain of barley above the base of the ring. Since the panel has been stood on the floor of the square cavity in the block at the ring's base, and since that cavity is precisely one digit deep, the midpoint of each panel will lie precisely three digits above the panel's base, where the midpoints of the mirrors lie. Thus, the centerpoint of each mirror will lie half a grain of barley higher than the face of the register.

[16]See notes 49 and 50, pp. 354-355, for an explanation of this point.

[17]See note 50, pp. 354-355, for the derivation of this figure.

[18]See notes 60 and 61, pp. 356-357, for a diagrammatic account of the points just made about the parallels and perpendiculars contained by the experimental apparatus when it is properly set up.

[19]*Ibn al-Haytham's Optics: A Study of the Origins of Experimental Science* (Minneapolis: Bibliotheca Islamica, 1977).

[20]See, however, notes 28 and 29, pp. 351-352.

[21]Although there is evidence for crucible smelting of iron in the Arab world of Alhacen's day, this process is a far cry from the sort of sophisticated casting that would have been needed to produce the mirror-forms Alhacen describes; see T. Rehren and O. Papakhristu, "Cutting-Edge Technology—The Ferghana Process of medieval crucible steel smelting," *Metalla* 7 (2000): 55-69.

[22]Bronze technology was advanced enough, even in Ptolemy's day, that he was able to produce satisfactory plane and concave mirrors for his reflection-experiments.

Until the Renaissance the best mirrors were formed of bronze or silver, both being malleable and relatively easy to polish. Nevertheless, the few extant mirrors we have from the period between Antiquity and the Renaissance are of fairly poor reflective quality, and even during the Renaissance, when glass mirrors with metal backing become increasingly common, the results were far from perfect because of imperfections in the formation of the glass. Problems with the quality of glass extended to the formation of lenses as well, and they became especially acute in the late sixteenth and seventeenth centuries when efforts to develop and perfect telescopy and microscopy became a central concern of opticians. For an account of pre-modern mirrors and their low quality, see Sara J. Schechner, "Between Knowing and Doing: Mirrors and their Imperfections in the Renaissance," *Early Science and Medicine* 10 (2005): 137-162. For an account of the development and use of lenses during the Middle Ages and Renaissance, see Vincent Ilardi, *Renaissance Vision from Spectacles to Telescopes*, forthcoming in *Memoirs of the American Philosophical Society*, 2006.

[23]That Alhacen was fully aware of this phenomenon is clear from his having written treatises on concave spherical and parabolic burning mirrors; see items 1 and 2 on p. xvii in Smith, *Alhacen's Theory*. Alhacen's treatise on parabolic burning mirrors was in fact translated into Latin and disseminated fairly widely in that form; for a critical edition of this version, along with a German translation, see I. L. Heiberg and E. Wiedemann, "Ibn al-Haitams Schrift über parabolische Hohlspiegel," *Bibliotheca Mathematica*, ser. 3, vol. 10 (1910): 201-237.

[24]Presumably, this scattered and absorbed light constitutes secondary light, but the model of dispersion and absorption here seems inconsistent with Alhacen's earlier account of how secondary light mixes with the color on the surfaces of things; see Smith, *Alhacen's Theory*, liv-lv.

[25]That the reflectivity of mirrors is due to something like physical hardness, but not to physical hardness itself, is clear from the fact that light reflects intensely from water (4.3.99, p. 321).

[26]If, however, the line of sight does not coincide with the the axis but lies above the cone and enters it through its vertex, the plane containing that line of sight and the axis will form two corresponding lines of longitude on opposite sides of the mirror's surface (4.5.34, p. 338).

[27]In this case, of course, if the line of sight coincides with the axis of the cone, all planes passing through the viewpoint along that axis will form lines of longitude on the exposed portion of the reflecting surface. From any other viewpoint, though, only one plane of reflection will form a line of longitude on that surface, as is the case with convex conical mirrors.

[28]See p. xliii.

[29]The same test can be applied to any point on arc KL between X and where circle AGB intersects KL to the right of X. Take point D in figure 16, p. 531, as an example, and let it be represented in figure 16e, p. 533, with A and B posed on their respective normals as before within the same circle KLHF. Draw circle ADB through the relevant points, connect AB, and bisect it at point M with diameter PMN so that arc AP = arc BP. When normal DG is extended, it intersects PN at point P on circle ABD, leaving

angle of incidence BDGP and angle of reflection ADGP subtended, respectively, by equal arcs BP and AP. Those angles are therefore equal, so D passes the test as a legitimate point of reflection.

[30]See esp. 5, 2.457-2.461, p. 470.

[31]As should be clear from this analysis, the two points of reflection that yield angles greater than LGH lie on the same side of arc KL as the point, whether object-point B or center of sight A, that is closer to center of curvature G. By the same token, the point of reflection that yields an angle less than LGH lies on the same side of arc KL as the other point on the normal that lies farther away from the center of curvature. Thus, if A and B were to switch places in figure 16, p. 531, the two points of reflection D and D′ that yield angles ADB and AD′B greater than LGH will lie on the same side as the center of sight and thus on the side opposite the object-point.

[32]"Dato speculo sphaerico convexo aut cavo, datisque punto visus et puncto rei vise, invenire in superficie speculi punctum reflexionis," *Oeuvres complètes de Christiaan Huygens*, vol. 20 (La Haye: M. Nijhoff, 1940), 265.

[33]This determination is made in proposition 38, pp. 458-459, which is analyzed on pp. lx-lxii.

[34]Case two of this determination, as analyzed on p. liv, is an apparent exception in that points A and B lie in the plane that passes perpendicular to axis XY through vertex-point X, which serves as a hypothetical point from which the form of A would reflect to B. But this exception is less exceptional than it may appear at first glance, because circles and points are both degenerate conic sections.

[35]For the actual demonstration of this point, see proposition 36, case 2, paragraph 2.344, p. 453.

[36]This, of course, is tantamount to forming angle D′T′N′ equal to half of angle BGA, as was the case in the construction for proposition 25, pp. 427-432.

[37]This same method can be applied to finding D when normals AG and BG are equal. Thus, as illustrated in figure 20m, p. 546, we start by cutting line D′M′ at point N′, which coincides with Q′, such that M′N′ (i.e., M′Q′):N′D′ (Q′D′) = BG:AG. At point D′ we form angle N′D′T′ (i.e., Q′D′G′) equal to half of angle LGH (leaving angle D′T′N, which = D′G′Q′, equal to half of angle BGA), and then from point B′ we pass a line through D′G′ to point Q′ so that B′G′:D′G′ = BG:DG (i.e., the radius of the circle). We then form angle BGD in the circle of the mirror equal to angle B′G′D′, and D will be the point of reflection. Accordingly, it is clear that the case in which normals BG and AG are equal is simply a special, limiting case that can be subsumed under the more general case in which normals AG and BG are to one another in any ratio we please.

[38]This same method applies in the case where normals AG and BG are equal, as illustrated in figure 20r, p. 548. Take some line D_2′M′, and cut it at point A′ such that A′M′:A′D_2′ = BG:AG. Since BG = AG, then A′M′ = A′D_2′, so Q′ and N′ in the previous figure will both coincide with A′. At D_2′ form angle M′D_2′T′ equal to half of angle LGH, and from A′ (i.e., Q′ in the previous figure) drop line A′G′ to line D_2′T′ such that A′G′:D_2′G′ = AG:D_2G (i.e., the radius of the circle). At point G in the circle form angle D_2GA equal to angle D_2′G′A′. D_2 will therefore be a point of reflection. The second point of reflection between D_1 and K can be found by reversing the process and form-

ing the relevant figure on point B rather than A. Thus, as in the previous analysis, this is simply a special, limiting case that can be subsumed under the more general case in which normals AG and BG are to one another in any ratio we please.

[39]This is equivalent to the stipulation that, when AG = BG, as in figure 20d, p. 544, angles D_1AG and D_1BG must both be acute if reflection is to occur from three points on arc KL. Otherwise, A and B will be too close to the center of curvature for circle ABG to intersect that arc, and, as we have seen, the two points where circle ABG cuts arc KL are reflection-points. The case in which AG ≠ BG is a bit more complicated, because A and B can be posed in such a way that, although circle ABG does intersect arc KL on both sides of midpoint X , there can only be one reflection from arc KL , and it will occur from some point D according to which reflected angle BDA < angle LGH. Take the case represented in figure 20n, p. 547, where circle ABG intersects arc KL on both sides of X, and yet A is close enough to G that angle XAG is obtuse rather than acute. According to our stipulation, there can be no reflection other than that from point D. If such reflection were possible, then, as we have already established in the previous section (pp. xli-xliii), the reflected angle would have to be greater than LGH, and the reflection would have to occur to the right of point X. That such reflection is impossible in this case can be established on the basis of the test applied in the previous section. Let the conditions in figure 20n be recapitulated in figure 20s, p. 549, so that A and B are equivalently disposed within circle KLHF, circle ABG passes through the same points on arc KL, and angle XAG is obtuse rather than acute. The issue in this case is whether there can be any reflection from segment EX of arc KL. Assume that point D in the top diagram of figure 20s is such a point. Applying the test, we pass circle ABD through D, connect AB, bisect it at M, and draw diameter PMN through it so that arc AB is bisected at P. Since the extension of normal DG does not intersect circle ABD at point P, it is clear that angle of incidence BDG and angle of reflection ADG are subtended by unequal arcs, so they are unequal. Likewise, if we apply the test to point E in the middle diagram, or to point X in the bottom diagram, it is clear that those two points cannot be points of reflection for the same reason: i.e., that normals EG and XG do not intersect their respective circles ABE and ABX at point P, so they cut unequal arcs from APB, leaving the angles of incidence and reflection unequal. There is thus no point between X and E at which the normal will intersect the circle at P and therefore no point at which the angles of incidence and reflection will be equal.

[40]Thus, as illustrated in figure 20v, p. 550, lines B′G′ and B″G″ can be projected through Q′ to form equiproportional segments: i.e., B′G′:G′D′ = B″G″:G″D′. Clearly, however, the resulting angles B′G′D′ and B″G″D′ are not equal. It should be noted that only two such lines can be projected through Q′ so as to fulfill the requisite conditions, so only two angles can be appropriately formed, from which it follows that only two points of reflection are possible.

[41]The construction in this case is given in figure 20y, p. 552, where points Q and N coincide, because MQ and QD are equal, given the equality of normals BG and AG. Thus, line BQG is perpendicular to DM, so the angle it forms with DG is simply half of angle BGA formed by the normals. This, of course, represents the limiting case for the more general situation in which AG and BG can be in any ratio we please.

[42]See Smith, *Alhacen's Theory*, xxv.

[43]Ibid.

[44]It is interesting that, like Alhacen, Ptolemy specifies iron as the material from which to form the test-mirrors. As pointed out in n. 22 above, bronze was the normal material from which metal mirrors were made in Ptolemy's day because, unlike iron, it is reasonably easy to work. However, forming plane, convex cylindrical, and concave cylindrical mirrors from strips of iron, as Ptolemy describes, would have been immeasurably less demanding at the practical level than forming the spherical and conical mirrors Alhacen describes.

[45]See the experiment detailed in book 4, paragraph 3.108, pp. 323-324, where the equal-angles law is confirmed for the reflection of secondarily illuminated color by viewing the color's image through the appropriate hole in the wooden ring.

[46]Note, however, that Alhacen's addition of mirrors to test was limited by his ability to analyze them mathematically; see pp. xx-xxi above.

[47]Smith, *Ptolemy's Theory*, 132.

[48]Ptolemy's construction in this case has H and Z lie on a line, AG, that falls between the mirror's surface and its center of curvature D. Radius BED of the mirror is normal to that line, and H and Z are situated at equal distances from intersection-point E. Alhacen, on the other hand, places the two points on diameters DL and DM so that their distances ZD and HD from the center of curvature are equal. Suffice to say, both methods of construction are equivalent insofar as Z and H are equidistant from both E and D.

[49]See the discussion on p. xli above.

[50]See paragraphs 5, 2.151 and 2.152 (pp. 417-418), 2.155 (p. 418), 2.208 (p. 429), 2.343 (p. 453), and 2.363 (pp. 455-456). The reference to the *Elements*, which is found in paragraph 5, 2.151, cites the book (3) but not the proposition.

[51]See paragraphs 5, 2.147 (p. 417), 2.160 (p. 419), and 2.165 (p. 420). The reference to the *Conics* (book 2, proposition 4) is found in paragraph 5, 2.47.

[52]For the two references to Campanus cited by Risner, see *Opticae thesaurus*, proposition 33, p. 144, and proposition 71, p. 168. That Risner was not attempting to pinpoint Alhacen's actual sources is reinforced by Sabetai Unguru's claim that Risner's citation of sources for Witelo's *Perspectiva*— which was published in tandem with the *De aspectibus* in the *Opticae thesaurus*—"should not . . . be taken to mean that Risner thought all of them were sources actually employed by Witelo" (*Witelonis* Perspectivae *liber primus*, Studia Copernicana, XV [Warsaw: Ossolineum, 1977], 28).

[53]See I. L. Heiberg, ed. and trans., *Sereni Antinoensis opuscula* (Leipzig: Teubner, 1896), 58-65.

[54]It bears noting that the Arabic version of the *De aspectibus* also contains explicit references to *Conics*, V.34 and V.61 that are missing in the Latin version; see A. I. Sabra, "Ibn al-Haytham's Lemmas for Solving 'Alhazen's Problem'," *Archive for History of Exact Sciences* 26 (1982): 310, 318.

[55]See Jan P. Hogendijk, *Ibn Al-Haytham's* Completion of the Conics (New York: Springer, 1984). In section 7.7 of his introduction to this edition, Hogendijk ties Alhacen's approach to the solution of "Alhazen's Problem" to certain of the problems he addresses in his attempt to reconstruct book 8 of the *Conics*; see pp. 105-113.

[56]See p. cix below.

[57]Aside from the few explicit references given by Alhacen, the figures given in this paragraph are based on the citations I interpolated into the English translation where I felt they were needed to clarify Alhacen's mathematical reasoning.

[58]See, e.g., *Conics*, I.11-I.13.

[59]See pp. xli-xlii above for a discussion of the ostensible vs. the ulterior intent of these two propositions.

[60]There is, however, a fragmentary Latin version of the *Conics* attributed to Gerard of Cremona and consisting of the definitions and some of the enunciations from book 1; see I. L. Heiberg, *Apollonii Pergaei Quae Graece Existant Cum Commentariis Antiquis*, vol. 2 (Leipzig: Teubner, 1893), lxxv-lxxx. For an English translation of this version, see Marshall Clagett, *Archimedes in the Middle Ages*, vol. 4 (Philadelphia: American Philosophical Society, 1980), 3-13. In addition, there are a couple of medieval Latin texts containing snippets on conic sections that are based on Apollonius, but there is nothing to indicate that they were circulated in any significant way during the Middle Ages; see, e.g., Marshall Clagett, "A Medieval Latin Translation of a Short Arabic Tract on the Hyperbola," *Osiris* 11 (1954): 359-366.

[61]See Clagett, *Archimedes in the Middle Ages*, vol. 1 (Madison: University of Wisconsin, 1964), 673-675, esp. 675. In fact, Jordanus appeals to "*figuram 19 quinti perspective*," which I have renumbered to proposition 20. Presumably, then, Jordanus based his citation on a manuscript within which the figure for the proposition was labeled "19"; see p. cviii below.

[62]For a brief discussion of the problematic dating of Jordanus' life and works, see the article by Edward Grant in *Dictionary of Scientific Biography*, ed. Charles Gillispie, vol. 7 (New York: Scribner's, 1973), 171-172.

[63]The Latin translation itself of the *De aspectibus* raises some intriguing questions about the level of mathematical development in the Latin West at the beginning of the thirteenth century. The most obvious is whether the translator actually understood the mathematical content of what he was translating or whether he was mindlessly following the Arabic lead. The rendering "Ablonius," which is the Latin transliteration of the Arabic transliteration of "Apollonius," suggests (weakly) that he may have been unfamiliar with the *Conics*. On the other hand, the very decision to undertake the translation must have been motivated by a recognition on either the translator's part or that of his commissioner that the work was worth translating. That recognition must have been based on a reasonably informed evaluation of its content.

[64]*De refractione optice partes libri novem* (Naples, 1593), 76.

[65]For more detailed accounts of Witelo's life, see D. C. Lindberg's introduction to the reprint edition of Risner's *Opticae thesaurus* (New York: Johnson Reprint, 1972), vii-xiii, esp. vii-ix, Sabetai Unguru, *Witelonis* Perspectivae *liber primus*, Studia Copernicana 15 (Warsaw: Ossolineum, 1977), 12-19; and Sabetai Unguru, "Witelo," in Thomas F. Glick, Steven J. Livesey, and Faith Wallis, eds, *Medieval Science, Technology, and Medicine: An Encyclopedia* (New York: Routledge, 2005), 520-522. Among the works that Moerbeke translated for Witelo is Hero of Alexandria's *Catoptrics*, which was finished in late 1269 and which Witelo cites specifically in book 5 of the *Perspectiva*; see A. Mark Smith, ed. and trans., *Witelonis* Perspectivae *liber quintus*, Studia

Copernicana 23 (Warsaw: Ossolineum, 1983), 17. Moerbeke translated other Greek mathematical works, many of them Archimedean, that may have been of use to Witelo; for details, see Marshall Clagett, *Archimedes in the Middle Ages*, vol. 2 (Philadelphia: American Philosophical Society, 1976).

[66]The following discussion of the mathematical content of book 1 of the *Perspectiva* is based on Sabetai Unguru's edition in *Witelonis liber primus*, henceforth cited as "Unguru."

[67]Surely the most blatant example of Witelo's zest for rigor is to be found in his attempt to prove the fifth postulate (i.e., the "parallel" postulate) of book 1 of the *Elements*; see *Perspectiva* I.14 (Unguru, 55).

[68]Propositions I.119-I.128 also deal with proportionality theory as applied to the cutting of lines according to specific proportions and the formation of proportional figures from proportional line-segments.

[69]The relevant propositions in Euclid are V.8, V.16, V.19, and V.27-29 (through Campanus' recension). *Elements*, VI.3 also enters in, as does VI, definition 5 (through Eutocius' *Commentary on the Sphere and Cylinder of Archimedes*—see Clagett, *Archimedes in the Middle Ages*, vol. 2, 15-21).

[70]The generation of a right cone from the rotation of an isosceles triangle can easily be inferred from various constructions in book 5 of the *De aspectibus*; see, e.g., paragraph 5, 2.527, p. 482. Alhacen explicitly describes the method for generating a cylinder from the rotation of a rectangle in paragraph 4, 5.21, p. 333.

[71]Unguru, 116. Witelo's description of the conic sections according to obliquity of angle is reminiscent of the pre-Apollonian description of those sections according to a perpendicular planar cut through a cone whose angle at the vertex is right (parabola), obtuse (hyperbola), or acute (ellipse); for a discussion of the pre- and post-Apollonian definitions of the conic sections, see Michael Fried and Sabetai Unguru, *Apollonius of Perga's* Conica: *Text, Context, Subtext* (Leiden: Brill, 2001), 74-90. It should be noted, however, that Witelo stresses the obliquity of the planar cut itself on the basis of a right cone whose vertex-angle is right.

[72]As Unguru observes in "A Very Early Acquaintance with Apollonius of Perga's Treatise on Conic Sections in the Latin West," *Centaurus* 20 (1976): 121-122, the Arabic terms based on *mukefi* are provided in the fragmentary Latin version alluded to in note 60 above. Clagett is convinced that Witelo in fact "depended significantly" on this version for his knowledge of conic sections; see Clagett, *Archimedes in the Middle Ages*, vol. 4, 64.

[73]It is worth noting (as in fact Unguru does in note 3 to proposition 91, p. 190) that Witelo's analysis of conic sections is based on right cones only and is thus far more limited than Apollonius' analysis, which is generalized to all cones, including oblique and scalene ones. This limitation on Witelo's part may have been due to his failure to understand the full implications of Apollonius' analysis, but it may also have been due (and I suggest this as the likelier alternative) to his having relied too heavily on Alhacen, whose concern with cones and conic sections in the *De aspectibus* was centered on the analysis of planar cuts through right conical mirrors, according to his description of the formation of such mirrors on p. xxi above.

[74]The main principles upon which the proof of II.8 rests are: that the line drawn tangent to a conic section at its vertex is parallel to the ordinates of that section (I.32);

that when such a tangent intersects the asymptotes of a hyperbolic section, it is bisected at the point of tangency (II.3); that a hyperbola can be erected at any point lying between two intersecting lines, which will form its asymptotes (II.4); and that the ordinate of any hyperbolic section is bisected by the diameter of that section, from which it follows that the diameter of a hyperbolic section will pass through the midpoint of the ordinates (II.7). The resulting proof would not be perfectly rigorous, to be sure, but it would get the point across with adequate clarity. Furthermore, if Witelo had restricted his proof of II.8 to hyperbolas generated in right cones, it would have been especially easy, albeit lacking in generality.

[75]Another instance that reveals some acquaintance on Witelo's part with the *Conics* is to be found in proposition I.131 of the *Perspectiva*, where the point Witelo makes is clearly based on *Conics*, II.16. In this case too, I suggest, Witelo's knowledge of that proposition and its point could have come from his reading of Alhacen.

[76]In arguing for Alhacen as the primary source for Witelo's knowledge of the *Conics*, I am parting ways somewhat with Unguru, who believes that Witelo gained that knowledge through an as-yet-undiscovered Latin translation of the *Conics* produced, perhaps, by William of Moerbeke; see Unguru, "Very Early Acquaintance," 122. Although Unguru's theory is not implausible, I find its suppositional basis needlessly complex. If, however, Witelo did have some version of the *Conics* at hand—other than the one alluded to in note 60 above—then it may have consisted only of the enunciations without the accompanying proofs. Whatever the case, Witelo's grasp of conic sections seems less sure than it should have been under the assumption that he had access to a complete version of books 1 and 2 of the *Conics*.

[77]In addition to these sources, Unguru includes Serenus' *On the Section of a Cylinder*, but I see nothing concrete in Witelo's analysis of cylinders to indicate that he actually used it. Not only does he not attempt to prove that an oblique section through a cylinder produces a true ellipse, but in proposition I.103 (Unguru, 120), where it would have been natural for him to mention this equivalency explicitly, he fails to do so. Instead, he concludes by saying that "we shall, therefore, call that section *conic* [*pyramidalem*] in cones and *cylindrical* [*columpnarem*] in cylinders. Even so, that section in cones has been called before . . . an acute-angled section, or ellipse." As we have seen, Alhacen draws the same distinction between *sectiones piramidales* and *sectiones columpnares* in his discussion of the two kinds of ellipse.

[78]See Unguru, 32-35.

[79]Comprising part 5 of his *Opus majus*, Bacon's *Perspectiva* was completed by no later than 1267, but it evidently drew on ideas developed much earlier, perhaps even from the 1240's. In tandem with the *De multiplicatione specierum* (c. 1262), the *Perspectiva* was an influential source for optical lore in succeeding centuries. John Pecham's primary contribution to optics, the *Perspectiva communis*, was completed sometime around 1280 and enjoyed widespread circulation as a standard text for teaching in succeeding centuries. For critical editions of the two Baconian works, see David C. Lindberg, ed. and trans., *Roger Bacon and the Origins of* Perspectiva *in the Middle Ages* (Oxford: Clarendon, 1996) and *Roger Bacon's Philosophy of Nature* (Oxford: Clarendon, 1983). For a critical edition of Pecham's *Perspectiva communis*, see Lindberg, *John Pecham and the Science of Optics* (Madison: University of Wisconsin, 1970).

[80]See, for instance, Bacon's brief, general account of reflection in concave spherical mirrors in *Perspectiva*, III.i.4 (Lindberg, *Bacon and the Origins*, 269-275, esp. 271). Note that Bacon's mathematical explanation of image-locations in concave spherical mirrors is based on virtually the same diagram (figure 42, p. 273), even down to the lettering, as Alhacen's explanation of the same thing in proposition 32 of the *De aspectibus* (cf. figure 5.2.32, p. 250).

[81]See Lindberg, *Bacon and the Origins*, xx-xxiii.

[82]That Bacon was apparently able to follow the intricacies of Alhacen's mathematical reasoning is evidenced by his assertion in *Perspectiva*, III.i.3 that, although images in convex spherical mirrors generally look smaller than their objects, they sometimes appear equal or larger (Lindberg, *Bacon and the Origins*, 267). This assertion is based on *De aspectibus*, 6, prop. 6 (Risner, *Opticae thesaurus*,190-197), where Alhacen offers an extraordinarily intricate mathematical justification of it. Bacon may, of course, have simply taken Alhacen's word in the "enunciation" of the proposition (*Quod autem forma in his speculis* [i.e., convex spherical] *aliquando videatur maior re visa*), although there is no mention at this point in the theorem of the possibility of equality. Or he could have skipped to the very end of the proposition, where Alhacen does raise that possibility (*Igitur in his speculis imaginem aliquando equalem rei vise, aliquando maiorem esse*). Bacon was convinced that mathematics held the key to a full understanding of nature and the traces of God's creative impulse in it, so it is difficult to believe that he could not, much less would not, have read the entire proposition with critical care.

[83]Lindberg, *Pecham*, 61.

[84]Ibid., 203.

[85]Note, however, that placing both points inside the mirror to satisfy the condition set here (i.e., having circle AGB cut arc CH in two points) flouts the condition set at the beginning of the proposition, where one of the points is to lie either at or beyond point H.

[86]In fact, without a firm grasp of the points established in 5.43-48, particularly 5.44-46, one can easily be confused or misled by Alhacen's summary in 5.49. It seems likely, therefore, that Pecham based the garbled account in II.47 on that summary with little or no understanding of the points established in the preceding theorems.

[87]Along with numerous articles on particular aspects of Perspectivist optics and its reception in the Latin West, David Lindberg's *Theories of Vision from Al-Kindi to Kepler* (Chicago: University of Chicago, 1976) still shapes our understanding of the Perspectivist tradition and its development from the late thirteenth to the early seventeenth century.

[88]See Lindberg, *A Catalogue of Medieval and Renaissance Optical Manuscripts* (Toronto: University of Toronto, 1975).

[89]For elaboration, see Lindberg's introduction to the reprint edition of Risner's *Opticae Thesaurus*, xxi-xxiii.

[90]This distillation process is most clearly exemplified in the development of *Sentences* commentaries according to a fairly rigid canon of *quaestiones* based on certain key points in Lombard's text. Much the same thing happened with commentaries on Aristotle's works, which increasingly focused on specific *quaestiones*.

[91]As Lindberg points out in his introduction to the reprint edition of Risner's *Opticae Thesaurus*, there is evidence for the continued use of Witelo's *Perspectiva* as a text in mathematics even to the late sixteenth century in Cambridge; see p. xxiii. Nevertheless there is no way of determining precisely how either Alhacen's *De aspectibus* or Witelo's *Perspectiva* would actually have been used for the teaching of Euclidean geometry. Were they closely analyzed from cover to cover, or were they dipped into at specific points? If the latter, then at what specific points? To these questions we have no definitive answer, nor does it seem likely that we ever will.

[92]It should be noted, however, that many of these commentaries are slanted toward issues in Aristotelian natural philosophy, so the optical questions raised in them are rather narrowly defined by that context. See Lindberg, *Theories*, 122-139. See also Smith, *Alhacen's Theory*, xciv-c.

[93]The conclusion that Euclid's *Catoptrics* and Ptolemy's *Optics* were more widely read than Alhacen's *De aspectibus* and Witelo's *Perspectiva* is based on the fact that the surviving manuscripts (both complete and incomplete) of the first two works significantly outnumber the surviving manuscripts of the second two; see Lindberg, *Catalogue*, 47-50, 74 vs. 17-18, 77-79. The increasing popularity of these works during the Renaissance is especially clear in the case of Ptolemy's *Optics*, the overwhelming majority of whose surviving manuscripts date from the sixteenth and seventeenth centuries.

[94]It bears noting that, despite its relative mathematical complexity, the analysis of parabolic burning mirrors did capture the imagination of medieval and Renaissance scholars, presumably because that analysis was seen to have practical applications, whereas the exact determination of reflection-points has little or none. The primary source for the study of burning mirrors in the Latin West was Alhacen's *De speculis comburentibus*, although Bacon also wrote a tract that bears the same title; see the list of respective manuscripts in Lindberg, *Catalogue*, 20-21, 39-40. Witelo also offers an analysis of paraboloidal burning mirrors at the end of book 9 of the *Perspectiva*; see Risner, *Opticae Thesaurus*, 392-403. In addition to the mathematical analysis of such mirrors, Alhacen's and Witelo's accounts include "practical" advice on how to manufacture them. Clagett finds in Witelo's discussion of paraboloidal burning mirrors a strong indication that Witelo had access to at least some of the *Conics*, although his understanding was confused; see *Archimedes in the Middle Ages*, vol. 4, 91-98; for a detailed account of the tradition of burning mirrors and the associated study of parabolas in the later Middle Ages and Renaissance see the entire volume.

[95]For publication-details of these early editions of the *Elements*, see George Sarton, *A History of Science*, vol. 2 (New York: Norton, 1970), 43-45.

[96]For publication-details of these editions of the *Conics*, see ibid., 96-97. Not until Edmund Halley's edition of 1710 did a Greek edition of the *Conics* appear in print.

[97]For publication-details, see Unguru, *Witelonis liber primus*, 41-42.

[98]See Risner's preface to the *Opticae Thesaurus*, folio 2r, l. 31-folio 2v, l. 37.

[99]*Opticae libri quatuor ex voto Petri Rami novissimo per Fridericum Risnerum* . . . . (Kassel, 1606).

[100]See Lindberg, introduction to the reprint edition of *Opticae Thesaurus*, xxiv-xxv.

[101]See Smith, *Alhacen's Theory*, lxxxii-civ. See also Smith, "What Is the History of Medieval Optics Really About?" *Proceedings of the American Philosophical Society* 148 (2004): 180-194.

[102]*Paralipomena*, chap. 3; for an English translation, see William Donahue, *Johannes Kepler, Optics*: Paralipomena to Witelo and Optical Part of Astronomy (Santa Fe: Green Lion, 2000), 75-91.

[103]For an English translation of the *Lectiones Opticae*, see H. C. Fay, trans., and A. G. Bennett and D. F. Edgar, eds., *Isaac Barrow's Optical Lectures, 1667* (London: The Worshipful Company of Spectacle Makers, 1987).

[104]See letter 1213 in A. R. Hall and M. B. Hall, ed. and trans., *The Correspondence of Henry Oldenbourg*, vol. 6, (Madison: University of Wisconsin, 1969), 42-46.

[105]For the Latin version of Huygens' solution, see *Philosophical Transactions (1665-1678)*, 8 (1673), 6119; for an English translation, see Hall and Hall, *Correspondence of Oldenbourg*, 7 (1970), 191-192. Based on figure 28, p. 555, which is taken directly from the one provided in the *Philosophical Transactions*, Huygens' method for solving the problem is as follows. Let the top circle DdDd centered on A represent a great circle within the sphere of the mirror. Let B and C represent the center of sight and object-point, and let circle ABC be drawn through them and centerpoint A of the mirror. Let point z be the center of that circle. Draw AER normal to BC. Cut AR at point N such that AR:AO = AO:AN, and through point N draw line MN parallel to BC. Cut AR at point I such that AI:AO = AO:4AE. Find point Y above A such that IY = IN, and through point Y draw line MY parallel to line AZ connecting the centers of the two circles. Finally, find points S and X on line AR such that IS = IX, and the square formed on either of them = one-half $AO^2 + AI^2$. Lines MY and MN will therefore form the asymptotes of a hyperbola whose opposite branches pass through X and S. Points D and d where each of those branches cuts the circle of the mirror will be potential points of reflection for B and C. According to Hugyens' placement of B and C in the figure, then, if the mirror is convex, reflection can only occur from point D on the convex surface facing B and C, so if a normal is dropped from A through that point D, it will bisect angle BDC. Suffice it to say that neither of the two points d can be a reflection-point because either ray Cd or ray Bd would have to pass through the surface to reach it. By the same token, if reflection occurs from the concave portion of the mirror facing B and C, reflection can only occur from point D, since B and C lie entirely outside the circle of the mirror. Thus, if a normal is dropped from A to that point D, it will bisect angle BDC. The same will not hold, of course, for either point d. On the other hand, if B and C are located inside the great circle of the mirror, as illustrated in figure 28a, p. 556, the same procedure can be followed to define the two asymptotes, MN and MY, as well as points S and X on them. The two branches of the hyperbola passing through those points will each cut the circle on the mirror at points D and d, and if a normal is dropped from A to the two points D and the two points d, it will bisect the respective angles BDC and BdC. Hence, the two pairs of points D and d will yield four reflections altogether. Since circle ABC in Hugyens' solution is the same as Alhacen's cutting circle, the number of possible reflections from the arc on the mirror that subtends angle BAC will depend on whether circle ABC cuts that arc to the left or

right of its midpoint, so it is possible for B and C to be placed in such a way that only one reflection will occur; see the discussion on pp. xli-xliv above.

[106]See letter 1489 in *Correspondence of Oldenbourg*, vol. 7., 73-81.

[107]See letter 1528 in ibid., 177-193. Oldenbourg evidently sent a copy of Hugyens' solution to John Wallis, who wrote back in July of 1669, admitting that "Mr. Hugen's optical Probleme I have not had time yet to consider of" but adding rather snidely that "it doth not seem, at first view, to be a matter of very great difficulty"; letter 1260 in ibid., 159-161.

[108]See letter 1548 in ibid., 246-256.

[109]These excerpts were published in two parts in *Philosophical Transactions (1665-1678)*, 8 (1673), 6119-6126, 6140-6146.

[110]The notion that Alhacen's method for finding reflection-points is needlessly unwieldy is reflected in lecture IX of Barrow's *Lectiones Opticae*, where he deals with reflection from spherical convex mirrors. After outlining his own method for finding the point of reflection in such mirrors, he provides a faithful but abbreviated version of Alhacen's method in proposition 5.25 of the *De aspectibus* in order to provide something "acceptable to [the] taste" of geometers while stripping it "of that horrible combination of prolixity and obscurity, and of . . . the uncouth barbarity of speech" so characteristic of the Alhacenian original; See Fay et al., *Barrow's Optical Lectures*, 118-121. For a survey of various approaches to Alhazen's Problem since the mid-seventeenth century, see J. A. Lohne, "Alhazens Spiegelproblem," *Nordisk Matematisk Tidskrift* 18 (1970): 5-35.

[111]Sarton's original sentence, which I found irresistably pithy, reads: "After that, the Apollonian tradition was lost in the new geometry of the time, like a river in the ocean" (*An Introduction to the History of Science*, vol. 2 [Baltimore: Williams & Wilkins, 1927], 28).

MANUSCRIPTS AND EDITING

In the previous edition of books 1-3 of Alhacen's *De aspectibus* I provided a detailed account of the available manuscripts and outlined my procedures for selecting particular representatives from among them for collation in the critical text. I concluded that the seventeen complete or virtually complete manuscripts could be broken into three family groups, the first consisting of six members (*F*, *P1*, *Va*, *V2*, *L2*, and *S*), the second of four (*Er*, *C1*, *O*, and *M*), and the last of seven (*E*, *P3*, *P2*, *L3*, *C2*, *L1*, and *V1*).[1] I also concluded that the first family lies closest to the *Urtext* and, furthermore, that among its members *F* is closest to the family progenitor. *F* was therefore the logical candidate to represent this family in the critical text, but since it lacks a considerable portion of books 1 and 2, I was forced to fall back on its nearest relative, *P1*. For the second family the choice was less clear, but I eventually decided on *Er*, thus bringing the total for collation to two. The third family was even more problematic, but I was finally drawn to *E*, *P3*, and *L3* as the most suitable choices for collation. To the resulting list of five manuscripts I added *S* and *C1*, because both seemed to form inter-family links, *S* between the first and second families, *C1* between the second and third. Altogether, then, I based my critical text on seven manuscripts—*P1*, *S*, *E*, *P3*, *L3* , *Er*, and *C1*—using *O* for occasional cross checking when necessary.

Over the course of editing the text on the basis of these seven manuscripts, I was led to modify my initial conclusions somewhat.[2] For one thing, it became clear to me that *E* and *P3* are so close as to be virtually identical, the latter having most likely been copied directly from the former. It was therefore obvious that *P3* added nothing of substance to the critical text. I also discovered that *P1*, which I initially took to be the arch-representative of the first family, was less reliable as a textual witness than its distant relative *S* and, furthermore, that *O*, which I had marginalized somewhat in my initial evaluation, would have been preferable to *Er* as a representative of the second family. On the basis of these insights, I decided in retrospect that, were I to do it all over again, I would substitute *O* for *Er* and ignore *P3*. And that is precisely what I have done here in the critical edition of books 4 and 5, exchanging *O* for *Er* and dropping *P3* from consideration. I have also added *F* to the mix, since it includes the entirety of books 4 and 5, thus bringing the revised list of manuscripts to be collated back to seven: *F*, *P1*, *S*, *E*, *L3* , *O*, and *C1*. Sample pages from these seven manuscripts are reproduced on pp. cxxi-cxxvii below, each page containing the incipit of proposition 32, pp. 141 (Latin) and 446 (English) along with the relevant diagram, which is not included in the Saint-Omer manuscript (see p. lxxi).

As before, so now, the process of establishing the critical text has led me to reconsider my already-reconsidered assumptions about the selected family-representatives and their place in the manuscript-tradition. Central to this reconsideration is that the text shifts quite early in book 5 from narrative explanation to mathematical demonstration. In narrative explanation, of course, two different, sometimes even contradictory, readings can make perfect sense in a given context. Choosing the "right" reading is thus dictated more often than not by consensus of manuscripts (appropriately weighted for authority) rather than by the logic of the passage. In mathematical discourse, however, there is little or no ambiguity, so the right reading is dictated more often than not by logic, not consensus. In many cases, in fact, consensus is simply wrong. Accordingly, as the text of book 5 unfolded, it became increasingly clear to me that manuscripts *F* and *P1* were even less authoritative and reliable than I had expected and, conversely, that *O* was commensurately more so.

This re-evaluation of *F*, *P1*, and *O* in light of the critical text of book 5 raises questions about the authenticity of *F* and *P1* as witnesses to the *Urtext*. It could be, of course, that the two manuscripts reflect flaws in the *Urtext* itself, flaws that were corrected as the text passed from hand to hand in its subsequent transmission. With their heavy burden of redactions, *O* and *E* in particular seem to bear this possibility out. It could also be that, although still closer to the *Urtext* than the rest of the manuscripts, *F* and *P1* suffered from the maladroit efforts of the original scribe to copy mathematical theorems that made little or no sense to him. Or it could be that *F* and *P1* represent a particular textual compilation, parts of which were more or less authentically tied to the *Urtext* and parts of which were not, a possibility already raised in the previous edition of books 1-3.[3] Whatever the case, it is by now evident that the textual tradition of Alhacen's *De aspectibus* is complicated enough to resist a simple, definitive reconstruction, which is hardly surprising, given the size and complexity of the text in question. Yet, despite such qualms about the textual tradition and its accurate reconstruction, I have no misgivings about my organization of the manuscripts according to families or my selection of representatives from those families. I am, in short, confident that the critical text is appropriately critical.

The Critical Text: The gross format for books 4 and 5 is clear and clearly stated at the beginning of each book, where the number of chapters (or parts), and a brief description of their content is given explicitly in most of the manuscripts. Accordingly, book 4 is properly divided into five chapters, book 5 into two. As far as book 4 is concerned, all but two of the manuscripts used in the critical text agree on both the number and placement of the chapter-breaks, and—as can be seen in Table 4, Appendix 2, in *Alhacen's Theory*, p. 658—the two exceptions, *L3* and *C1*, diverge only by splitting the fifth chapter into two, the new chapter opening with the phrase "in speculis autem columpnaribus."

Book 5 is a different matter altogether. For a start, it consists of only two chapters, the first of which occupies a single paragraph. In addition, despite the claim at the beginning of the book that it "is divided into two parts," a claim repeated in all but two of the manuscripts (*F* and *P1*), chapter 2 is subdivided in several different ways in the relevant manuscripts. A look at Table 5 of the appendix just cited shows that *C1* leads the way by opening a new chapter (designated as 2b) with the phrase "restat iam ut loca ymaginum." This is followed in *O* by chapter 2c, whose incipit is "in speculis exterioribus pyramidalibus," after which *S*, *P1*, *L3*, and *F* interpolate chapter 2d, beginning with "in speculis spericis concavis." Then comes chapter 2e in *P1*, *L3*, *O*, and *F*, its opening phrase being "in speculis columpnaribus concavis." In *C1*, finally, a weak break that signals chapter 2f occurs at the phrase "in speculis pyramidalibus concavis."

Barring the stray chapter-breaks, which are evidently spurious, the Latin text of these two books presents a relatively blank face broken into two main segments, i.e., books 4 and 5, and seven lesser segments consisting of the five chapters in book 4 and the two in book 5. What remains is a succession of long, unrelieved swaths of text whose internal punctuation is as sporadic as it is haphazard. Some of these swaths, moreover, are dauntingly long, the most egregious example being chapter 2 of book 5, which occupies well over half the combined text of books 4 and 5. Left in this rather sterile format, the critical text would have been as faithful as possible to the original, to be sure, but it would have been essentially inaccessible to contemporary readers, who are accustomed to a far more punctuated format than their medieval forebears. The trick is to impose such punctuation without traducing the intent of the original. Fortunately, this task is made easy by Alhacen's rigorously systematic approach, according to which books 4 and 5 are further broken, albeit implicitly, into two clear sub-levels of organization.

The first of these is topical and is determined by an analytic passage through the seven mirrors discussed earlier in the introduction. Thus, in both books, Alhacen analyzes reflection and its various aspects according to a specific order, starting with plane mirrors, passing to convex spherical, convex cylindrical, and convex conical mirrors, and ending with concave spherical, concave cylindrical, and concave conical mirrors. Ultimately, I decided not to punctuate book 4 according to this topical format because it is short enough and the organizational principles are clear enough to need no reinforcement. Not so for book 5, however. Not only is it more than half again as long as book 4, but virtually all of it occupies a single chapter, i.e., chapter 2, whose inordinate length is matched by its analytic complexity. In order, therefore, to break this textual segment into manageable chunks, I imposed strong breaks between topical sections. These breaks, in fact, occur precisely where the various interpolated chapters do in the manuscripts, so in a sense I was merely following their lead.

The second sub-level of organization pertains to book 5 alone and is based on the fact that the vast majority of it consists of geometrical constructions and proofs. On the face of it, dividing the text into its constituent propositions should be simple enough, given Alhacen's penchant for ending proofs with phrases such as "quod est propositum" or "et ita propositum." Moreover, several of the manuscripts key the diagrams by number to their appropriate theorems. The problem is that not all of the manuscripts do number the diagrams and, worse, that among those that do, the numbering is inconsistent. Another problem is that, in several instances, what could be construed as individual propositions in succession can also be construed as specific cases falling under the head of a single, more general proposition.

An obvious solution to these problems would be to follow the format of the Arabic text, but unfortunately there is as yet no critical edition of that text available, although one has been in the works for some time now.[4] Nor do the Latin manuscripts offer much hope, since they are so obviously inconsistent. I was therefore forced to decide on my own how to parse the text by propositional elements. After some trial and error, I eventually decided that the most appropriate breakdown results in a set of 54 propositions, a few of which are so long and involved that they need further subdivision into cases and even subcases. Accordingly, between individual propositions I have inserted fairly strong spacing-breaks, whereas between cases and subcases within a given proposition I have inserted weaker spacing-breaks. Each proposition is further demarcated by a numerical designation (e.g., [PROPOSITIO 1]). Altogether, then, I have organized the text according to an order of spacing-breaks from strong to weak: strongest between chapters, less strong between topical segments, weaker yet between propositions, and weakest between intra-propositional elements—all with the hope of making the complex analytic structure of the text as transparent as possible.

At the lowest level of external punctuation, division into paragraphs, I had to fall back on my own devices for lack of an appropriate guide either in the manuscripts themselves, where such punctuation is random at best, or in a critical Arabic edition. Hence, unlike the previous text and translation of books 1-3, this one is not keyed to the Arabic version in its paragraph-structure. Nevertheless, I have followed the convention established in my previous edition of numbering the paragraphs for easy internal reference. Regrettably, the lack of coordination between Latin and Arabic texts at this level of punctuation will make future comparison of the two versions more difficult for books 4 and 5 than for the preceding three.

As to internal punctuation, finally, I have tried insofar as feasible to break the text up according to both the sense and syntactic structure of the Latin. At times, of course, the syntax of the Latin is so convoluted that I have had no real choice but to break sentences up into more digestible chunks. I have also taken liberties with the punctuation itself, following modern conventions by capitalizing words at the beginning of sentences, inserting commas, and so forth. In addition, I have taken the liberty of capitalizing letter-designations in the text, so that readings such as "linea ab" or "angulus gnd" in the actual manuscripts have been transformed to "linea AB" or "angulus GND" in the critical text.

Both honesty and admiration compel me to close my discussion of the critical text with a few remarks about Friedrich Risner's 1572 edition of the *De aspectibus*. As I pointed out earlier, Risner's aim in creating this edition was not to make it critical by modern historical standards. It was, rather, to upgrade the work to contemporary, late-Renaissance standards by revising the grammar and vocabulary (albeit fairly lightly), breaking the text into propositional elements, and inserting commentary when he deemed it necessary.[5] Nowhere is this latter modification clearer than in book 5, where Risner explains virtually every propositional conclusion either by providing specific citations to the appropriate mathematical source, mostly Euclid's *Elements*, or by explaining in detail the logical steps leading to that conclusion. So deep was Risner's understanding of the work and its analytic structure, in fact, that at one point he was able not only to recognize that a brief passage had been improperly transposed in the manuscripts but also to restore the passage to its rightful place.[6] It is therefore without shame that I acknowledge my debt to Risner. Without his clear and virtually inerrant guidance, I would have been far harder pressed than I was to make sense not just of certain propositional elements, but of the overall analytic structure of the second chapter of book 5.

Diagrams: Perhaps the knottiest problem I faced as an editor was how to handle the diagrams accompanying the text. For one thing, the number of diagrams varies widely among manuscripts. In the fourth book, for example, *F* has no diagrams whatever, *C1* three, *L3* four, *E* fourteen, *S* fifteen, *O* eighteen, and *P1* twenty-five. In the fifth book the number of diagrams is considerably greater, although the variation in number among manuscripts is commensurately less. Hence, the first and most obvious issue I had to address was which, if any, of the diagrams in the two books should be included as an integral part of the text and which should be regarded as mere ancillary illustrations and, therefore, treated as marginal glosses. In fact, I had already faced this problem on a much smaller scale in my previous edition of books 1-3, and the criterion I followed there is the one I followed here: if the letter-designations in the diagram reflect equivalents in the text, then that diagram is to be considered integral.[7] On that basis, I effectively eliminated all the diagrams in book 4, since they are clearly meant to illustrate technical descriptions not couched in specific geometrical format—i.e., where, instead of saying "if line EA is extended from center of sight E through vertex A of the cone," the text simply says "if a line is extended from the center of sight through the vertex of the cone." Book 5 is entirely different, in that most of it consists of geometrical propositions with specific letter-designations for points in the construction described. In this case, then, the obvious choice was to include all diagrams that reflect the letter-designations given in the propositions. According to that standard, I was able to pare the number of relevant diagrams to 83, still sizeable but significantly smaller than it would have been had I taken an all-inclusive approach.

Having resolved the problem of quantity, I was faced with the more vexing issue of quality. In a few cases, the requisite diagrams are straightforward enough that, when produced with ruler and compass, they adequately reflect the conditions specified in the proposition (e.g., parallelism or equality of lines, equality of angles, etc.). These diagrams actually "look" like what they are meant to represent. In most cases, though, the structure of the theorems and the constructions on which they are based are too complex for the simple expedients of compass and ruler, especially when those constructions are three-dimensional. The interrelationships among angles, lengths, and areas within the diagram are so intricate and exacting as to defy such easy or straightforward reproduction. Consequently, most of the figures provided in the manuscripts misrepresent, often grossly, what they are intended to illustrate.

Take, for example, FIGURE 5.2.25, p. 581, which is adapted directly from the drawing provided on folio 64v of manuscript *O*. This diagram is intended to illustrate "Alhazen's Problem" as applied to convex spherical mirrors: i.e., given a convex spherical mirror with centerpoint G, and given point-source B of radiation and center of sight A, to find point D of reflection on the mirror's surface. According to the construction given in the theorem, which has already been discussed on pp. xclvii-xlviii of the introduction, the resulting figure ought to reflect a variety of key conditions. For instance, the length of MF compared to that of FK should be equivalent to the length of BG compared to that of AG. Angle FKC should be precisely half of angle BGA. Line BZ should be perpendicular to line DI. Line DZ should be the same length as line ZI, and, as a result, triangle BDZ should be identical in size and shape to triangle BZI. Most important of all, angle of incidence BDE should equal angle of reflection EDA. Clearly, none of these conditions has been met in the figure. The closest the diagram comes to representative accuracy is a gross approximation of equality between DZ and ZI, and even in that case the degree of accuracy falls short of 75%.

One more example should suffice. In FIGURE 5.2.31, p. 587, which is adapted from a more complicated diagram on folio 67r of manuscript *O*, point G represents the vertex of a convex conical mirror with circle DEZ as its base and GZ and DZ as its outer edges. Point T represents the center of the cone's base-circle, TG the axis of the cone, TR the normal to point E on the base-circle, and GE a line of longitude on the cone's surface. MGN represents a plane parallel to that of the base circle, leaving source-point A of radiation and center of sight B positioned below it. KCF, finally, represents a line normal to line of longitude GE.

The intent of this construction is to illustrate that, under the conditions just specified, C is the appropriate point of reflection for A and B. But, as actually represented, point T is nowhere near the center of base-circle DEZ and, consequently, axis GT of the cone is badly misplaced. This, in turn, renders the diagram virtually useless as an aid to understanding why in this particular case C must be the point of reflection for A and B, because the proof for this claim depends on TER's being normal to circle DEZ—which it is obviously not in the diagram—from which it follows that angles HER and AER should be equal—which they are obviously not in the diagram.

Whether and how such misrepresentations might have affected a medieval reader's ability to make sense, or at least immediate sense, of the propositions they purport to illustrate are open questions. In ancient and medieval times diagrams were not always provided in mathematical texts, but we cannot be certain whether or to what extent such omission might have been intentional. If intentional, then the reader would have been expected to construct the diagrams for himself, either mentally or physically, and he would have been expected to do so unerringly from the verbal description. This interpretation is borne out to some extent by book 4 of the *De aspectibus*, where the geometrical descriptions provided by Alhacen are so detailed and punctilious as to render illustrative diagrams almost superfluous. Nevertheless, the very fact that such diagrams are supplied in some of the manuscripts indicates that more than a few medieval scholars regarded them as useful, if not necessary adjuncts to the descriptions they illustrate.

Unlike those of book 4, the mathematical constructions in book 5 are often so intricate and involved that it is diffcult to believe Alhacen expected anyone to make sense of them without the aid of diagrams. The issue here, however, is not whether Alhacen actually provided diagrams, much less whether he provided the diagrams (or their equivalents) that appear in the Latin manuscripts. The definitive resolution of this issue awaits the appearance of A. I. Sabra's long-expected edition of the Arabic text of books 4 and 5.[8] The relevant question is whether the diagrams in the Latin manuscripts constitute an integral part of the *Urtext*. Were they, in other words, explicitly produced, either through copying from the Arabic exemplar or through original construction, to accompany the initial Latin translation of the *Kitab al-Manazir*?

The remarkable consistency among figures across the spectrum of manuscripts suggests strongly that they were. Not only is there general agreement among the manuscripts about the point-by-point letter-designations in individual figures, but there is also general agreement about how those figures should be configured and presented.[9] Such consistency indicates (not surprisingly) that, like the text itself, the figures were simply copied from manuscript to manuscript rather than made to order for each manuscript. Furthermore—and again, not surprisingly—the figures seem in most cases to have been produced apart from the text, scribe and illustrator working independently rather than in collusion.[10]

With these points in mind, and having determined which figures to include with the text, I was left with one final issue. What form should those figures take? I had three clear options. One was to include a set of generic figures appropriately keyed to the text. These I had at hand in the form of diagrams I had laboriously reconstructed as I worked my way through the text, proposition by proposition. The problem with this choice is that I had taken great pains, with the aid of a fairly versatile drawing program, to make the reconstructed diagrams as accurate and true-to-description as possible—which means that they misrepresent the representations provided in the manuscripts. The second option was to reproduce the figures directly, by scanning, from one or more of the manuscripts. This option has the signal advantage of presenting the figures warts and all, with minimal editorial intervention on my part. But there are disadvantages as well. One concerns clarity. The scans would have to come from microfilm copies of the manuscripts, and the best examples available in that form are too light and nebulous to be useful. Another disadvantage is that, as they appear in the microfilms, many of the figures are so small or so awkwardly oriented that it is difficult to reconcile them with the text. Such figures could, of course, be magnified or reoriented but at the sacrifice of clarity and ease of reading.[11]

The option I finally lit on was to trace the figures directly from the scanned versions, re-orienting the abstracted diagrams as needed, and relettering them suitably, the result being crisp, clear, and eminently readable. I chose manuscript *O* as the representative basis for this process both because it includes almost all of the requisite figures and because those figures have not been editorially adjusted, as is sometimes the case in other manuscripts, *E* and *P3* in particular. On rare occasions I had to look beyond *O* to *P1* and *P3* for what I needed, but the lion's share of the adapted diagrams come directly from *O*. As far as basic configuration and lettering are concerned, therefore, the figures accompanying the Latin text are absolutely faithful to the originals, although they do not necessarily reflect the orientation of those originals. I should add that not all of the figures included with the Latin text are abstracted from originals. Only those that are too complex to have been accurately, or at least adequately, produced by compass and ruler have been treated this way. The rest come from my own stock of reconstructed diagrams. I should also add that, in the case of FIGURES 5.2.31, 5.2.31a, and 5.2.31b on pp. 587 and 588, I have actually abstracted each diagram from the composite figure provided on folio 67r of ms. *O*.[12] Generally speaking, though, I have treated the adapted figures with a relatively light editorial hand.

The Critical Apparatus: Since the conventions I used for the critical apparatus in this edition are precisely the same as those I used in the edition of books 1-3, I will not repeat them here. They can be found in *Alhacen's Theory*, pp. clxii-clxiv.

The Translation and Commentary: As far as my basic approach to translation is concerned, I have nothing to add to my discussion in the previous edition of books 1-3.[13] But a few words about mathematical notation are in order. Much of Alhacen's mathematical reasoning in book 5 is based on the analysis of ratios and proportions given in the fifth book of Euclid's *Elements* and subsequently applied to triangles and parallelograms in the sixth. Ratios and proportions are, of course, readily convertible to fractions and equations, so that, for instance, the expression $a:b :: c:d$ translates almost automatically to $a/b = c/d$. But consider the implications of such conversion. The first expression calls for comparison (i.e., magnitude a compares in size to magnitude b in the same way that magnitude c compares in size to magnitude d). The second calls for division (i.e., magnitude a divided by magnitude b leaves the same quotient as magnitude c divided by magnitude d). Not only do the two expressions invoke different operations (comparison vs. division), but the operative terms are different. In the first expression it is the magnitudes themselves that are being manipulated; in the second it is the numerical quantity of the magnitudes, abstracted from the magnitudes themselves, that is being manipulated.

To see how fundamentally incompatible the two expressions are at both the operational and conceptual level, we need look no further than *Elements* V.16, where Euclid demonstrates that, if $a:b :: c:d$, then, by alternation, $a:c :: b:d$. In fractional form, this means that, if $a/b = c/d$, then $a/c = b/d$, which is arrived at by multiplying both sides of the equation by b/c. So far so good, but suppose that *a* and *b* represent areas, while *c* and *d* represent lengths. In fractional form this poses no problem whatever, since areas can be divided by lengths (and vice-versa), leaving the quotients equal. But it poses an insuperable problem as a statement of proportionality, because there is no meaningful comparison between lines and areas. The absurdity of such a comparison becomes even clearer if *a* and *b* are taken to represent angles. For, while it makes sense to say that angle *a* compares in size to angle *b* in the same way that line *c* compares in size to line *d*, it makes no sense whatever to say that angle *a* compares in size to line *c* in the same way that angle *b* compares in size to line *d*. At bottom, then, the language of ratios and proportions is as different from that of fractions and equations as, say, Attic Greek is from modern English. Each has its distinct grammar, syntax, and vocabulary.

Not only did Alhacen think in the language of ratios and proportions; he "spoke" it fluently. I have therefore resisted the urge to recast his discourse in modern algebraic form, not just because such translation would be inauthentic and misleading, but because it would stand in the way of a proper appreciation of the ingenuity and elegance with which Alhacen manipulated ratios and proportions in his analysis of reflection. I have, however, streamlined the expression of proportions in the text, rendering such verbal statements as "erit proportio BN ad NO sicut proportio BM ad MO" in the symbolic form BN:NO = BM:MO, the equal sign substituting for the double colon so that the expression should be taken as "BN is to NO as BM is to MO" rather than "BN to NO is equal to BM to MO." Likewise, in denominating the square created from a given length, I have used the exponent for the sake of convenience (i.e., "quadratum AG" is rendered "AG^2"), but the result is meant to be understood as "the square composed from side AG" rather than "AG squared" in the algebraic sense. As to rectangular areas, finally, I have chosen to denominate them by placing an unspaced comma between the constituent sides, so that "ductus BD in DG" is rendered BD,DG and is to be construed as "the rectangle formed by sides BD and DG" rather than "BD times DG."

Aside from streamlining mathematical expressions, I have taken a few other liberties with the English translation. For one thing, I have reinforced the organizational breaks discussed earlier by interpolating parenthetical headings, such as [CASE 1] or [SUBCASE 1a], to clarify the analytic structure of the text. In addition—and here I must again acknowledge my debt to Risner—I have provided external and internal citations at appropriate points in the translation, inserting them parenthetically and setting them in brackets. In a few instances I have done no more than fill in the blanks. For example, where the text reads "angle GAN = angle GBA, as Euclid demonstrates in the third," I have appended the specific reference parenthetically (i.e., " . . . third [book of the *Elements*, prop. 32]"). Likewise, when the text adverts to earlier internal conclusions (for instance, toward the beginning of paragraph 2.201, p. 273, when it reads "according to what we established earlier," I have inserted the appropriate reference parenthetically (i.e., " . . . established [in proposition 24, lemma 6]"); and at various points in the theorems I have inserted "by previous conclusions" after certain claims to indicate that they have already been justified and, therefore, do not come out of thin air. But most of the interpolated citations amount to commentary and, as such, are intended to steer the reader to the specific theorem in Euclid (or, far more rarely, in Apollonius or Serenus) that justifies the conclusion at that point in the text. Thus, for example, at the end of paragraph 2.72, p. 238, I have given the particular theorem upon which the conclusion rests, so the resulting passage reads, "Therefore AG:DG = AE:ED [by Euclid, VI.3]." In some cases the conclusion and its basis struck me as so obvious, even to a reader unschooled in Euclidean geometry, that it required no annotation at all, but such cases are fairly limited

In addition to these source-citations, I have also inserted brief explanations at spots in the text where I considered such parenthetical interruptions short enough not to interfere unduly with the logical flow. The remainder of the commentary I have remanded to endnotes. At a few places in the translation, where the same letter is used to designate different points in the Latin version, I have distinguished the letters in the English translation by adding prime or double-prime signs and adjusting the designations in the accompanying diagrams to match (see, e.g., proposition 38, pp. 458-459 and its accompanying figure on p. 260). Moreover, I have done my best to make the diagrams that go with the English translation reflect the actual conditions specified in the construction and proof. They are, in short, meant to look as much like what they represent as possible. Accordingly, I have felt free to add letter-designations when necessary and even to add lines to the originals, all with the aim of making the analysis illustrated by the figure as easy to follow as possible. As a result, the diagrams included with the English text are interpretations of the originals in much the same way that the English text itself is an interpretation of the Latin original. To distinguish the figures adapted from the Latin text from those reconstructed for the translation, I have designated the former in capital letters (i.e., FIGURE 5.2.25), the latter taking the form figure 5.2.25. In book 4, the numbering of figures is determined by book and chapter, so that figure 4.2.1 refers to the first figure pertaining to chapter 2 of book 4. In book 5, most of the figures are determined by book, chapter, and proposition, so that figure 5.2.25 is keyed to proposition 25, which occurs in the second chapter of book 5. Because the first six figures in book 5 illustrate points that are not keyed to propositions, I have designated them neither by chapter nor proposition. I have simply listed them in order from 5.1. to 5.6.

So numerous are the diagrams used in this edition (well over 300) and, in some cases, so involved that inserting them directly into the text in appropriately reduced form struck me as too problematic to be worthwhile. I therefore decided to include them separately, the result occupying a total of 191 pages. In order to ease the burden of referring to them, though, I have put all the diagrams pertaining to the English translation and commentary at the end of the volume that contains the introduction and the Latin text. Conversely, I have placed virtually all the diagrams pertaining to the Introduction and Latin text at the end of the volume that contains the English translation and commentary. In a few cases, however, diagrams at the end of the volume containing the introduction are referred to in the introduction. Enabled thus to consult the diagrams continually in the one volume while progressing through the theorems in the other, the reader will be spared the inconvenience and irritation of flipping back and forth from page to page in an effort to reconcile text and figure.

The reference-aids provided in this edition are the same as those provided in the previous one. Accordingly, I have included a detailed topical synopsis at the beginning of each book in the English translation. In addition to the bibliography and general index, I have provided a Latin-English index keyed to technical terms in both Latin text and English translation. I have also provided an English-Latin glossary for cross referencing. In all, I have done my best throughout this edition to strike a balance between the needs and demands of experts in the field of medieval optics and mathematics and those of the larger community of interested scholars who require guidance to navigate that field effectively. Perhaps I have leaned too far in this latter direction, offering more guidance than necessary, but I took that risk to ensure that even a relatively inexpert reader could follow the intricacies of Alhacen's analysis and, in the process, gain a true appreciation of its ingenuity, rigor, and elegance.

NOTES

[1]For a complete description of the manuscripts and my method for grouping them according to families, see Smith, *Alhacen's Theory*, clv-clxxi. The actual manuscripts designated by the listed sigla are as follows: ***F*** Florence, Biblioteca Nazionale Centrale, ms II.III.324; ***P1*** Paris, Bibliothèque Nationale, ms Lat. 7247; ***Va*** Vatican, Biblioteca Apostolica, ms Palat Lat. 1355; ***V2*** Brugge, Stedelijke Openbare Biblioteek, ms 512; ***L2*** London, British Library, ms Sloane 306; ***S*** Saint-Omer, Bibliothèque Municipale, ms 605; ***Er*** Erfurt, Wissenschaftliche Bibliothek, ms Ampl F.392; ***C1*** Cambridge, Trinity College, ms 0.5.30; ***O*** Oxford, Corpus Christi College, ms 150; ***M*** Munich, Bayerishe Staatsbibliothek, ms CLM 10269; ***E*** Edinburgh, Royal Observatory, Crawford Library, ms Cr3.3; ***P3*** Paris, Bibliothèque Nationale, ms Lat. 7319; ***P2*** Paris, Bibliothèque Nationale, ms Lat. 16199; ***L3*** London, Royal College of Physicians, ms 383; ***C2*** Cambridge, University Library, ms Peterhouse 209; ***L1*** London, British Library, ms Royal 12.G.7; and ***V1*** Vienna, Österreichische Nationalbibliothek, ms 5322.

[2]See ibid., clxvii-clxix.

[3]See ibid., clxv.

[4]A. I. Sabra has in fact been working on this edition for well over 20 years, as indicated by the analysis of six propositions from book 5 that he published, along with an English translation, in "Ibn al-Haytham's Lemmas," 299-324, esp. 315-324. These six propositions correspond to the ones numbered 19-24 in the present Latin edition.

[5]See Smith, *Alhacen's Theory*, clx-clxi.

[6]The passage in question occurs at the beginning of paragraph 2.451 but is found at the end of paragraph 2.445 in all the manuscripts.

[7]See Smith, *Alhacen's Theory*, clxxvi.

[8]It is worth noting that the diagrams Sabra adapted from the Arabic originals in "Lemmas for Solving 'Alhazen's Problem'" correspond in all essential respects to their counterparts in the Latin manuscripts.

[9]The figures in Risner's edition are somewhat misleading in this regard, because, although most of them are based on the manuscripts he used (within the family represented by *E*), he felt free on occasion to reletter them; see, for example, p. 132 of Risner's edition for his version of figure 5.2.1, p. 220, where all the letters have been unaccountably changed. Risner also included many of the figures provided in the manuscripts for book 4. On the other hand, he made significant strides toward improving the accuracy of the more involved figures in the manuscripts; see, e.g., p. 150 of Risner's edition for his version of Figure 5.2.25, p. 239. In several instances I found his improved figures to be indispensable as I worked my way through Alhacen's mathematical analysis.

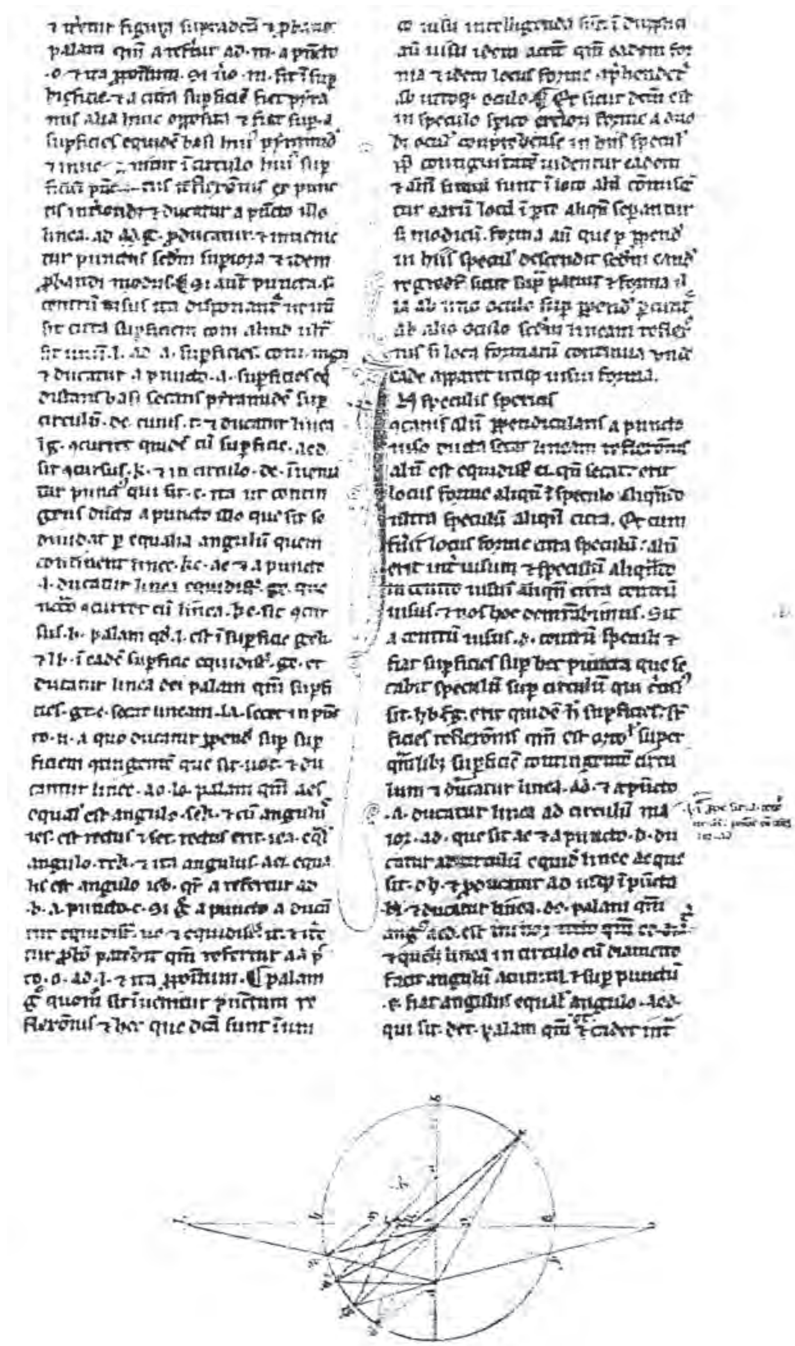
[10]The production of illustrated and/or rubricated manuscripts tended to be a small-scale assembly-line operation, each phase of which, from writing to illuminating and

illustrating, was carried out by a separate specialist. The resulting lack of communication between scribe and illustrator certainly helps explain the divergences in letter-designations between text and diagrams in so many of the manuscripts. It also helps explain why, in such manuscripts as *O* and *E*, extensive textual correction was necessary.

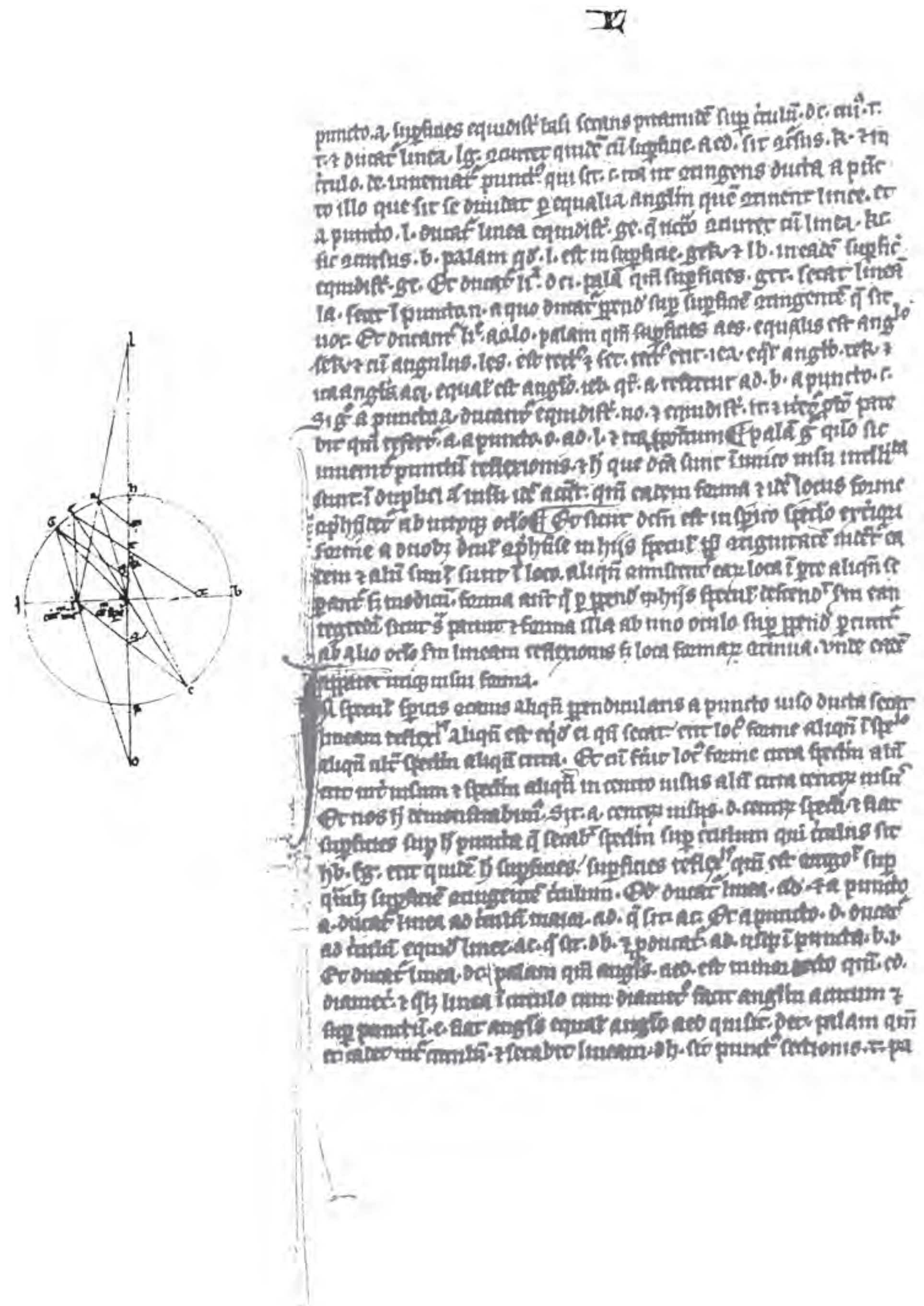
[11]On the one hand, when such digitally scanned diagrams are magnified, the lines tend to fragment and blur. On the other hand, when those diagrams are reoriented, the accompanying letters are reoriented as well, so, if the diagram is inverted, the letters will also be inverted. These problems can, of course, be resolved in various ways by enhancement and cut-and-paste options, but, given the number and complexity of the diagrams involved, I rejected that option as too time consuming to be worthwhile.

[12]See figure 5.6, p. 219, for this composite figure, the portion highlighted by heavier lines forming the basis for FIGURE 5.2.31, p. 587. Note that, in addition to abstracting the figure from its context in the composite diagram, I inverted it to stand the cone upright on its base and adjusted the lettering to conform with this inversion.

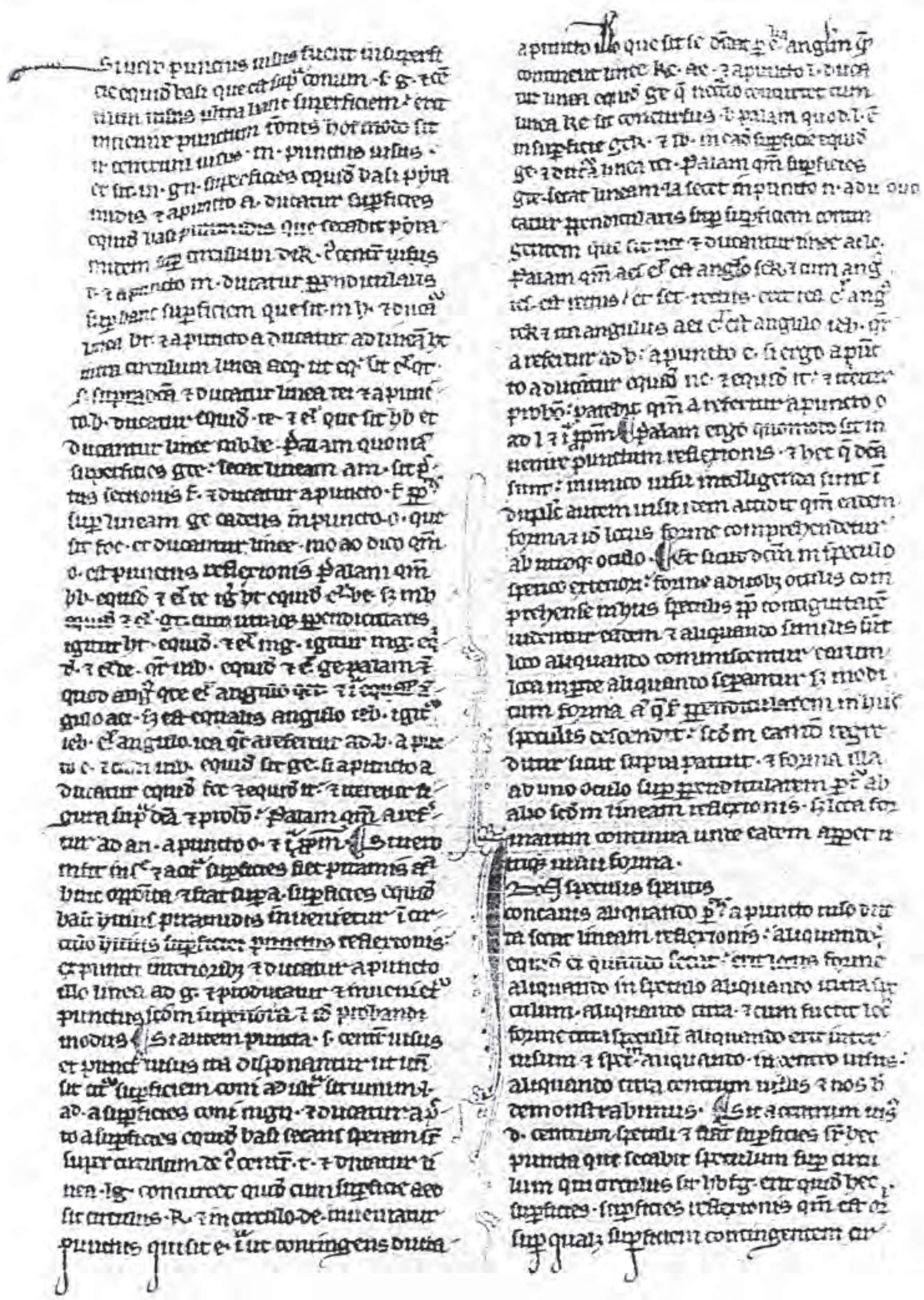
[13]See Smith, *Alhacen's Theory*, clxxiv-clxxv.



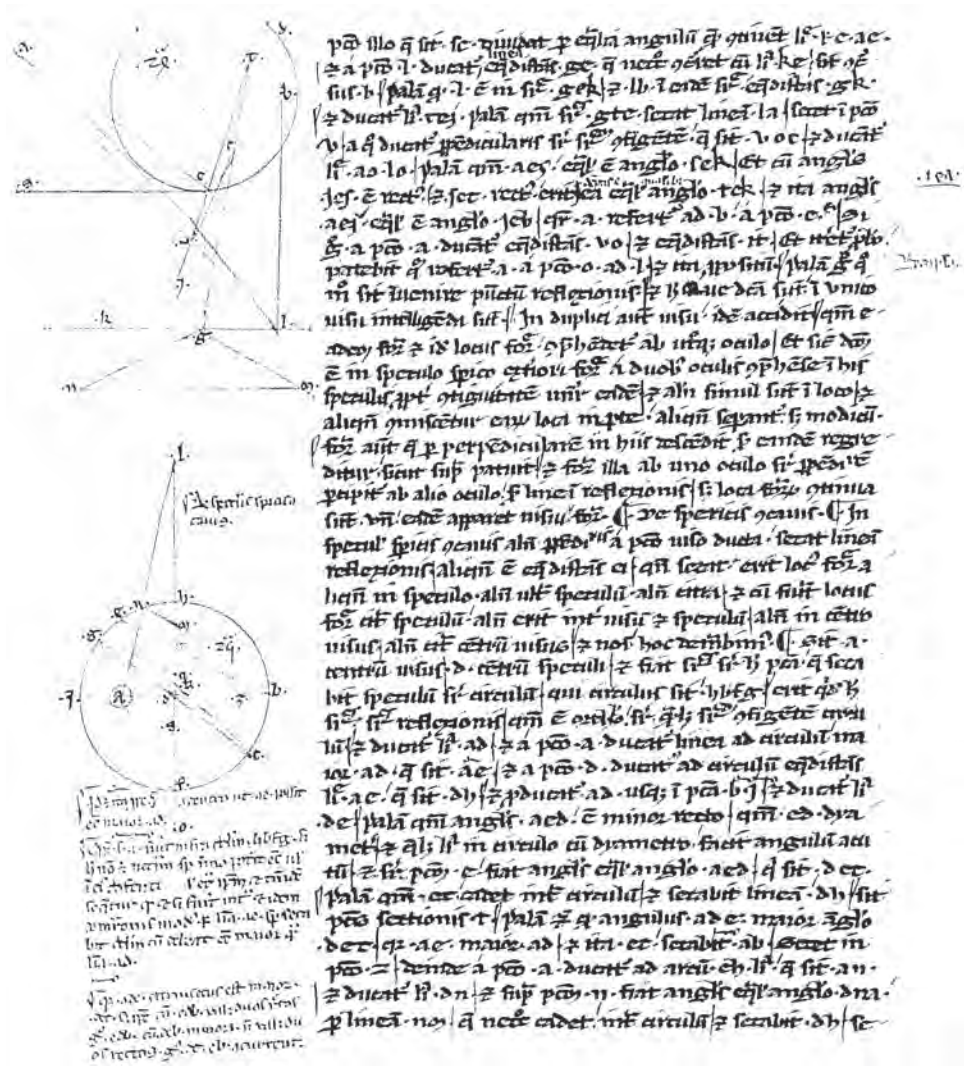
F Florence, Bibl. Naz. Cent., MS II.III.324, f 72 (71)v



P1 Paris, Bibl. Nat., MS Lat. 7247, f. 129v



S Saint-Omer, Bibl. Munic., MS 605, f 96r



E Edinburgh, Crawford Library, Royal Observatory, MS Cr3.3, f. 122v

L3 London, Royal College of Physicians, MS 383, f. 77r

O Oxford, Corpus Christi College, MS 150, f. 68v

C Cambridge, Trinity College, MS 0.5.30, f. 105v

ALHACEN'S
DE ASPECTIBUS
LATIN TEXT

[QUARTUS TRACTATUS]

Liber iste dividitur in quinque partes. Pars prima proemium libri; secunda in declaratione quod lucis accidit reflexio ex politis corporibus; tertia in modo reflexionis forme; pars quarta in ostensione quod comprehensio forme ex corporibus politis non est nisi ex reflexione; pars quinta in modo comprehensionis formarum per reflexionem.

[CAPITULUM 1]

[1.1] Iam explanavimus in libris tribus modum comprehensionis formarum in visu cum fuerit directus, et enumeravimus singula que in rebus visis comprehendit visus. Sed diversificatur adquisitio visus tripliciter: aut enim directe, sicut diximus; aut per reflexionem in politis corporibus; aut per penetrationem, ut in raris, quorum non est raritas sicut raritas aeris. Nec potest diversificari visus nisi hiis modis tribus.

[1.2] Et hiis duobus modis posterioribus comprehendit visus in rebus visis ea que supra exposuimus et quorum adquisitionem in visu directo patefecimus. Et forsitan visus in hiis incurret errorem aut consequitur veritatem. Et nos assignabimus in hoc libro quomodo per reflexionem fiat formarum adquisitio, et quomodo erit reflexio, et quis linearum reflexarum situs. Et preponemus quedam antecedentia preponenda.

1 *post* prima *add.* est *C1R*; *inter. O* 2 *ante* in *add.* est *R*/lucis: luci *C1R*; lucibus *O*/accidit: accidet *R*/ex: a *R*/politis *corr. ex* polititis *S* 3 *post* tertia *add.*est *R*/pars *om. R*/quod *mg. a. m. F*; *om. P1* 5 pars *om. R*/*post* quinta *add.* est *R*/reflexionem *corr. ex* flexionem *O*; *a. m. S* 6 libris tribus *transp. SOL3C1E*/modum: modos *C1* 7 directus *corr. ex* ductus *O* 10 *post* raris *add.* corporibus *C1* 11 sicut raritas *mg. L3*/*post* sicut *add.* est *C1*/raritas *om. O*/ nec *corr. ex* non *E*; et non *R*/*post* nisi *add.* in *ER*/tribus . . . modis (12) *om. P1* 13 ea *om. R*/ exposuimus *corr. ex* exposuerimus *C1* 14 forsitan: forte *FP1*/incurret *corr. ex* curret *a. m. S*; *alter. ex* intret *in* incurrit *a. m. E*; incurrit *R*/*ante* errorem *add.* in *R* 15 consequitur *alter. in* consequetur *C1*/quomodo *corr. ex* quando *a. m. E*; quando *R* 17 reflexarum *mg. F*/reflexarum situs *transp. P1*/antecedentia *corr. ex* accedentia *S*; accedentia *E*; accidentia *OL3R*

[CAPTITULUM 2]

[2.1] Planum ex libro primo quoniam lux a corpore lucido luce ei propria vel accidentali dirigitur in omne corpus ei oppositum, et eodem modo color cum in eo lux fuerit mittitur. Igitur corpore polito opposito corpori lucido, mittitur ad ipsum lux lucidi mixtim cum colore, et reflectitur lux cum colore, sive fuerit fortis sive debilis, sive prima sive secondaria.

[2.2] Et quod fiat in luce forti reflexio patere potest opposito luci forti speculo ferreo, et etiam oppositus sit paries speculo; et descendat super ipsum lux declinata non recta. Videbitur in pariete lux fortis reflexa, que quidem non videbitur super eundem locum si speculum auferatur, nec videbitur super eundem locum si speculum moveatur; immo secundum motum speculi mutabitur locus lucis reflexe in pariete. Quare palam reflexionem fieri in luce forti.

[2.3] In luce debili patere potest defacili. Intra domum aliquam per foramen unicam a terra elongatum, sed non multum, descendat lux diei, non solis, super aliquod corpus. Et circa corpus statuatur speculum ferreum, et circa speculum corpus aliquod album. Apparebit in secundo corpore albo lux maior quam sine speculo, et augmentum illius lucis non est nisi ex speculi reflexione, quoniam ablato speculo, sola lux secundaria debilis apparebit in corpore albo.

[2.4] Amplius, si diligens figatur intuitus in lineis per quas a corpore primo lux in speculum mittitur, perpenditur quidem linearum illarum declinatio super speculum et super idem linearum reflexionis declinatio eadem. Et est proprium reflexioni ut eadem sit declinatio et idem angulus linearum venientium et reflexarum. Quod si moveatur

19 *post* planum *add.* est *R*; *inter. a. m. S*/*post* ex *add.* hac *E*/libro primo *transp. FP1*/primo *inter. O*/ quoniam: quod *R*/ei *om. ER* 20 dirigitur: dirigatur *R*/in . . . mittitur (21) *mg. a. m. C1* (color *om. C1*) /*post* omne *add.* ad quod reflectantur luces vel colores ad (ad^1, ad^2 *inter.*) *O*/ei *corr. ex* enim *O* 21 lux fuerit *transp. FP1*/igitur: itaque *R*/polito *corr. ex* posito *OL3*/corpori *corr. ex* corporis *C1* 22 lucidi *om. R*/mixtim: mixtum *FP1*; *corr. ex* mixtum *L3C1* 23 sive^2: si *E* 24 secondaria: secundaria *R* 25 in *om. FP1*/patere potest *transp. ER* 26 et etiam: si *R*; et cum *inter. a. m. E*/sit *om. FP1*; fuerit *R*/descendat: descenderit *R* 28 *post* si *add.* moveatur *O*/*post* speculum *inter.* vel *O* 29 auferatur *alter. in* movetur *a. m. E*/nec . . . speculum *om. R*/speculum moveatur *transp. O*/*ante* moveatur *add.* vel *R*/moveatur: removeatur *C1* 30 secundum *inter. a. m. E*/*post* speculi *add.* et *L3C1*/*post* locus *scr. et del.* speculi *O* 31 *post* palam *inter.* est *O*; *mg. a. m. SC1*/luce forti *transp. S* 32 defacili: facile *R*/*ante* intra *add.* si *R*/unicam: unicum *R* 34 *post* diei *add.* ut *O*/et: sed *C1*/circa *alter. ex* citra *in* contra *a. m. C1*/speculum ferreum (35) *transp. L3C1* 35 circa: citra *C1*/ferreum . . . speculum *rep. FP1* 36 secundo *inter. L3*/illius lucis (37) *transp. P1* 37 lucis *mg. L3*; *om. ER* 39 *post* figatur *add.* ?? *C1* 40 perpenditur: perpendetur *R* 41 illarum: aliarum *O*/*post* linearum *add.* punctum *R* 42 reflexioni: reflexionis *R*/eadem sit *transp. R*/sit *om. C1* 43 quod *corr. ex* et *a. m. E*

corpus album a loco reflexionis in alium locum, tamen circa speculum, non videbitur in eo lucis augmentum, nec videri poterit nisi in illo situ tantum. Quare planum proprium esse reflexioni hunc situm.

[2.5] Hoc idem poteris videre secundaria luce si predictum speculum sit argenteum et corpus tertium album sit ex alia parte speculi. Apparebit quidem super corpus tertium lux secundaria, et super corpus secundum lux maior illa, et palam huius maioritatis causam solam esse reflexionem. Patebit autem lucis reflexio in omni loco ubi super corpus descendat per foramen aliquod lux fortis, adhibito luci speculo et ei corpore albo opposito modo supraposito.

[2.6] Verum locum reflexionis proprium et linearum situm explanabimus. Iam patuit in libro primo quod lux reflexa sequitur rectitudinem linearum, quare ex corporis politis fit reflexio secundum processum rectitudinis in situ proprio.

[2.7] Amplius, planum ex superioribus quod lux secunda a corpore illuminato accidentali luce procedens secum fert colorem corporis. Ab omni igitur corpore illuminato seu lucido color mixtus cum luce ad corpora opposita polita mittitur, et mixtim in partem debitam reflectitur.

[2.8] Et huius rei fides poterit fieri si intra domum unius foraminis tantum descendit lux solis super corpus forti et specioso colore. Et statuatur circa ipsum speculum ferreum, et circa speculum corpus concavum ad ciphi modum intra quod sit corpus album, et aptetur hoc vas in loco reflexionis ut lux reflexa incidat in corpus album. Apparebit quidem super faciem albi corporis color illius in quo fit descensus lucis, quod quidem non accideret si extra proprium reflexionis situm statuatur corpus album. Et secundum diversas colorum species hec probatum invenies, velut colori celesti, rubore, viriditate, et huiusmodi. Quare planum colorem mixtum cum luce remitti, et certior est coloris reflexi apparentia si speculum fuerit argenteum.

44 circa: citra *C1* 45 nisi *mg. a. m. F*/illo situ *transp. SOL3C1E* 46 *post* planum *add.* est *R*/ reflexioni: reflexionis *R*/*post* situm *scr. et del.* h *P1* 47 poteris: poterit *FP1SR*/videre: videri *R*/ *post* videre *inter.* in *a. m. L3E*; *add. C1R* 50 illa *corr. ex* illarum *E*/et *om. E*/causam: causa est *O* 51 autem *inter. a. m. L3*/lucis reflexio in omni loco: in omni loco lucis reflexio *R*/ubi *corr. ex* visi *L3* 52 descendat *alter. in* descendit *C1E*; descendit *R*/*post* aliquod *scr. et del.* super *C1*/luci *corr. ex* lucis *E* 53 opposito *om. O*; *mg. L3*; *inter. E*/supraposito: superposito *L3*; supradicto *OR* 54 proprium *om. R* 55 *post* lux *scr. et del.* et *C1* 56 corporis: corporibus *R* 58 *post* planum *add.* est *R* 59 ab omni igitur (60): igitur ab omni *C1* 60 igitur: ergo *O*/mixtus: mixtim *SOL3R*; mixtum *P1*; *corr. ex* mixtum *F*; *alter. ex* mixtum *in* mixtim *C1E*/ad *corr. ex* et *S* 61 *post* corpora *scr. et del.* ad *S*/mittitur *corr. ex* mittere *O*/mixtim: mixtum *L3*; *corr. ex* mixtum *C1*/ partem: parietem *P1* 62 huius: huic *R* 63 descendit: descendat *FP1SR*/solis *om. R* 66 ut *corr. ex* et *a. m. C1* 67 *post* illius *scr. et del.* corporis *S*/quo: quod *OR*/*post* fit *add.* primo *R* 68 accideret: accidet *R*/reflexionis situm *transp. R* 69 hec: hoc *L3C1R* 70 velut: ita in *R*/colori: colore *OER* 71 mixtum: mixtim *O*/cum *om. L3*/certior: certiorem *ER*/est: esse *SL3ER*; *corr. ex* esse *FO*; *alter. ex* esse *in* esset *C1* 72 apparentia: apparentiam *ER*/argenteum *corr. ex* argentumteum *F*

[2.9] Quare autem non appareat hec probatio—scilicet, quod comprehendatur color reflexus cuicumque corpori opponatur speculum et ei adhibeatur album—hec est ratio. Sicut supradictum est, colores debiles, licet simul cum luce mittantur, non sentiuntur, quia forme que reflectuntur debiliores sunt formis a quibus reflexio oritur. Et hoc in luce potest patere, quoniam luce forti in speculum cadente et reflexa in pariete, debilior videbitur lux parietis quam speculi, et notabilis est inter eas proportio.

[2.10] Idem patebit in luce debili pari modo. In domo prima dispositione prima, si corpus tertium album ponatur loco speculi ferrei vel circa ipsum, maior apparebit lux super hoc corpus quam super secundum, quod non accideret nisi reflexio lucem debilitaret.

[2.11] Sed dicet aliquis causam huius rei esse nigredinem speculi ferrei, que admixta luci in speculum cadenti ipsam obumbrat, et inde, reflexa in corpus secundum, debilis et fusca apparet. Sed in corpus tertium loco speculi vel circa positum, non descendit lux nisi a corpore primo nulli admixta nigredini. Verum quod hoc non sit in causa palam eo quod, loco speculi ferrei argenteo posito, eadem accidet probatio.

[2.12] Pari modo reflexus color debilior erit colore a quo fit reflexio, quod in domo reflexionis coloris patere poterit, si corpus album loco speculi ponatur vel circa. Fortior apparebit in ipso color quam in corpore albo intra vas posito. Et idem patebit si in loco ferrei argenteum ponatur speculum. Igitur reflexio debilitat et luces et colores, sed colores amplius quam luces secundum utrumque speculum. Et est quoniam colores accedunt debiliores quam luces, unde facile efficiuntur in reflexione debiliores.

[2.13] Amplius, color debilis, cum ad speculum pervenit, miscetur colori eius, quare reflexus apparet debilis et tenebrosus; et forme

73 appareat hec: apparet hoc *L3C1E* 74 opponatur *corr. ex* opponati *L3* 75 et: in *FP1*; sed *R* 76 debiles *inter. a. m. E*/non sentiuntur *om. FP1*/quia: in *O*; *inter. L3*; *om. ER*/*post* forme *inter.* enim *a. m. E*; *add. R*/que *alter. in* et *O* 77 debiliores: debilioris *F*; delioris *P1*; deliores *S*/hoc *inter. a. m. E*/hoc in *transp. L3*/hoc in luce (78): in luce hoc *S*/in *inter. a. m. C1*/*post* in *add.* hoc *ER* 78 in[2] *inter. O* 81 *post* modo *add.* ut *R*/*post* domo *add.* in *R*/prima dispositione (82) *inter. L3*; *inter. a. m. O* 82 prima *om. R*/*post* corpus *add.* tersum *R*/tertium *corr. ex* tersum *a. m. E*; *alter. ex* tirsum *in* tersum *a. m. C1*/ponatur: ponamus *ER* 84 accideret: accidet *P1*/reflexio *corr. ex* reflexionis *C1* 85 huius: huiusmodi *C1E* 86 inde *corr. ex* in *a. m. E*; *om. R* 87 sed *corr. ex* si *a. m. E* 88 tertium *alter. in* tersum *a. m. C1*/vel circa *om. R*/circa: citra *C1*/*post* positum *scr. et del.* d *F*/non *corr. ex* duo *F* 89 primo: proprio *O*/*post* palam *add.* ex *OR* 90 *post* eo *add.* est *R*/argenteo: argento *FP1C1*/*ante* posito *scr. et del.* posito *F*/accidet *om. FP1*; accidit *ER* 91 fit *inter. a. m. E*/reflexio: reflexionis *E* 92 reflexionis coloris: et vase ut antea *R*/poterit: potest *R* 93 ponatur vel circa: vel circa ponatur *O*/*post* in[1] *add.* ea *FP1*/quam: quem *FP1* 94 si *om. FS*; *inter. P1O*/in *om. R*/*post* ferrei *add.* speculi *R*/argenteum: argentum *O* 96 *post* luces *scr. et del.* unde facile efficiuntur in reflexione *O*/quoniam: quia *O* 97 accedunt debiliores *transp. OC1*/debiliores: debiles *S*; *inter. a. m. L3*/unde . . . reflexione (98) *mg. O* 99 cum . . . (eius: speculi) . . . aliquis (103) *mg. S*/ad speculum pervenit: pervenit ad speculum *R* 100 apparet: apparebit *ER*/*post* forme *scr. et del.* enim *O*

debiliores sunt reflexe quam in loco reflexionis, et reflexio causa est debilitatis.

[2.14] Poterit aliquis dicere non esse debilitatem formarum in reflexione nisi ex elongatione earum a sua origine. Sed explanabitur quod, licet ab ortu equaliter elongentur lux directa et lux reflexa, tamen debilior erit reflexa.

[2.15] Intret radius solis domum aliquam per foramen, et opponatur foramini in aere speculum ferreum minus foramine. Et lux foraminis residua cadat in terram super corpus album, et lux a speculo reflexa cadat in corpus album elevatum. Hoc observato ut eadem sit elevati et iacentis a foramine longitudo, videbitur quidem super elevatum lux minor quam super iacens. Et huius minoritatis non potest assignari causa nisi reflexio sola. Idem accidet si speculum fuerit argenteum.

[2.16] Idem in colore potest patere, luce solis in domum aliquam per foramen descendente super corpus coloris fortis cui circa adhibeatur speculum, et aliud corpus concavum intra quod sit corpus album in quod cadit reflexio. Et statuatur in domo aliud corpus album eiusdem modi cum eo quod est in concavo, et sit elongatio huius albi a corpore colorato in quod cadit lux foraminis eadem cum elongatione albi quod est in concavo ab eodem, et elongatione speculi ab eodem. Perpendi quidem poterit color debilior in albo quod est intra concavum quam in eo quod est extra, licet equidistant ab actu suo—id est a corpore colorato. Et in causa est reflexio colorem debilitans.

]2.17] Amplius, lux reflexa fortior est luce secundaria, licet eiusdem sit elongationis ab origine sua. Luce etenim reflexa cadente in corpus aliquod, si aliud eiusmodi corpus ponatur extra locum reflexionis, et

101 sunt *inter. O*/*post* reflexe *scr. et del.* ?? *C1*/quam *corr. ex* quasi *F*/reflexio *corr. ex* reflexionem *O*/causa est *transp. O*/causa *corr. ex* causam *L3*/est *om. L3* 103 formarum *om. P1*; *alter. ex* foraminis *in* forme *O* 105 quod *om. FP1*; *inter. a. m. E*/*post* quod *add.* solam *mg. L3*/*post* ortu *add.* suo *O*; sui *mg. a. m. C1*/elongentur: elongetur *O*/tamen *om. FP1O* 107 *post* solis *add.* in *R*/ et *inter. O* 108 ferreum *corr. ex* ferenum *E* 109 cadat[1]: cadit *SL3*; *corr. ex* cadit *a. m. E*/a speculo reflexa: reflexa a speculo *FP1*/cadat[2]: cadit *L3* 110 album *alter. in* aliud *inter. a. m. C1*/ *post* album *add.* a terra *O*/*post* et *add.* latentis *FP1E* 111 iacentis *mg. a. m. F*; *om. P1E*/a *inter. a. m. C1*/*post* foramine *scr. et del.* et *L3C1*/super elevatum *corr. ex* elevatum super *E* 113 accidet *corr. ex* accidit *P1*; accidit *ER*/fuerit *om. E*; sit *R*/argenteum *corr. ex* arteum *mg.F* 114 idem: item *L3C1*/luce: lux *R*/*ante* solis *add.* enim *R*/domum: domui *O* 115 descendente: descendat *R*/circa: contra *C1*; *om. R*/adhibeatur: adhibeantur *L3*; *corr. ex* adhibeantur *C1* 116 aliud: quod *O*/*post* aliud *inter.* sit *O*/corpus[1] *om. E*/sit *corr. ex* si *S* 117 cadit: cadat *O*/statuatur: statuatuatur *S*/aliud *corr. ex* aliquid *O*/eiusdem: eius *E* 118 cum eo *inter. O*/huius: huiusmodi *C1*/a *mg. a. m. E*/*post* corpore *scr. et del.* s *O* 120 ab[1] *corr. ex* cum *L3E* (*a. m. E*)/eodem[1]: eadem *E*/*post* et *add.* cum *R*/elongatione: elongatio *OL3C1E*/eodem[2]: eadem *L3E*/perpendi *alter. ex* perpendicularis *in* erit *O*/perpendi quidem poterit (121): tunc comprehendetur *R* 121 quidem poterit *transp. E* 122 equidistant: equidistent *OL3C1ER*/actu *alter. in* arcu *L3*; ortu *C1ER*/a: in *FP1* 124 fortior est *transp. O*/est *inter. SOC1* (*a. m. C1*); *om. L3E* 125 sit: sint *R*/etenim: enim *R* 126 eiusmodi: huiusmodi *O*; eiusdemmodi *L3C1*

sit cum eo eiusdem elongationis a speculo, videbitur super ipsum lux minor quam in illo.

[2.18] Idem etiam planum erit in domo, si deprimatur in terra in directo foraminis speculum quod accipit totam foraminis lucem. Erit lux fortior super corpus in loco reflexionis positum quam super aliud eiusdem modi extra hunc locum tantumdem a speculo elongatum.

[2.19] Eodem modo, si excedat lux foraminis quantitatem speculi, et cadat circa speculum lux in terram aut corpus album a quo aliud corpus tantum elongatur quantum corpus reflexionis a speculo, debilior apparebit in eo lux quam super reflexionis corpus.

[2.20] Similiter accidit in colore. Si corpus aliquod tantum distet a speculo extra situm reflexionis quantum aliud ei simile quod est in situ, apparebit quidem super corpus quod est in situ reflexionis color reflexus; super aliud forsitan nullus. Si enim ferreum fuerit speculum aut fere nullus videbitur, aut omnino nullus; si vero argenteum fuerit speculum, apparebit super ipsum color aliquis, sed valde debilis, et longe debilior quam in corpore quod est in situ reflexionis.

[2.21] Et iam igitur planum quod forme lucium et colorum reflectuntur ex corporibus politis et in reflexione debilitantur. Et erit forma directa fortior reflexa cum eadem earum origo et equalis ab ea elongatio. Et reflexa fortior secundaria cum idem vel equalis earum ortus et par elongatio.

[CAPITULUM 3]

Pars tertia: in modum reflexionis formarum in corporibus politis

[3.1] Politum est lene multum in superficie, et lenitas est quod sint partes superficiei continue sine pororum multitudine. Lenitas intensa

129 etiam *corr. ex* est *O*; et *L3* / deprimatur: deponatur *R* / terra: terram *R* 130 quod *alter. in* et *inter. O* / accipit *alter. in* accipat *O*; accipiat *R* 132 tantumdem *corr. ex* tandtumdem *F* / a speculo *mg. L3E* (*a. m. E*) / a speculo elongatum: elongatum a speculo *ER* 133 foraminis *om. O* 134 cadat: cadit *SL3*; *corr. ex* vadat *O* / in terram *corr. ex* interior *a. m. E* / *post* aut *inter.* in *a. m. C1* 135 quantum: quam *E* 139 *post* situ[1] *add.* reflexionis *R* / apparebit . . . situ[2] *om. P1* / color reflexus (140) *om. O* 140 forsitan: forte *FP1* / nullus: vel *O* 141 aut . . . speculum (142) *om. O*; *mg. a. m. L3* / fere nullus *om. R* / *ante* videbitur *add.* modicus *R* 142 sed: et *O* 143 corpore *corr. ex* corde *F* 144 et[1] *om. OC1* / *post* forme *scr. et del.* lucis *P1* / reflectuntur (145): inflectuntur *FP1*; *om. ER* 145 *post* politis *add.* reflectuntur *ER* 146 *post* cum *inter.* sit *O* / *post* eadem *add.* fuerit *R* / *post* ea *add.* origine *R* 147 *post* cum *add.* est *R* / vel *inter. a. m. E* / equalis earum *transp. C1* / earum: eorum *O*; *inter. a. m. L3*; *om. ER* / *post* ortus *add.* est *C1* / *post* et[2] *scr. et del.* p *S* 149 pars . . . politis (150) *om. FP1S* 151 lene: leve *R* / lenitas: levitas *R* / quod: ut *R* 152 pororum *inter. a. m. E* / lenitas: levitas *R*

est ubi multa partium superficiei continuitas et pororum parvitas et paucitas, et finis lenitatis est privatio pororum et privatio divisionis partium. Igitur politas est politiva continuitas partium superficiei cum poris raris et exiguis, et finis politive est vera continuitas partium et privatio pororum.

[3.2] In omnibus politis superficiebus, licet diversis subiaceant figuris, accidet reflexio, et idem reflexionis modus et eadem proprietas: et est quod in omni polita superficie ab omni puncto fit reflexio; et sumpto quocumque puncto in superficie a quo fiat reflexio, linea accessus forme alicuius ad illum punctum et linea reflexionis in eadem superficie erunt cum linea perpendiculari super illud punctum erecta; et tenebunt hee linee eundem situm respectu perpendicularis et equalitatem angulorum. Et volo dicere perpendicularem que sit perpendicularis super superficiem tangentem corpus politum in illo puncto, et due linee cum perpendiculari sunt in eadem superficie ortogonaliter cadente super superficiem corpus politum in puncto a quo fit reflexio tangentem.

[3.3] Si autem linea per quam accidit ad speculum forma cadat perpendiculariter super illud, fiet reflexio forme per ipsam, non per aliam, et hoc est proprium in omni reflexione in omni polito corpore. Si ergo corpus politum fuerit planum, superficies tangens punctum reflexionis erit una et eadem cum superficie corporis. Si vero fuerit columpnare speculum interius aut extra politum, erit contactus superficiei speculi et superficiei contingentis linea tantum secundum longitudinem speculi intellecta. Idem in speculo piramidali intus vel extra polito. In sperico, sive concavo interius sive exterius polito, contingens superficies tangit in solo puncto.

153 est *om. SL3*; *alter. in* et *inter. O* / ubi *corr. ex* ut *L3* / *post* ubi *add.* est *R* 154 lenitatis: levitatis *R* 155 igitur: itaque *R* / politas: politio *R* / est: vel *O* / politiva *alter. in* politura *a. m. C1* / *post* politiva *scr. et del.* igitur politas *F*; *inter.* est *O* 156 politive *corr. ex* polite *O*; *corr. ex* politionis *a. m. E*; politionis *R* 157 *post* pororum *add.* omnium *C1* 158 *post* politis *scr. et del.* licet *O* 159 reflexio et idem *om. FP1* / et[1] *inter. a. m. E* / *post* proprietas *add.* est *R* 160 in: ab *R* / polita: politi *ER* / *post* superficie *add.* et quolibet eius *R* / ab omni *om. R* / ab . . . superficie (161) *mg. a. m. S* 161 *post* superficie *scr. et del.* a quo in superficie *P1* / quo: qua *R* / fiat: fit *ER* 162 alicuius *om. ER* / illum: illud *R* / *post* eadem *scr. et del.* lo *F* 163 superficie *inter. O* 164 respectu: respecti *F* 165 angulorum *corr. ex* angelorum *F* / sit *inter. a. m. E* 167 cum *inter. C1* 168 *post* superficiem *add.* visi *O* / in *inter. OL3E* (*a. m. L3*) 170 per quam *om. O* / accidit: accedit *C1R* / *post* speculum *inter.* ad *O* / cadat: cadet *E* 172 ergo: igitur *R* 174 et *om. OL3E* / superficie *corr. ex* super forme *O* / *post* vero *scr. et del.* corporis *S* / *post* fuerit *add.* columna *F* 175 extra: exterius *R* / erit: erunt *FP1SL3E*; *corr. ex* erunt *O* / *post* superficie *scr. et del.* et *F* / speculi: speculum *O* 176 et superficiei *om. O* / contingentis *alter. in* tangentis *a. m. C1* / linea *corr. ex* lineam *SL3*; lineam *C1* 177 intellecta: intellectam *OE* / speculo piramidali *transp. O* 178 concavo *om. R* / *post* concavo *add.* sive *C1* / interius *om. O*; *inter. L3*; *inter. a. m. E* / tangit *corr. ex* tangi *O* 179 puncto *corr. ex* speculo *O*

[3.4] Quomodo autem ad oculum pateat hic modus reflexionis in speculis omnibus explanabimus. Accipe tabulam eneam spissam ut firmior sit, eius longitudo non minus quam 12 digitorum, et sit latitudo sex digitorum. Et fiat linea equidistans extremitati longitudinis et circa illam extremitatem. Et super punctum huius linee medium ponatur pes circini, et fiat semicirculus cuius semidyameter sit latitudo tabule.

[3.5] Et extrahatur a puncto quod est centrum linea ortogonaliter super dyametrum iam factum. Et erit linea illa semidyameter dividens semicirculum per equalia. Et in hoc semidyametro sumatur mensura unius digiti et, posito pede circini super centrum, fiat semicirculus secundum quantitatem partis residue semidyametri, residue scilicet secundum semidyametrum quinque digitorum.

[3.6] Et dividantur semicirculi primi medietates in quot libuerit partes ita quod sibi respondeant in qualitate prima—scilicet prima prime, secunda secunde, et sic de aliis—et protrahantur linee a centro ad puncta divisionum.

[3.7] Deinceps in semidyametro mensura digiti signetur, et ex parte centri, et super punctum signatum protrahatur linea equidistans dyametro semicirculi, sive tabule extremitati, quod idem est. Et secetur ex tabula quod interiacet hanc lineam et semidyametrum usque ad centrum et lineas primas ad divisiones semicirculi protractas—id est ad lineas tales semidyametro propinquiores.

[3.8] Post secetur tabula circa semicirculum maiorem ut solum remaneat semicirculus. Et secetur tabula sub centro ubi centri locus acuatur quasi punctus hoc tamen modo ut in eadem superficie plana remaneat cum semicirculo et aliis lineis.

[3.9] Post sumatur tabula lignea plana excedens eneam in longitudinem duobus digitis, et sit quadrata, et eius altitudo, sive spissitudo,

180 quomodo: quoniam *FP1S*/autem: et *E*/*post* autem *add.* etiam *R* 181 explanabimus: declanabimus *P1*/*post* spissam *add.* paululum *O* 182 *post* longitudo *inter.* sit *O*; *add.* sit *R*/*post* non *inter.* sit *a. m. C1*/minus: minor *R*/quam *om. R*/12: 22 *P1*/digitorum: digitis *R*/et *om. OL3ER*/*post* sit[2] *inter.* que *a. m. E*; *add.* que *R* 185 circini: circuli *FP1*; *inter. E*/semicirculus *corr. ex* semicircularis *O*/semidyameter: semidyametri *E* 187 factum: factam *R* 188 semicirculum *corr. ex* circulum *L3C1* (*a. m. C1*)/hoc: hac *R*/semidyametro: dyametro *O*/sumatur *corr. ex* sumsit *F* 189 circini: circuli *FP1*; *om. C1* 190 *post* semidyametri *inter.* huius *a. m. S*/residue *inter. a. m. S*/residue scilicet *om. R*/scilicet *mg. a. m. C1*; *inter. a. m. E* 191 *ante* semidyametrum *scr. et del.* d *C1*/semidyametrum *corr. ex* semi *L3E* (*a. m. L3*)/quinque: 2 *O* 192 dividantur: dividentur *E*; dividiantur *R*/semicirculi *corr. ex* semidantur *F*/medietates: medietatis *FP1SOL3* 193 quod: ut *R*/respondeant: remaneant *FP1SL3*/qualitate: quantitate *O*; equalitate *R*/prima *scr. et del. O*; *om. C1R* 196 digiti *corr. ex* digitis *OL3* 198 quod item est *inter. L3*; *om. OC1ER* 200 semicirculi *corr. ex* semicirculis *S*/protractas: promotas *O*; pertractas *C1* 201 *post* tales *scr. et del.* dyametri *L3C1*/propinquiores *om. P1* 202 *post* post *add.* illud *C1* 203 et *inter. O*/ubi: ut *C1ER* 204 quasi: qua *L3*/punctus: punctum *R*/plana *om. R* 205 *post* remaneat *inter.* centrum *a. m. C1* 206 longitudinem (207): longitudine *R* 207 et sit quadrata *inter. O*/eius: ei *O*

septem digitorum. Signetur ergo in hac tabula punctum medium, et super ipsum fiat circulus excedens maiorem circulum tabule enee super quantitatem digiti magni. Et fiat super idem centrum circulus equalis circulo minori tabule enee.

[3.10] Et dividatur circulus maior in partes in equalitate respondentes partibus semicirculi tabule enee, ut scilicet prima respondeat prime, secunda secunde, et sic de aliis. Et secetur circumquaque tabula lignea ut solum remaneat maior circulus, et erit hec sectio usitato secandi modo. Secetur etiam pars tabule circulo minori contenta, et modus sectionis erit ut huic tabule associetur alia tabula ita ut linea a centro huius ad centrum illius transiens sit perpendicularis super illam. Et adhibito tornatili instrumento centris earum, fiat sectio partis circularis iam dicte. Est autem alterius tabule associatio, ut fixa stet in sectione.

[3.11] Igitur restabit tabula quasi anulus circularis cuius latitudo duorum digitorum, longtitudo 14, altitudo septem, et sit hec altitudo optime circulata ad modum columpne. Remanent autem in latitudine huius anuli linee dividentes circulum eius secundum divisionem semicirculi tabule enee.

[3.12] A capitibus harum linearum producantur linee in superficie altitudinis exterioris perpendicularis super superficiem latitudinis, et poterit hoc modo fieri. Queratur regula bene acuta cuius capiti linee adhibeantur, et regula moveatur donec tangat superficiem altitudinis in qualibet parte acuminis. Signa eius capita, et fac lineam, quoniam illa erit perpendicularis quam queris. Et eadem sit operatio secundum quamlibet dividentem lineam.

[3.13] Aliter poterit hoc idem fieri. Ponatur pes circini super terminum linee dividentis, et fiat semicirculus secundum altitudinem anuli, qui dividatur per equa. Et protrahatur a puncto ad punctum

208 signetur: significetur *L3* 209 super (210): secundum *O*; *alter. in* secundum *L3*; *om. R* 210 quantitatem: quantitate *R*/et *om. O* 212 et . . . enee (213) *mg. a. m. S* 213 ut *om. O*/respondeat (214) *corr. ex* respondet *C1* 214 secetur circumquaque *transp. ER* 215 erit: fiet *R*/usitato: usitati *P1* 216 secetur: seceat *O*/etiam: et *O*/circulo minori: minore circulo *R*/contenta: contempta *O*; *corr. ex* contempta *L3* 217 alia *corr. ex* talia *O* 218 huius *om. P1*/super *corr. ex* per *O*/illam: ipsam *L3C1* 220 dicte: dicto *O*/*ante* est *scr. et del.* autem *O*/autem *inter. O* 222 quasi: sicut *FP1*; ?? *O*/*post* latitudo *add.* erit *R* 224 optime *om. P1*/columpne: calumpne *F*/remanent: remanet *FC1* 225 anuli: anguli *E* 227 *post* capitibus *add.* autem *R*/harum linearum *transp. R*/linee: linea *R* 228 perpendicularis *alter. in* perpendiculares *O*; perpendiculares *R* 229 regula *om. FP1*/bene: lene *S*/*post* cuius *scr. et del.* s *F* 230 adhibeantur: adhibeatur *FP1O*/regula: reliqua *FP1*; *alter. in* ita *a. m. E*/*post* regula *scr. et del.* in *S*/tangat: transeat *ER* 231 signa *inter. O* 232 illa *inter. a. m. E*/queris *corr. ex* quamvis *O*/et . . . lineam (233) *om. R* 234 circini: circuli *FP1* 235 *post* dividentis *add.* circulum *R*/*post* altitudinem *scr. et del.* anguli *P1* 236 equa: equalia *SR*/ad: in *R*

linea, et ita in singulis. Pari modo, a terminis illarum dividentium protrahantur perpendiculares ex parte interioris altitudinis.

[3.14] Amplius, sumatur in altitudine interiori ex parte faciei non divisa altitudo duorum digitorum, et in perpendicularibus fiat signum. Et in signis illis fiat circulus equidistans faciei anuli hoc modo. Tabula aliqua plana fiat circularis equalis circulo minori tabule enee, et secetur ex ea pars aliqua usque ad centrum quasi triangulus ex duobus semidyametris et arcu circuli secundum quod libuerit ut possis tabulam cum manu imponere et locis assignatis aptare. Apta ergo locis illis ut sit equidistans faciei anuli, et fac circulum secundum ipsam.

[3.15] Sumatur etiam in hunc circulum altitudo medietatis grani ordei, et fiant signa, et in punctis signatis fiat circulus per aptationem tabule. Et in hoc circulo postremo fiat circularis concavitas, et sit unius digiti eius profunditas et altitudo tamquam altitudo tabule enee. Et sit altitudo hec intra altitudinem duorum digitorum ut eadem sit postremi circuli et concavitatis superficies.

[3.16] Aptetur autem huic concavitati tabula enea, que quidem intrabit concavitatem usque ad circulum minorem, cum distantia minoris a maiori sit unius digiti, et concavitas similiter. Igitur circulo postremo et tabule enee communis erit superficies, et linee perpendiculares in altitudine anuli tangunt lineas divisionis tabule enee, et cadent perpendiculariter super tabulam eneam. Sit autem facies tabule enee divisa ex parte faciei anuli divise.

[3.17] Amplius, in exteriori altitudine anuli signetur punctus a longitudine duorum digitorum, et posito pede circini super punctum signatum, fiat circulus secundum quantitatem unius grani ordei. Et instrumento ferreo cuius similiter latitudo sit quantitas unius grani ordei perforetur foramine columpnari. Et baculus ligneus foramini aptetur,

237 et *inter. E*/in: de *R*/*post* singulis *inter.* sectionibus *L3*; *add.* sectionibus *C1*/pari *alter. ex* ponatur *inter.* eodem *O*/*post* dividentium *add.* circuli *C1* 238 perpendiculares: particulares *E* 240 divisa: diverse *O*; divise *R* 242 plana: plena *L3*/equalis circulo *transp. OL3C1R*/*post* minori *add. circulo O* 243 ea: eo *FP1SO*/triangulus: triangulis *FP1L3*; triangus *S*; triangulum *R*/duobus: duabus *R* 244 quod *inter. a. m. E* 245 *post* assignatis *add.* imponere et *E*/ergo: igitur *O*/locis *inter. a. m. S* 246 ipsam: ipsum *C1* 247 etiam: ergo *L3C1E*/in: infra *R*/circulum: modum *E*/medietatis grani *transp. C1* 248 punctis: ?? *O*/signatis: assignatis *OL3C1ER* 249 circulo postremo *transp. R*/*post* circulo *add.* possumus *P1*/*post* fiat *add.* circulus *P1*/concavitas *corr. ex* cavitatis *O* 250 *post* digiti *inter.* et sit *a. m. E*/eius *inter. a. m. E*/et altitudo *inter. O* 251 altitudo hec *transp. R* 252 circuli: speculi *E*/concavitatis *corr. ex* continuitatis *L3*/superficies: species *R* 253 intrabit *alter. ex* intrat *in* intret *E*; intret *R* 254 *post* minorem *add.* et *R*/cum *inter. a. m. C1* 257 anuli: anguli *FP1*/tangunt: tangent *R* 258 facies: superficies *R* 259 *post* divisa *scr. et del.* ex parte *O* 260 altitudine *corr. ex* planitudine *a. m. E*; planitudine *R*/signetur: signatur *SOE*; significatur *L3*; *corr. ex* signatur *C1*/punctus *om. O*/a *om. C1* 261 longitudine *corr. ex* altitudine *a. m. C1*/circini: circuli *FP1* 262 secundum *corr. ex* secundus *S*/et . . . ordei (263) *mg. a. m. E* 263 *post* cuius *scr. et del.* ci *F*/unius: cuius *FP1* 264 perforetur: perfororetur *F*/*post* foramine *scr. et del.* circulari *O*

qui quidem, cum transierit ad interiorem concavitatem, tanget tabule enee superficiem. Pari modo, super singulas exterioris altitudinis perpendiculares similia et equalia efficiantur foramina in quantitate et altitudine.

[3.18] Deinde sumatur tabula lignea quadrata cuius latus est equale dyametro anuli, et protrahatur in eius superficie linea dividens quadratum per medium equidistans lateribus. Et ab una parte sumatur longitudo duorum digitorum, et fiat signum. Post sumatur longitudo semidyametri minoris circuli tabule enee, et posito pede circini, fiat circulus transiens per signum, qui circulus erit equalis minori circulo tabule enee et concavitati anuli.

[3.19] Deinde supra centrum huius circuli sumatur longitudo duorum digitorum, et infra centrum similiter, et signentur puncta. Ab utroque in utramque partem protrahatur linea equidistans lateribus quadrati, et in utraque harum linearum signetur longitudo duorum digitorum ex utraque parte puncti signati. Et a punctis unius linee signatis protrahantur linee equidistantes ad puncta alterius linee signata, et fiet quadratum quatuor digitorum. Fodiatur hoc quadratum secundum altitudinem unius digiti, et concavationis latera efficiantur plana et ortogonalia, et fundus similiter planus.

[3.20] Deinde aptetur hec tabula faciei anuli ita ut circulus minor applicetur foramini anuli, et extremitas eius extremitati. Et firmetur hec applicatio cum clavis ut immota maneat tabula. Nota quod in omnibus predictis duorum digitorum mensura certa debet esse et determinata, et ob hoc in linea aliqua fiat immutabili ne ex mutatione mensure error accidat.

[3.21] Amplius, fiat columpna ferrea concava plana aliquantulum spissa ut nec statim intret nec immutari queat, et sit quantitas dyametri circuli eius unius grani ordei. Et ponatur columpna in foraminibus, que quidem cum ad interiora anuli pervenerit, continget lineas in tabula

265 cum . . . singulas (266) *mg. O*/tanget: tangit *L3C1E* 266 *post* modo *add.* si *FP1SC1*; *inter.* si *L3*/super: per *C1* 267 et equalia *om. O* 269 *post* tabula *scr. et del.* ignea *C1*/est: fit *O*; sit *L3C1ER* 271 quadratum *om. O*/quadratum per medium: per medium quadratum *ER*/per medium *inter. L3E*; *a. m. E*/sumatur *corr. ex* sumuatur *F* 272 *post* signum *add.* et *R* 273 circini: circuli *FP1* 274 transiens . . . circulus[2] *om. P1*/*post* qui *add.* quidem *R* 276 supra: super *E* 277 duorum *inter. E*/infra: infixa *L3*/signentur: assignentur *L3* 278 *post* partem *add.* et *R*/protrahatur *corr. ex* pertrahatur *S* 279 *post* utraque *scr. et del.* istarum *P1*/harum *corr. ex* secundarum *L3*/duorum: et *S* 280 parte puncti signati: puncti signati parte *O* 281 protrahantur linee equidistantes: protrahatur linea equidistans *O*/equidistantes *corr. ex* equidistan *F* 282 fiet: fiat *L3ER*/fodiatur: cavetur postea *R*/*post* fodiatur *scr. et del.* in *E* 283 *post* altitudinem *scr. et del.* illius *P1*/latera *corr. ex* litera *O* 286 anuli: eius *R* 287 nota: notandum *R*/*ante* quod *add.* vero *R* 288 predictis: punctis *O*/duorum: dictorum *R*/certa *inter. a. m. E*/et *inter. C1E*; *a. m. C1* 289 et ob hoc *mg. a. m. E*/*post* ob *scr. et del.* bet *O*/hoc *om. FP1*/linea: lignea *SL3E*/fiat: fit *L3E*/immutabili: immutabuli *F*; *corr. ex* immutabuli *P1S*/ne: nec *P1* 292 nec[1] *om. ER*/et *om. E* 294 quidem *corr. ex* idem *L3*

enea factas. Et erit operis eius complementum si linea tabule enee sit contingens circulo columpne in puncto linee altitudinis anuli perpendicularis super tabulam eneam et transeuntis per centrum circuli columpne.

[3.22] Fiat autem in capite columpne anulus aut repagulum quod non permittat columpnam intrare nisi ad locum determinatum. Fit autem huiusmodi longitudinis columpna ut procedens supra tabulam eneam attingat lineam equidistantem dyametro tabule intra quas facta est sectio. Et est linea illa equidistans basi trianguli tabule enee.

[3.23] Amplius, fabricentur septem specula ferrea quorum unum planum; duo sperica, unum concavum intra politum, aliud extra; duo columpnaria, unum concavum, aliud in superficie politum; duo piramidalia, unum politum in facie, aliud in concavitate. Speculum autem planum fit circulare, et sit eius dyameter longitudinis trium digitorum.

[3.24] Speculum columpnare politum in superficie sit lucidum et perfecte politum, et sit dyameter circuli longitudinis sex digitorum, qui circulus est basis eius. Longitudo autem columpne sit trium digitorum. In base columpne sumatur corda longitudinis trium digitorum. Similiter in base eiusdem columpne opposita sumatur equalis huic corda et ei opposita ut linee a capitibus unius corde ad capita alterius producte sunt recte. Et secetur hec columpna secundum harum linearum processum ut restet nobis pars columpne cuius capita portiones cordarum earum, aut altitudo axis portionis remanentis minus quam dimidii digiti. Axem autem dico lineam a medio puncto arcus ad medium corde punctum productum.

[3.25] Columpne concave longitudo sit trium digitorum, et dyameter basis eius sex digitorum, et in ea sumatur corda trium digitorum, et

295 factas: stans *FP1O*/eius *om. L3*; *mg. a. m. C1*/complementum *corr. ex* completum *S*/sit *om. R* 296 contingens circulo: contingat circulum *R* 297 transeuntis *corr. ex* transeuntes *E* 300 permittat *corr. ex* permittit *E*/fit: sit *R* 1 autem *om. FP1*/huiusmodi: huius *SL3C1ER*/supra: super *FP1ER* 2 *post* tabule *add.* linee *P1*/intra: inter *OE*/quas: quam *R* 3 est[1] *inter. F*/*post* et *add.* hec *R*/*post* illa *scr. et del.* et est *P1*/basi *mg. a. m. C1*/*post* basi *scr. et del.* linearum *C1*/trianguli *corr. ex* triangulum *O*/*post* trianguli *scr. et del.* linea tria *P1*/tabule enee *transp. C1*/enee *inter. a. m. S* 4 *post* unum *scr. et del.* speculum *O* 5 intra . . . extra *inter. a. m. E*/*post* extra *add.* duo piramidalia unum politum in facie aliud in concavitate *R*/duo . . . politum (6) *om. E* 6 superficie: specie *FP1S*; *corr. ex* specie *O*/duo . . . concavitate (7) *om. R* 8 fit: sit *R*/sit: fit *FP1O*/dyameter *corr. ex* diametre *S*; *corr. ex* diametri *C1*/longitudinis *om. R* 9 *post* digitorum *add.* in base columpne sumatur corda longitudinis trium digitorum *mg. a. m. E* 10 superficie: facie *O* 12 *post* columpne *scr. et del.* en *O* 13 in . . . digitorum (14) *scr. et del. E*; *ante* in *add.* similiter *E*; *post* columpne *add.* eiusdem opposita *E*; corda *om. E*/base: basi *R* 14 base: basi *R* 15 equalis huic *transp. R*/equalis huic corda: corda equalis huic *L3C1*/opposita: opposite *O* 16 sunt: sint *SOER*; *alter. in* sint *C1*/secetur: sececetur *F* 17 restet: restat *L3*; *corr. ex* restat *OC1E* 18 *post* capita *add.* sint *R*/earum aut *om. R*/aut: autem *O*/*post* altitudo *scr. et del.* basis *L3*; *add.* autem *R*/portionis remanentis *transp. R* 19 minus: minor *R*/*post* quam *add.* altitudo *R*/autem *om. O* 20 productum: productam *R* 21 *post* columpne *add.* autem *FP1*/sit *inter. a. m. E* 22 et[1] . . . digitorum *om. O*/ea: eo *FP1*

fiat sectio sicut in prima. Et erit altitudo axis partis remanentis minus quam dimidii digiti. Sit autem in hiis omnibus politura exquisita et equalitas omnimoda.

[3.26] Speculum piramidale queratur dyameter basis cuius quantitas sit sex digitorum, et corda trium, et longitudo piramidalis quatuor digitorum et dimidii. Et fiat sectio secundum lineas rectas, et axis portionis altitudo minor quam dimidii digiti, et hoc in unoquoque piramidali intellige.

[3.27] Speculum spericum sit portio sperica cuius dyameter sit sex digitorum, et dyameter basis huius speculi trium digitorum, et erit axis minus quam dimidii digiti. Item operare in speculo sperico concavo.

[3.28] Deinde facias septem regulas ligneas planas quarum latera equidistantia et ortogonalia super capita equidistantia in fine possibilitatis, et sit longitudo regularum sex digitorum, latitudo quatuor. Postea quadrato concavo adaptetur alia regularum ita ut ortogonaliter cadat super inferiorem concavi quadrati superficiem, et vide ut facile intret quadratum ne comprimens immutetur.

[3.29] Cadat igitur super faciem lateris regule acumen tabule enee, et ubi continuabitur ei fiat signum, et a puncto assignato producatur in extremitates regule linea equidistans lateribus regule ut sit linea illa linea longitudinis regule. Deinceps in longiori parte illius linee circa punctum sumptum sumatur altitudo medii grani ordei, et fiat punctum. Dico quod ille est punctus medius regule, qui etiam centro foraminum opponitur recte.

[3.30] Probatio: quoniam centra foraminum elongantur super superficiem tabule enee in medii grani quantitate, et distant a superficie anuli per duos digitos, igitur punctus ille distat ab eadem per duos

23 partis *inter. L3C1E* (*mg. a. m. C1*; *a. m. E*)/minus: minor *R* 24 *post* quam *add.* altitudo *R*/politura: politia *O*/*post* omnimoda *add.* in *R* 26 speculum piramidale: speculo piramidali *R* 27 piramidalis *om. R*/*post* quatuor *add.* sit *O* 29 *post* altitudo *scr. et del.* timor *F*; *add.* sit *R*/*post* quam *add.* altitudo *R*/hoc *inter. L3E* (*a. m. E*); hec *R*/in *om. E*/unoquoque: unoque *S*; utraque *L3C1ER* 30 intellige *corr. ex* intelligere *L3* 31 superius *O* 36 quatuor:7 *O* 37 concavo *om. R*/alia: aliqua *L3R*/*post* alia *scr. et del.* rum ita *F*/regularum *corr. ex* regularium *L3* 38 concavi quadrati *transp. O*/superficiem *corr. ex* super faciem *O* 39 comprimens: compressa *R*/immutetur: immittetur *E* 40 cadat *alter. in* cadet *O*/super faciem: superficiem *FP1*; superficie *O*/faciem *inter. a. m. E*/*post* faciem *scr. et del.* in *F* 41 ubi *inter. O*/a *om. C1*/assignato: signato *O* 42 *post* extremitates *scr. et del.* enee *P1*/regule[1] *alter. in* linee *a. m. E*/ut: et *FP1O*/*post* linea[2] *scr. et del.* longitudinis regu *F* 43 linea *om. O*/linea longitudinis: linealis *FP1*; *inter. L3*/longiori: longiore *R* 44 sumptum *om. C1* 45 ille *om. O*; illud *R*/est *om. FP1SO*; *inter. L3E* (*mg. L3*; *a. m. E*)/punctus medius: punctum medium *R*/*post* regule *inter.* puncto *O*/qui: quod *R*/etiam: in *O*/centro: centris *R* 46 opponitur *corr. ex* opposita *O*/recte: directe *O* 47 probatio *om. OL3C1ER*/quoniam: quam *FP1*/*post* quoniam *add.* enim *R* 48 distant: distans *O*/*post* superficie *add.* tabulee enee superficies (superficies *inter.*) *O* 49 punctus ille: punctum illud *R*

digitos. Et regula in quadrato concavo, per digitum unum. Quare ab extremitatibus regule ad punctum sunt tres digiti, quare punctus ille est medius. Super hunc medium punctum producatur in utramque partem linea secundum latitudinem equidistans extremitatibus. Et medietates linee longitudinis super quam hec est perpendicularis dividantur per equalia per lineas latitudinis perpendiculares extremitatibus equidistantes. Et ita divisa erit regula in quatuor equales partes. Similis fiat in aliis regulis operatio.

[3.31] Hiis completis, adaptetur speculum planum uni regularum. Et est ut sit regula cavata secundum altitudinem speculi ita ut superficies speculi sit in eadem superficie cum superficie regule, et ita ut medium superficiei speculi punctum directe supponatur medio superficiei regule puncto, et ita ut linea dividens superficiem regule in duo equalia dividat etiam superficiem speculi per equalia, et ut continuentur partes speculi cum linea dividente. Et observetur in possibilitatis fine.

[3.32] Deinde speculum columpnare politum in facie applicetur alicui regule ita ut medius eius punctus cadat super medium regule punctum, et ita ut linea in longitudine speculi sumpta dividens ipsum per equalia continuetur cum partibus linee longitudinis superficiei regule eque dividenti, et ut media longitudinis speculi linea sit in superficie regule. Et hoc sic fieri poterit utriusque basis speculi arcus per equalia dividantur et a puncto divisionis signato ad oppositum signatum linea producatur, et linee medie longitudinis regule aptetur et continuetur.

[3.33] Speculum columpnare concavum aptetur regule ut media longitudinis eius linea secundum equalem arcuum basium divisionem sumpta equidistans sit medie linee longitudinis regule, et etiam ut

50 et *inter. L3*/regula: reliqua *FP1*; ita *E*; *om. R*/*post* regula *inter.* est *L3*; *add.* est *C1*/ab: pro *E* 51 regule: tabule *O*/punctus . . . medius (52): punctum illud erit medium *R* 52 est: erit *E*/hunc: hoc *R*/medium punctum *transp. FP1* 53 secundum latitudinem *mg. a. m. E*; secundum altitudinem vel latitudinem *inter. L3*/latitudinem: altitudinem *O* 54 quam: quas *C1*/hec est *transp. C1ER*/perpendicularis *inter. a. m. O*; *corr. ex* perpendiculariter *L3* 55 perpendiculares *corr. ex* propter *a. m. L3* 56 *post* ita *scr. et del.* et ita *S*/in *inter. O* 58 completis: expletis *O*/planum *inter. a. m. E*/uni *inter. L3* 59 regula *om. O*/ita *corr. ex* prima *S*/ita . . . speculi (60) *inter. L3*; ita . . . superficie[1] (60) *mg. a. m. C1*/*post* ita *scr. et del.* s *F* 61 *post* speculi *add.* sit in eadem superficie speculi *C1* 62 regule[1] *mg. a. m. C1*/ut *om. E*/regule[2] *inter. a. m. E* 63 etiam: et *OL3E*/et *om. E*/continuentur: continuetur *P1*; *corr. ex* continentur *OL3* 64 *post* et *add.* hoc *R* 66 ita ut *corr. ex* ut ita *E*/medius: medium *R*/eius punctus *transp. L3C1R*/punctus: punctum *R*/*post* punctus *add.* eius *E* 67 punctum *om. C1*/ut *inter. L3*; *om. E* 68 continuetur *corr. ex* continetur *L3*/superficiei: superficiem *O* 69 regule *inter. a. m. E*/*post* regule *add.* per *O*/dividenti: dividentis *O* 70 sic *om. OL3*; sicut *E*/*post* poterit *add.* ut *O*; *inter.* sic *a. m. L3*; *inter.* ut *a. m. C1* 71 a: ex *O*/signato: signata *FP1* 72 *post* signatum *add.* punctum *R*/regule *om. R* 74 columpnare: columpnarem *S* 75 longitudinis *om. O*/arcuum *corr. ex* arcu *C1*/arcuum basium *transp. R* 76 *post* equidistans *add.* linee *E*/medie linee *transp. R*/linee *om. E*/*post* longitudinis *scr. et del.* linea *C1*

utriusque arcus corda cum lineis longitudinis extremis sint in superficie regule.

[3.34] Piramidale speculum extra politum applicetur regule ut acumen eius sit in termino medie longitudinis regule linee, et linea dividens portionem piramidis per equa—que scilicet a cono ad medium arcus basis punctum producitur—sit in superficie continuata cum parte restante medie linee longitudinis regule.

[3.35] Speculum piramidale concavum applicetur regule ita ut acumen eius sit in directo medie linee longitudinis regule; corda vero arcus basis sit in superficie, scilicet regule. Linea a cono ad medium arcus basis punctum ducta sit equidistans medio linee longitudinis regule. Cum autem longitudo piramidum sit quatuor digitorum et dimidius, restabunt ex longitudine regule digitus et medius.

[3.36] Adaptandum regule speculum spericum extra politum, fiat in regula circulus secundum quantitatem trium digitorum. Eius sit centrum medium regule punctum. Et cava et apta speculum ut medium superficiei eius punctum sit in superficie regule et in medio puncti medie linee longitudinis regule, quod quidem sciri poterit per applicationem alterius regule acute equalis huic in longitudine et divise per equalitatem et applicate medie linee longitudinis regule ita ut medius huius regule acute punctus tangat medium speculi sperici punctum.

[3.37] Spericum concavum, facto in regula circulo secundum quantitatem trium digitorum cuius centrum medius regule punctus, cavato circulo, imponatur ita ut circulus basis speculi sit in superficie regule, et punctum medium concavitatis speculi directe oppositum medio regule puncto. Et dyameter basis speculi continuetur medie linee regule, quod ita perpendetur. In regula acuta punctus signetur, et ab illo puncto

77 corda *inter. a. m. E*/lineis: linee *R*/sint: sunt *P1* 79 piramidale speculum *transp. L3C1*/*post* piramidale *scr. et del.* s *F* 80 *post* termino *add.* linee *R*/*post* longitudinis *scr. et del.* linee *C1*/ regule: eius *O*/linee *corr. ex* lignee *a. m. C1*; *om. R* 81 piramidis: piramidalis *OL3ER*/cono: vertice *R* 82 sit: fit *FS* 83 medie linee *transp. R* 84 piramidale *inter. L3*/acumen (85): arcuum *L3* 86 sit: fit *FP1S*/scilicet *om. L3C1R*/scilicet regule *om. O*; *scr. et del. E*/regule *inter. L3*/*post* regule *add.* et *R*/*post* linea *scr. et del.* sit *C1*/cono: vertice *R*/ad *inter. a. m. E* 87 punctum *corr. ex* puncta *O*/medio: medie *R* 88 autem longitudo *corr. ex* a longitudine *a. m. E*/ piramidum: pyramidis *R*/dimidius: dimidii *OL3C1ER* 89 medius: dimidius *R* 90 fiat *corr. ex* fifiat *C1* 91 regula: respectu *FP1*; *inter. a. m. E*/secundum: secundus *F*/trium *om. O*/ sit: fit *FP1S*/sit centrum (92) *transp. ER* 92 et cava *om. R*/apta: aptetur *R*/ut *mg. a. m. E* 93 puncti: puncto *OC1R* 94 *post* regule *add.* per *L3C1*; *scr. et del.* per *E*/quidem *om. O*/*post* quidem *add.* est *L3C1*/sciri poterit: scire poteris *O*/poterit *om. FP1* 95 *post* applicationem *scr. et del.* alterius *O* 96 medius (97): medium *ER* 97 acute *inter. a. m. E*/punctus *corr. ex* pres *O*; punctum *ER* 99 *post* concavum *add.* aptatur *R*/in regula: integro *O* 100 medius: medium *R*/punctus: punctum *R* 101 ut *om. S*/basis *om. L3C1*; *inter. a. m. E* 102 speculi: speculum *FP1*/*post* speculi *add.* sit *R*/*post* directe *add.* sit *S* 103 continuetur *corr. ex* continetur *L3*I/*post* regule *scri. et del.* o *F* 104 quod: que *R*/perpendetur *corr. ex* perpendeatur *O*/punctus: punctum *R*/signetur: significetur *L3C1*

longitudo semidyametri basis speculi notetur ex utraque parte. Et ita hec acuta regula medie linee regule applicetur ut punctum signatum in ea directe opponatur medio concavitatis speculi puncto et dyameter in ea factus similis sit cum basis dyametro.

[3.38] Hiis peractis, in semidyametro tabule enee triangulum per equalia dividente signetur ab acumine eius longitudo equalis axi huius speculi concavi, et fiat punctum. Axis autem sic dinoscitur. Regula acuta superficiei applicetur ut acuitas directe sit super mediam longitudinis lineam, puncto eius super medium concavi punctum directe statuto. Deinde acus recta et subtilis secundum illud regule acute punctum perpendiculariter cadat in speculum. Descendet quidem super medium concavi punctum. Signetur autem in acu punctum quod post eius descensum tangit acuitas regule sive punctum signatum, et sit modicum declinata regula ut certius possit fieri in acu signum. Postea secundum longitudinem acus a puncto signato in ea metire ab acumine tabule enee in linea triangulum dividente, et fac punctum.

[3.39] Deinceps hanc regulam facias intrare quadratum concavum ita ut acumen tabule enee descendat supra speculum; et adhibeatur regula acuta ut signetur punctum in linea dividente triangulum, quem tetigerit ex ea regula acuta, cum acumen trianguli descenderit usque ad superficiem speculi concavi. Signa igitur punctum.

[3.40] Erit autem hoc secundum punctum minus distans ab acumine quam primum, superficies enim tabule enee distat a superficie anuli sive tabule in qua est quadratum concavum per duos digitos minus medietate grani ordei. Punctus autem medius regule directe est oppositus medio speculi sperici concavi puncto, qui quidem distat ab

105 longitudo *om. O*/semidyametri *corr. ex* semedyametri *O* 106 *post* applicetur *add.* et *L3*; *scr. et del.* et *C1*/ut: et *O* 107 directe *corr. ex* directa *F* 108 factus: facta *R*/similis *corr. ex* simul *F*; *alter. in* simul *O*; simul *R*/sit *om. FP1S*; *inter. a. m. E* 109 tabule:tunc *P1*; *om. S*; *corr. ex* regule *OL3* 110 signetur: signatur *FP1*; significetur *L3*/eius: est *O* 111 sic *inter. a. m. E*/ dinoscitur: dignesserit *O*; dignoscitur *R* 112 *post* acuta *inter.* speculi *a. m. C1*/*post* superficiei *add.* speculi *R*/sit *om. FP1* 113 lineam: ferream *O*/super: similiter *FP1S*; supra *OL3C1*/*post* concavi *add.* speculi *R*/*post* punctum *scr. et del.* medium *F* 114 secundum *alter. in* super *C1*; *alter. ex* contra *in* super *E*; super *R*/illud: aliud *O* 115 in *mg. F* 116 concavi: concavum *FP1*/acu: actu *E* 117 post *corr. ex* potest *S*/eius: suum *R*/descensum: decessum *OL3*; recessum *E*/tangit: tangat *FP1SR*/acuitas: concavitas *ER*/*post* regule *add.* sue *C1*/sive *corr. ex* sue *L3E*/sive . . . signatum *om. R* 118 sit: sic *L3*/*post* ut *add.* cuius *FP1* 119 *post* signato *add.* et *O* 120 tabule: regule *OL3E*/enee *corr. ex* ?? *E* 122 *post* tabule *scr. et del.* tabule *F*/supra: super *L3C1* 123 linea: lineas *L3*/triangulum *om. O*/quem: *corr. ex* quoniam *S*; quam *O*; quod *R* 124 descenderit: descendet *C1* 125 superficiem *corr. ex* semidyametrum *L3*/speculi *inter. a. m. S*; *om. OL3C1ER*/*post* igitur *add.* erit hoc autem secundum *OL3C1* (hoc autem *transp. O*); *add.* hoc vero secundum *R* 126 erit . . . hoc *om. C1*/erit . . . punctum *om. FP1OL3R*/punctum *om. E* 127 *ante* quam *scr. et del.* minus *E*/enee *corr. ex* eene *F* 129 punctus: punctum *R*/medius: medium *R*/regule directe *corr. ex* recte regule *L3* 130 oppositus: oppositum *R*/*post* medio *add.* vel *FP1*/speculi *om. OL3C1E*/*post* speculi *scr. et del.* or *P1*/sperici *om. P1R*/qui: quod *R*

eadem superficie tabule per duos digitos. Cum ergo acumen tabule enee ortogonaliter descendat, non cadet super medium concavi, qui est terminus axis, sed in puncto altiori, quare propositum.

[3.41] Signetur vero in speculo concavo punctum in quod accidit acumen tabule enee, et extracto in puncto illo foramine ortogonaliter descendente et modico ad hanc quidem mensuram ut in eo descendat acutum donec acuitas regule adhibite contingat punctum linee dividentis triangulum primo signatum. Quod cum fuerit, erit quidem acumen tabule enee in eadem superficie cum termino axis speculi, que superficies sit equidistans superficiei regule. Et erit linea a termino axis ad acumen ducta perpendicularis super superficiem tabule enee. Axis autem speculi in eadem superficie cum centris foraminum, quoniam distantia eorum a superficie anuli duorum est digitorum, et medius terminus axis similiter.

[3.42] Hiis cum diligentia preparatis, poterit videri quod promisimus. Immitatur anulo regula super quam est speculum planum donec acumen tabule enee cadat super speculum, et infigatur regula quadrato concavo, et in eo subtus regulam aliquid opponatur quod ei firmitatem conserat ne vacillet. Deinde opponatur pargamenum foraminibus, et cum digito fiat impressio ut obturentur et impressionem percipere possis. Et signum foraminis fiat in pargameno cum incausto, vel aliquo alio. Unum autem foraminum relinquatur apertum declinatum non super regulam mediam, et adhibeatur radio solis foramen apertum. Certioratum autem erit huius rei comprehensio si adhibeatur radio solis per foramen domus intranti.

131 tabule[2] *corr. ex* regule *O* 132 enee *om. R*/*post* descendat *scr. et del.* descendat *S*/cadet: cadat *L3*; *alter. ex* cadat *in* cadit *E*/*post* concavi *add.* punctum *R*/qui: quod *R* 133 *post* est *scr. et del.* punctus *P1*/terminus: terminis *S*/puncto altiori: punctum altius *R*/*post* quare *add.* patet *R*/propositum: propo *F*; propter *P1*; proprium *O*; *corr. ex* proprium *L3*; liquet *C1* 134 signetur *corr. ex* servetur *L3*/vero: ergo *C1*/concavo *corr. ex* concavum *S*/quod *corr. ex* quo *C1*/accidit: accidat *OL3*; *corr. ex* accidat *C1*; incidit *R* 135 extracto: extra *FP1*; extracta *S*; *corr. ex* extracta *C1E*/*post* extracto *scr. et del.* regula *L3*/*post* illo *scr. et del.* cavent *L3* 137 acutum: acumen *C1ER*/regule: tabule *L3*; *corr. ex* tabule *a. m. C1*; *alter. ex* tabule *in* linee *a. m. E*; linee *R* 138 cum *inter. L3*; *inter. a. m. E*/fuerit: fuerint *E* 139 enee *inter. C1*; *inter. a. m. L3*/superficie: sic *P1* 140 sit: est *R*/equidistans: distans *FP1* 141 ducta perpendicularis: dicta perpendiculariter *P1* 142 *post* eadem *add.* est *O*; *add.* erit *R* 143 distantia: distantie *FP1*/a *inter. a. m. S* 144 medius *om. OC1R*/terminus *om. L3*; *inter. a. m. C1* 147 enee cadat *transp. S*/*post* et *add.* sit *R*/infigatur: infixa *FP1SR*; *corr. ex* infixa *a. m. E* 148 aliquid: aliquo *L3*/*post* aliquid *add.* deinde *C1*/opponatur: apponatur *C1ER*/ei *inter. O*/*post* ei *add.* conferat *R*/firmitatem: confirmitatem *FP1*; firmitudinem *O*/firmitatem conserat (149) *transp. E* 149 conserat *om. R*/opponatur: apponatur *C1*/pargamenum: pergamenum *R* 151 pargameno: parchameno *FP1*; pergameno *R*/*post* incausto *scr. et del.* longitudo *L3*/vel *inter. L3*/aliquo alio (152) *transp. C1* 152 alio; albo *O*; *inter. S*; *om. L3E*/relinquatur: relinquetur *FP1* 153 non *om. OL3*; *corr. ex* nec *E*/*ante* super *add.* mediam *E*/regulam mediam *transp. R*/*post* regulam *add.* non *OL3*; *inter.* non *a. m. C1*/mediam *om. E*/et *inter. O* 154 certioratum: certior *OR*; certiorata *E*/autem . . . intranti (155) *inter. a. m. E*; huius rei: huiusmodi *E*; *post* si *add.* hoc *E* 155 intranti: intrare *O*

[3.43] Cum igitur radius foramen intrans ad speculum pervenerit, videbis ipsum reflecti ad foramen illud respiciens super lineam tabule enee equalem angulum continentem cum linea triangulum per equa dividente et angulo quam tenet linea a foramine discooperto cum illo tabule semidyametro. Si vero foramen in quod fit reflexio discoopertum opponas radio priore cooperto, videbis reflecti radium in coopertum.

[3.44] Si vero foramini imponatur columpna ferrea concava, quam ad quantitatem foraminum fieri precipimus (ut firmius stet modicum cere circa eam apponatur), descendet lux per columpne concavitatem sicut descendit per foramen. Et reflectetur in foramen respiciens, et super lineas tabule enee erit descensus et reflexio pari modo, ut prius. Et si ad secundum foramen columpnam transtulerimus, in primum lucem reflexam videbimus. Erit autem debilior lux per columpnam descendens quam sine columpna per foramen. Erit autem videre eundem reflectendi modum in debiliori luce.

[3.45] Obturetur foramen cum cera ut modicum circa centrum eius restet vacuum, et videbitur lucis reflexio in foramine, sive circa centrum. Pari modo, si concavitatem columpne cum cera obturaveris ut remaneat quasi terminus solius axis, descendet lux super axem columpne et reflectetur ad centrum foraminis similis. Eodem modo, alterata columpna imposita, cum descenderit lux super axem unius foraminis, reflectetur super axem similis, centrum enim foraminis directe axi opponitur. Et cum lucis reflexio cadat in centrum nec moveatur nisi per lineam rectam, oportet ut procedat secundum axem.

[3.46] Obturatis autem foraminibus singulis preter medium quod directe super tabulam eneam incidit, fiat baculus columpnaris ad quantitatem foraminis, et extremitas eius acuatur ut remaneat solus terminus axis eius. Et descendat per foramen, et signa punctum speculi in quod ceciderit. Deinde descendat radius solis per foramen illud. Cadet quidem super punctum signatum, et circa ipsum efficiet circulum.

157 videbis: videbit *FP1SOL3*; ?? *C1* 158 continentem: continente *O*/*post* cum *scr. et del.* pe *C1* 159 et: ei *OL3C1ER*/angulo: triangulo *FP1*/quam: quem *R*/illo: eadem *R* 160 quod: quo *O* 161 opponas *corr. ex* opponis *a. m. C1*/videbis: videbit *OL3*/reflecti radium *transp. R* 162 concava *om. O* 163 foraminum *om. FP1*/precipimus: precepimus *SO*/*post* precipimus *add.* que *R*/firmius *corr. ex* firmus *C1* 164 cere *corr. ex* scere *L3*; cera *scr. et del. O*/*post* eam *inter.* cera *O*/apponatur: opponatur *FP1*/descendet: descendat *E* 165 respiciens: inspiciens *E*; sibi respondens *R* 166 erit *om. O* 170 debiliori: debiliore *R* 171 eius: ei *R* 172 videbitur *corr. ex* videtur *a. m. E*/sive: simili *L3C1R*; simili *inter. a. m. E* 173 *ante* pari *inter.* eius *a. m. E*/cum *om. O* 174 solius *corr. ex* solus *L3* 176 alterata: altera *R*/columpna imposita: columpne imposite *O*/unius . . . axem (177) *inter. L3* 177 reflectetur *corr. ex* reflectitur *E*/reflectetur . . . foraminis *mg. a. m. S* 180 quod: g *E* 181 incidit: incedit *C1* 183 descendat *corr. ex* descendet *E*/*post* foramen *add.* ad speculum *R*/signa: signetur *R*/*post* signa *add.* tabule *OE*; *scr. et del.* tabule enee *L3*/*post* punctum *add.* tabule enee *FP1S*; *add.* enee *OE*/speculum *om. R* 184 quod *corr. ex* quo *C1*/ceciderit: cecideret *L3* 185 super: superi *P1*/*post* super *add.* quod *L3*/efficiet *corr. ex* efficit *S*

[3.47] Signetur autem in fine huius lucis circularis punctum, et secundum quantitatem linee interiacentis puncta signata fiat circulus. Erit quidem circulus iste maior circulo foraminis, quoniam processus lucis per foramen ingredientis est in modum piramidis. Verum in nullo foraminum videbitur lucis reflexio, unde palam quod lux descendens per axem reflectitur super eundem. Verumptamen apparebit lux circularis circa basem interioris foraminis maioris quidem capacitatis radio, maioris etiam lucis interioris circulo.

[3.48] Et palam hanc lucem apparentem esse per reflexionem, verum non per reflexionem lucis super axem descendentis, quod ex hoc poterit patere. Obturata utraque foraminis base ut quasi sola remaneat axis via, et radio solis per viam axis descendente, non apparebit lux illa circularis circa inferiorem basem foraminis, quare non procedebat ex reflexa luce axis.

[3.49] Amplius, supra quamdam regulam supposuimus ut ortogonaliter caderet in quadratum concavum. Si aliquantulum ex eis auferatur ut regula declinetur ita ut extremitas a quadrato remotior sit dimissior radio descendente super foramen medium, non cadet perpendiculariter supra speculum, et apparebit lux reflexa a foramine medio remota. Et quanto maior erit declinatio maior erit lucis reflexe a foramine remotio. Si vero ad rectitudinem regula reducatur, lux reflexa circa inferiorem foraminis basem, ut prius, videbitur.

[3.50] Palam igitur quod luce super speculum perpendiculariter cadente, regreditur ad foramen per quod ingressa est. Cum vero lux axis declinata ceciderit, reflectitur non ad foramen, sed apparebit super lineam superficiei anuli perpendicularem super tabulam eneam et descendentem per centrum foraminis medii.

186 autem: igitur *R*/huius: eius *L3*/*post* huius *scr. et del.* s *C1* 187 signata: signa *L3*/fiat circulus *om. FP1S*/circulus: circulum *O*/erit . . . circulus (188) *inter. L3* 188 quidem: que *S*/*post* circulus *scr. et del.* punctum *S*/lucis *om. P1* 189 in[1]: per *ER*/verum . . . reflexio (190) *om. R* 191 *post* eundem *add.* axem *C1* 192 basem interioris: basim interiorem *R*/*post* capacitas *add.* luce incidente vel *R* 193 *post* radio *add. et R*/lucis *om. O*/*post* interioris *add.* lucis *R* 194 *post* palam *add.* est *R*/apparentem *om. R*/*post* per *scr. et del.* in *S*/reflexionem *corr. ex* inflexionem *a. m. E* 196 *ante* patere *scr. et del.* com *C1*/base: basi *R* 197 non *om. FP1*; *inter. L3*; *inter. a. m. E* 198 inferiorem basem: interiorem basim *R*/procedebat: precedebat *FP1S*; procedat *C1* 199 luce axis: lucis axe *R* 200 *post* amplius *add.* ut *ER*/quamdam: quidem *R*/regulam *om. ER*/*post* ut *add.* regula *R* 201 ex eis: inde *R* 203 dimissior: demissior *R*/descendente: descente *S*/cadet: cadat *OL3*; *corr. ex* cadat *E* 204 supra: super *ER* 205 *post* quanto *add.* erit *C1*/maior: maiorem *S*/declinatio: declaratio *FP1S*/*post* declinatio *add.* tanto *ER* 206 regula *om. S*; linea *OL3E*/*post* regula *add.* cum speculo *C1* 207 inferiorem: interiorem *R*/basem: basim *R* 208 *post* igitur *scr. et del.* luce *P1*/*post* luce *add.* quod *C1*/super: supra *E* 209 per . . . foramen (210) *rep. E* (axis[1] *inter. a. m.*; reflectitur non[1,2] *transp.*) 210 reflectitur non: non reflectetur *R*/*post* foramen *scr. et del.* per *P1*; *add.* per quod ingressa est *R*/*post* sed *add.* tamen *R*/*post* apparebit *add.* centrum lucis semper *R* 211 superficiei: superficie *O*/*post* superficiei *add.* concave *R*/perpendicularem: perpendiculariter *E*

[3.51] Quecumque autem dicta sunt in duobus foraminibus primis declinatis intellige in singulis. Et quod dictum est in speculo plano, luce per foramen declinatum seu medium descendente, regula recta seu declinata, in aliis speculis intellige.

[3.52] Si autem regula in qua fuerit speculum columpnare extra politum declinetur in quadrato ita ut non ortogonaliter cadat super quadratum sed declinetur super partem dextram vel sinistram, apparebit tamen lux reflecti super foramen simile eius descensui, et medium lucis super medium foraminis, sicut visum est regula non declinata.

[3.53] Regulam in quam situm est columpnare concavum impones, et descendat acumen tabule enee donec tangat superficiem speculi, et declinabis hoc speculum secundum latus suum sicut declinasti extra politum.

[3.54] Idem in speculis piramidalibus concavis operaberis.

[3.55] Spericum concavum aptetur donec descendat acumen tabule enee in foramen speculi factum secundum acuminis descensum.

[3.56] Spericum extra politum sic imponatur ut acumen tabule enee sit in superficie regule et in eadem superficie cum medio speculi puncto, quod sic fieri poterit. Adhibeatur regula acuta regule et puncto speculi medio, et descendat acumen tabule enee quousque sit in directo acuitatis regule. Et tunc cogatur sistere.

[3.57] In speculis columpnaribus videbis reflexionem hoc modo. Aptetur speculum, sicut dictum est, et per foramen medium descendat baculus columpnaris, sicut factum est in speculis planis. Cadet quidem baculus super mediam longitudinis speculi lineam, et erit eius terminus in superficie regule. Super mediam lineam signetur punctum in quod cadit, et ab hoc puncto in superficie regule sumatur longitudo semidyametri circuli facti in regula ad discernendum circularem lucis casum. Et ex alia parte puncti sumatur longitudo eadem, et habebitur linea equalis dyametro predicti circuli. Videbitur autem lux cadens

213 dicta sunt *transp. FP1S* 214 in[1]: de *C1*/et *om. S*/dictum est *L3C1*/*post* plano *add.* de *R* 215 *post* foramen *add.* seu *R*/seu: sive *FP1*/regula . . . declinata (216) *inter. a. m. E*/*post* regula *add.* seu *R* 216 seu *om. L3*/declinata: declanata *R* 218 politum: positum *O*; *corr. ex* punctum *a. m. C1*/non *om. O*/cadat super: cadet supra *FP1* 219 super: secundum *O*/dextram *corr. ex* destram *S*; *corr. ex* dexteram *O* 220 reflecti: reflexa *O* 221 *post* est *inter.* in *E*; *add.* in *R* 223 regulam: regula *C1*/quam: qua *L3C1R* 224 et: ut *R*/descendat *alter. in* descendet *E* 225 declinabis: declinabit *FP1OL3E*; *corr. ex* declinabit *C1*/declinasti: declinati *O* 227 idem: item *L3*/operaberis *om. P1* 230 enee *om. FP1* 231 regule *inter. a. m. E*/*post* regule *add.* enee *E*/et: est *O* 233 acuitatis *corr. ex* acuitas *O* 234 regule *corr. ex* regula *E* 235 videbis: videbit *FP1OL3* 236 *post* aptetur *add.* hoc *P1* 237 factum *inter. E* 238 longitudinis *corr. ex* lineis *P1*/speculi *inter. a. m. E*/et . . . lineam (239) *mg. a. m. E* 239 *post* mediam *scr. et del.* longitu *P1*; *add.* igitur *R* 241 semidyametri *corr. ex* diametri *a. m. C1*/circuli *corr. ex* circulis *L3* 242 parte *inter. O*/et[2] *inter. a. m. E* 243 equalis *inter. OL3*; *inter. a. m. E*/dyametro: semidyametro *O*/predicti: predicto *FP1SL3C1*

extendi supra predictam lineam tantum, et reflectitur ad foramen medium. Et circa eius basem inferiorem videbitur lux circularis maior circulo inferiori, sicut in speculis planis visum est.

[3.58] Idem in speculis piramidalibus videre poteris.

[3.59] Pari modo in speculis spericis, luce per foramen medium descendente, fiat circulus in superficie regule ad quantitatem circuli iam dicti. Et videbitur lux extendi super hunc circulum et reflecti ad foramen medium modo iam dicto. Et apparebit in hiis omnibus rectis reflexionibus linea perpendicularis in interiori superficie anuli secare lucem circularem reflexam et dividere circulum eius per medium.

[3.60] Quod dictum est de luce naturali videri poterit in luce accidentali. Domus unici foraminis opponatur parieti in quam descendit solis radius, et applicetur instrumentum foramini cum intraverit lux accidentalis per foramen non medium. Videbitur reflecti per eius oppositum, et si aptetur instrumentum ut intret per duo foramina, reflectetur per duo similia.

[3.61] Verum ut possis perpendere lucem cum intraverit directe et ad ipsam transierit, appone superius pargamenum album, et inclina instrumentum donec videas lucem cadentem super pargamenum. In speculis etenim non plene comprehenditur lucis accidentalis casus propter debilitatem eius. Idem autem in hac luce patebit quod in naturali patuit, non enim est diversitas in earum natura nisi quod una fortis et alia debilis.

[3.62] Palam ergo quod luces propter diversas lineas ad specula accidentes per diversas reflectuntur lineas. Et si eadem parte ad speculum venerit, in eandem gradiuntur partem, et declinatio linearum reflexionis equalis declinationi erit linearum accessus. Et planum quod linee lucis reflexe et advenientis sunt in eadem superficie ortogonaliter

244 supra: super *R*/reflectitur: reflectetur *OR* 245 basem inferiorem: basim interiorem *R* 246 inferiori: interiori *R*/*post* in *add.* speris *L3*/speculis *corr. ex* speris *a. m. O*; *inter. L3*; speris *E* 247 speculis: speris *FSL3*; *corr. ex* speris *a. m. O*; spericis *P1*/*ante* piramidalibus *inter.* speculis *L3* 248 *post* in *add.* speris *L3*/speculis *corr. ex* speris *O*; *inter. L3* 252 in *inter. S*/interiori: interiore *R* 253 et *om. SOL3C1E* 254 *post* quod *add.* autem *R*/de: in *O*/videri: videre *L3C1*/poterit: poteris *C1* 255 parieti *om. C1*/quam: qua *L3*; quem *R*/descendit: descendat *OL3C1* 256 solis radius *transp. R*/*post* cum *add.* ergo *R* 258 oppositum *corr. ex* oppositis *P1*; *corr. ex* oppositionem *a. m. E* 260 verum *om. S*/et . . . transierit (261) *om. R* 261 pargamenum: pergamenum *R*/inclina: declina *E* 262 cadentem: carentem *FP1SC1*/pargamenum: pergamenum *R* 263 speculis *corr. ex* peris *L3*; *corr. ex* speris *SE* (*a. m. S*)/etenim: enim *R*/plene *corr. ex* bene *a. m. E* 264 autem: aut *P1*/*post* quod *scr. et del.* qi *P1* 265 *post* quod *scr. et del.* in *E* 266 *post* fortis *add.* est *R* 267 propter: per *C1R*/ad *om. C1*/specula: speras *FOL3E*; *alter. ex* speram *in* speras *P1*/*corr. ex* speras *a. m. S*; speculi *C1* 268 accidentes: accedentes *C1R*/per: in *inter. L3*/et si *inter. a. m. E*/si: ab *C1*; *corr. ex* sub *OL3* (*a. m. O*)/si . . . partem (269) *om. R*/*post* si *add.* sub *E* 269 *ante* et *add.* quod secundum rectam perpendicularem incidentes secundum eandem regrediuntur *R*/et . . . equalis (270) *inter. a. m. E*/*post* et *add.* quod *R* 270 *post* reflexionis *add.* est *R*/*post* equalis *add.* est *C1E*/erit *om. OL3C1ER* 271 ortogonaliter: ortogonali *R*

super superficiem politi et contingenti punctum a quo fit reflexio. Et si super perpendicularem venerit, reflectetur super perpendicularem, et in quemcumque punctum cadit reflectitur in superficie perpendiculari super superficiem tangentem illud punctum.

[3.63] Et semper linea reflexa cum perpendiculari super illud punctum equalem tenet angulum angulo quem includit linea veniens cum eadem perpendiculari. Et huius rei probatio est quia palam quod, si descendat lux quecumque per foramen aliquod, reflectitur per ipsum respiciens. Et si constringatur foramen ut restet quasi solus axis, reflectitur per axem respicientis foraminis. Et si fiat alteratio descensus lucis, reflectitur lux per lineas per quas prius descenderat. Et palam quod foramina se respicientia eundem habent situm respectu medii, et cum non procedat lux nisi per lineas rectas, planum quod reflectitur per lineas eiusdem situs respectu medii cum lineis descensus.

[3.64] Unde cum accidit per ortogonale, per eam reflectitur solam, quare semper linee reflexionis eundem servant situm cum lineis descensus respectu superficiei contingentis punctum reflexionis. Et hoc substantiale sive in substantiali sive in accidentali luce, sive forti sive debili, et generaliter in omni.

[3.65] Et nos ostendemus ydemptitatem situs. Iam scimus quod superficies regule cadit super tabulam in qua quadratum fecimus ortogonaliter. Igitur linea media tabule ortogonaliter est super lineam communem ei et regule, et est super lineam latitudinis regule. Et tabula equidistans enee tabule, et linea eius media equidistans linee medie tabule enee, et est linee a centro tabule enee producte et dividentis arcum per equalia.

272 super *corr. ex* et *L3*/*post* et[1] *add.* aut superficiem contingentem *R*/*post* punctum *add.* politi *R*/*post* si *add.* lux *R* 273 venerit . . . perpendicularem *om. O*/reflectetur: reflectitur *ER* 274 quemcumque: quodcumque *OC1R*/cadit *corr. ex* cadat *O*; ceciderit *R*/reflectitur: reflectuntur *FP1SL3C1* 276 et *inter. O*/semper . . . punctum (277) *om. O* 278 eadem perpendiculari *transp. L3C1*/rei *om. FP1*/est *om. FP1*/quia: quoniam *O*/si *inter. O* 279 *post* per[2] *add.* aliud *R* 280 si *om. FP1SOE*; *inter. L3*/solus: solis *O*; *corr. ex* solis *L3* 281 respicientis: respiciens *O*; *corr. ex* respiciens *S*/foraminis *om. OL3C1ER*/si *inter. a. m. E* 282 lux *om. ER*/*post* per *scr. et del.* per *F*/descenderat: descendebat *S* 283 se *om. FP1SL3*; *inter. a. m. E*/*post* respicienia *inter.* se *L3*/habent *inter. a. m. E*/*post* medii *add.* cum lineis *C1* 283 et . . . medii (285) *mg. a. m. L3*; *inter. a. m. E*/lineas rectas *transp. OER*/planum: palam *ER*/*post* quod *add.* non *E*/reflectitur: reflecti *S* 285 eiusdem *inter. E*/*post* medii *add.* foraminis *ER*/*post* lineis *scr. et del.* respectu *C1* 286 *post* unde *scr. et del.* cum *C1*/accidit: accedit *SC1R*/ortogonale: ortogonalem *SC1R*/per eam reflectitur: reflectitur per eam *O*/reflectitur *inter. L3*; *inter. a. m. E* 287 *post* quare *add.* linee *C1* 288 *post* hoc *add.* est *C1* 289 substaniale: verum est *R*/in[1] *om. FP1* 291 ostendemus: ostendimus *O* 293 *post* tabule *inter.* enee *L3*; *add.* quadrati ortogonalis *R* 294 communem *corr. ex* que *O*/communem . . . lineam *inter. a. m. E*/*post* communem *add.* sectioni *E*; *add.* sectionis ipsius *R*/ei *om. R*/*post* et[2] *add.* illa res *C1*/est *om. R*/*post* regule[2] *scr. et del.* et est linea acutus tabule enee *C1*/tabula *corr. ex* regula *L3E* (*a. m. E*)/*post* tabula *inter.* est *a. m. C1* 295 equidistans: quadrati equidistat *R*/enee tabule *transp. L3C1*/et . . . linee (296) *inter. a. m. E*/*post* eius *add.* id est tabule quadrate *E*; *add.* id est tabule quadrate concave *R*/*post* media *add.* est *C1*/equidistans: equidistat *R* 296 et[1]: que *R*/et . . . enee[2] *om. C1*/linee: linea *R*/*post* linee *add.* enee medie *E*/producte: producta *R*/dividentis: dividens *R*/arcum: semicirculum *R*

[3.66] Linea autem communis tabule enee et regule, que est linea latitudinis regule, est equidistans linee communi tabule et regule, quare linea media tabule enee cadit perpendiculariter super lineam communem regule et tabule enee. Et regula perpendicularis est super superficiem quadrati, et superficies quadrati equidistans superficiei tabule, quare superficies tabule ortogonaliter super superficiem regule.

[3.67] Et similiter superficies tabule enee ortogonaliter super eandem, et linea media longitudinis regule est perpendicularis super latitudinem eius, quare linea media tabule erit perpendicularis super mediam longitudinis regule lineam, cum cadit super eam; et similiter linea media tabule enee erit perpendicularis super eandem. Et ita media linea tabule enee est perpendicularis super superficiem regule et super mediam longitudinis eius lineam, et ita est perpendicularis super superficiem speculi plani et super mediam longitudinis eius lineam.

[3.68] Amplius, superficies tabule enee est equidistans superficiei descendenti per centra foraminum, quoniam longitudo centrorum a superficie tabule enee eadem—id est medietatis unius grani ordei—et dyameter foraminis est unius grani ordei. Similiter latitudo superficiei columpne est unius grani, que superficies descendens per centra foraminum secat columpnam per medium. Et ita axis columpne est in superficie illa, et columpna descensu suo tangit lineam in tabula enea cui quidem equidistat axis, quoniam axis est equidistans cuilibet linee superficiei columpne.

[3.69] Et axis columpne cadit in punctum superficiei regule, a quo puncto linea ducta ad centrum tabule enee est perpendicularis super

298 *post* communis *add.* superficiei *R*/*post* enee *add.* superficiei *R*/*post* regule *add.* in *R*/que: qua *R*/est *inter. a. m. E* 299 regule[1] *om. C1ER*/*post* linee *scr. et del.* com *S*/*post* communi *add.* concave *R*/regule[2] *corr. ex* linee *L3* 300 cadit: cadet *ER*/perpendiculariter *corr. ex* perpendicularis *O* 1 et regula *mg. L3* 2 et superficies quadrati *mg. L3*/*ante* equidistans *add.* est *C1* 3 *post* tabule[1] *inter.* enee *a. m. S*; *add.* enee *R*/*post* tabule[2] *inter.* est *L3*; *scr. et del.* enee *C1*; *add.* enee *R*/ortogonaliter: ortogonalis *R*/*post* ortogonaliter *inter.* est *O*; *add.* est *C1R*/regule: tabule *L3*/*post* regule *inter.* ortogonaliter *a. m. E* 4 et . . . eandem (5) *om. R*/similiter *corr. ex* super *C1*/*post* enee *inter.* est *a. m. C1* 5 longitudinis *inter. a. m. E*; latitudinis *R*/*post* longitudinis *add.* tabule *O*; *scr. et del.* tabule *L3*/est: erit *E*/perpendicularis *alter. in* perpendiculariter *a. m. E* 6 latitudinem . . . super *inter. a. m. E*; *om. R* 7 *post* longitudinis *add.* tabule vel *L3*; *scr. et del.* eius *C1*/cum . . . eam *om. R* 8 erit: est *L3C1R*/perpendicularis *alter. in* perpendiculariter *a. m. E*/ super . . . perpendicularis (9) *mg. SL3* (*a. m. S*)/perpendicularis *alter. in* perpendiculariter *a. m. E* 10 longitudinis *alter. in* eius *a. m. E*/eius *om. E*/et *om. R*/ita *om. E*/ita est *transp. FP1*; est ergo *R* 12 est *om. S* 13 descendenti *mg. F*/quoniam: nam *R*/*post* a *add.* et *FS* 14 *post* enee *inter.* est *a. m. C1*; *add.* est *R*/*post* eadem *inter.* est *a. m. L3*/medietatis: medietatem *FP1S*/ordei *corr. ex* ei *a. m. C1*/et . . . ordei (15) *inter. a. m. E* 15 ordei *om. O* 16 est *corr. ex* sibi *L3*/*post* est *add.* igitur *L3*/que: et *R* 17 medium: mediam *L3* 18 descensu *corr. ex* descensuo *P1*/*post* descensu *add.* illo *E* 19 est *om. O*/linee *corr. ex* ille *a. m. C1* 21 et axis columpne *om. O*/ *post* axis *scr. et del.* et *P1*/punctum superficiei *transp. L3C1* 22 perpendicularis *alter. in* perpendiculariter *a. m. E*

tabulam eneam, quoniam, per quodcumque foramen descendat columpna, axis eius cadit super mediam longitudinis regule lineam, et omnes ille perpendiculares sunt equales.

[3.70] Et linea protracta a puncto regule in quem cadit axis per centrum foraminum est equidistans linee protracte a centro tabule enee ad terminum dyametri foraminis. Quoniam linea inter punctum illud et centrum est ortogonaliter super superficiem tabule enee, cum sit pars linee medie longitudinis regule, et etiam super axem. Et huic linee interiacenti centrum tabule enee et punctum est equidistans linea anuli transiens per centra foraminum et perpendiculariter cadens in superficiem tabule enee, quare equidistantes erunt linee cadentes in terminos linee anuli et longitudinis regule equalium et equidistantium.

[3.71] Pari modo in singulis foraminibus, quare linee a puncto regule in quem cadit axis producte ad centra duorum foraminum se respicientium sunt equidistantes duabus lineis a centro tabule enee ad extremitates dyametrorum eorumdem foraminum protractis, quare hee due linee equalem tenent angulum cum illis lineis.

[3.72] Et si a termino axis erigatur linea ad centrum foraminis, erit in superficie per centra descendente, et erit equidistans medie linee tabule enee. Quoniam linea inferior interiacens capita eorum est perpendicularis super tabulam eneam et equalis superiori eadem capita interiacenti et super tabulam eneam perpendiculari. Et est equidistans ei, quare linea a centro foraminis medii ad terminum axis columpne est equidistans medie linee tabule enee, et illa est perpendicularis super regulam, quare et ista. Igitur hec linea et latera alterum angulum continentia sunt equidistantes medie linee tabule enee et alteri linearum

23 quodcumque: quocumque *O* 24 columpna *corr. ex* columpnat *E*/cadit: cadat *C1*/*post* super *add.* lineam *C1*/regule *corr. ex* eius *L3C1* (*a. m. C1*)/lineam *om. C1*/et . . . equales (25) *om. R* 26 *post* regule *add.* enee *O*/quem: quam *FP1O*; quod *R*/per . . . foraminum (27) *om. O*/centrum (27): centra *R* 28 quoniam linea *mg. F*/et *inter. P1* 29 ortogonaliter: ortogonalis *R*/super: in *E*/superficiem *corr. ex* superficie *E* 30 *post* medie *scr. et del.* foraminis quoniam linea *O*/et[1]: est *E*/etiam . . . et[2] *om. R* 31 *post* centrum *inter.* et *O*/et *om. O*/punctum *corr. ex* ipsum *a. m. C1* 32 centra: centrum *E*/in: super *R* 33 superficiem: superficie *E*/*post* quare *scr. et del.* d *F*/in . . . equidistantium (34) *om. R* 34 anuli: anguli *FP1SOL3E*/anuli . . . regule *mg. a. m. C1*/longitudinis *corr. ex* longitudinibus *P1*/*post* regule *scr. et del.* et anguli *C1*/*post* equidistantium *add.* sunt equales *C1*; *add.* a puncto regule ad centra foraminum lineis a tabule enee centro ad terminos diametrorum eorundem foraminum in superficie tabule ductis *R* 35 *post* puncto *scr. et del.* in *F*; *scr. et del.* regula *P1* 36 quem: quod *R*/ad: et *O*/centra *corr. ex* centrum *E*; centrum *R*/respicientium (37) *corr. ex* respicientes *L3* 37 *ante* sunt *add.* tantum *OL3*/sunt *om. FP1SR*/duabus: duobus *S* 38 quare . . . linee (39) *om. R* 39 tenent . . . lineis: cum his lineis tenent angulum *R* 40 si *inter. a. m. E*/a *inter. C1* 41 centra: centrum *R*/descendente: descendentia *O* 42 linea *om. P1*/eorum: earum *R* 43 *post* super *scr. et del.* capi *F*/tabulam *corr. ex* capita *F*/equalis: equali *FS*/*post* superior *add.* et *O* 44 perpendiculari *corr. ex* perpendiculariter *P1* 45 *post* ei *add.* et *R*/quare *inter. a. m. E*; similiter *R*/*post* foraminis *scr. et del.* et *C1* 46 illa est *transp. ER* 47 *ante* regulam *scr. et del.* lineam *E*/latera: laterum *E*; altera *R*/latera alterum *transp. OL3E*/alterum *om. R* 48 continentia sunt equidistantes: continentes equidistant *R*/alteri: alterus *O*; alterum *C1*/linearum: linee *R*

in tabula enea angulum continentium, quare quasi partiales sibi oppositi sunt equales.

[3.73] Igitur linea media tabule enee dividit angulum suum per equalia, quare linea a centro foraminis medii dividit angulum suum per equalia. Et cum certum sit quod lux foramen declinatum intrans per illas lineas angulum continentes moveatur, planum quod lux omnis reflectitur per lineas que cum lineis descensus sunt in eadem superficie ortogonali super superficiem reflexionis et angulum equalem facientibus cum perpendiculari cum lineis descensus.

[3.74] Et lux perpendiculariter descendens reflectitur per perpendicularem. Et hoc generale in omni luce.

[3.75] Si autem declinetur regula non in latus suum sed in caput ut axis foraminis medii non sit perpendicularis super regulam, reflectitur lux, et videbitur super lineam altitudinis anuli perpendicularem et per centrum foraminis transeuntem. Et quanto maior fuerit declinatio, maior erit lucis reflexe a foramine vel axe elongatio. Et si diminuatur declinatio, diminuetur elongatio, et ita donec situs regule ad rectitudinem regrediatur, super perpendicularem illam reflectitur lux.

[3.76] Quod autem in hac declinatione axis foraminis medii et linea reflexionis sunt in eadem superficie ortogonali super superficiem reflexionis planum per hoc quoniam axis foraminis medii est perpendicularis super latitudinem regule—id est super lineam communem superficiei regule et superficiei per centra foraminum descendentis—et media linea tabule anuli est equidistans huic axi et equidistans medie linee tabule enee.

[3.77] Et media linea tabule enee est perpendicularis super latitudinem regule, et est super lineam communem superficiei regule

49 *post* enea *add.* reliquum *R* / *post* angulum *inter.* reliquum *L3* / continentium: continenti *R* / quasi: anguli *R* / partiales: percipiales *FP1* / oppositi *corr. ex* opposi *E* 51 linea *om. F* / linea media *transp. P1OC1* / media tabule enee: tabule enee media *R* / *post* dividit *scr. et del.* angulum *F* 52 quare . . . equalia (53) *inter. a. m. L3* / angulum suum *transp. L3C1* 53 per: super *FP1* / *post* lux *add.* per *P1* 54 illas lineas *transp. O* / lux omnis *transp. L3C1* 55 eadem *om. R* 56 super *om. O*; *inter. L3* 57 facientibus: facientes *R* / *post* cum[1] *add.* linea *R* / *post* perpendiculari *add.* angulo quem continet perpendicularis *R* / cum: tamen *F*; *alter. in* et *O*; *corr. ex* et *L3* 58 *post* et *add.* quod *R* / descendens: descensus *FP1* / per *om. FP1*; *inter. OL3* 59 generale: generaliter *E* / *post* generale *add.* est *FP1R* 60 si autem *corr. ex* suam *O* / *post* regula *add.* cum *L3* / caput: capite *E* 61 medii: medium *FP1S* / reflectitur: reflectatur *S*; reflectetur *OR*; *alter. ex* reflectatur *in* reflectetur *C1* 63 transeuntem: transeuntes *O* 64 *ante* maior *add.* tanto *ER* / *post* diminuatur *scr. et del.* elo *O* 65 diminuetur elongatio *mg. L3* / *post* elongatio *add.* lucis a foramine *C1* 66 regrediatur: egrediatur *O*; aggrediatur *L3*; *corr. ex* egrediatur *a. m. E* / *post* regrediatur *add.* et *FP1R* / super *corr. ex* per *a. m. S*; *inter. L3E* (*a. m. E*) / reflectitur: reflectatur *R* 67 *post* in *add.* declinatio *E* / declinatione: declaratione *FP1*; *corr. ex* declaratione *L3* 68 *post* reflexionis *scr. et del.* in *F* / sunt: sit *P1*; sint *R* 69 *post* planum *add.* est *C1* / *post* quoniam *add.* enim *R* 70 super[2] *inter. O* 71 superficiei[2] *inter. a. m. E* 72 *post* tabule *inter.* scilicet *a. m. E*; *add.* scilicet *R* / anuli *inter. a. m. E* 73 *ante* equidistans *add.* est *C1* / equidistans *corr. ex* quidem *L3* 75 est *om. R*

et superficiei tabule enee, quare superficies in qua sunt media linea tabule enee et axis foraminis medii ortogonalis est super superficiem regule. Et in hac superficie est linea perpendicularis in altitudine anuli, quoniam transit per terminos equidistantium—scilicet medie tabule enee et axis foraminis medii.

[3.78] Palam igitur quod lux reflexa que apparet in perpendiculari altitudinis anuli reflectitur per lineam que cum axe per quem fit descensus est in superficie ortogonali super superficiem regule. Luce ergo descendente in speculum planum, fit reflexio secundum lineas quarum eadem declinatio super superficiem speculi, et ipse cum perpendiculari in superficie ortogonali super superficiem speculi.

[3.79] In speculo columpnari exteriori eadem penitus probatio que est in plano—scilicet quod acumen tabule enee cadit super lineam longitudinis speculi ortogonalis, et similiter columpna descendens super eandem. Et pars illius linee super hos casus est ortogonaliter super tabulam eneam. Et semper, sive per foramen medium sive per declinatum descendet lux, reflexio eius cum descensu erit in eadem superficie ortogonali super superficiem contingentem lineam longitudinis speculi.

[3.80] In piramidali vero exteriori, cum superficies regule sit in eadem superficie cum linea media longitudinis piramidalis, sicut in columpnari, erit idem situs linearum superficierum et idem reflexionis modus, sicut in speculo plano, et eadem penitus probatio.

[3.81] In speculo autem columpnari concavo descendit acumen tabule enee usque ad lineam longitudinis eius mediam, et super eandem cadit axis cuiusque foraminis. Et linea pars illius inter hos casus est ortogonalis super superficiem tabule enee, et axis foraminis et media linee tabule enee sunt ortogonales super superficiem tangentem speculum illud in linea longitudinis, que est locus reflexionis, et equidistantes superficiei regule.

76 enee *om. FP1*/*post* linea *scr. et del.* regule et *C1* 77 enee *inter. a. m. E*/*post* medii *add.* et *E*; *add.* etiam *R*/est *om. SOE*; *inter. L3* 78 et *om. S*/*post* hac *scr. et del.* in *E*/est *om. FP1S* 81 apparet *corr. ex* apparent *S* 82 *post* anuli *scr. et del.* e *C1*/reflectitur: reflectetur *S*/fit: sit *L3*/descensus (83): de census *R* 83 est *inter. a. m. C1*/ergo: igitur *FP1* 84 descendente: descente *O*/fit: sit *SL3* 85 *post* ipse *add.* sunt *C1ER*/cum *inter. a. m. L3* 86 superficiem speculi *transp. ER* 88 quod *inter. a. m. E*/cadit: cadat *R* 89 ortogonalis: ortogonali *O*; ortogonaliter *C1ER* 90 eandem: eadem *O*/super[1] *corr. ex* per *L3C1* (*a. m. C1*); inter *R*/ortogonaliter: ortogonalis *R* 91 sive[1] *corr. ex* fuit *L3*; *mg. a. m. C1*/medium: mediam *FP1S* 92 descendet *corr. ex* descendit *a. m. C1*; descenderit *R*/*post* descensu *scr. et del.* erit *E*/superficie . . . super (93) *mg. O* 93 *ante* superficiem *scr. et del.* superficiebus *O* 94 vero *om. O*/superficies: superficie *FP1SL3*/*post* regule *scr. et del.* regule *O* 95 media *om. FP1SR*/piramidalis: pyramidis *R*/sicut: sit *FP1* 96 *post* idem[1] *scr. et del.* ?? *E*/superficierum: superficiei *L3ER* 97 speculo plano *transp. R* 98 autem *om. OL3C1ER*/descendit *corr. ex* descendet *O*; *alter. in* descendet *a. m. C1* 100 linea pars illius: pars illius linea *R* 102 linee: linea *L3C1ER*/super *om. FP1* 103 et *om. C1*/ equidistantes: equidistantem *OER*; *alter. in* equidistantem *C1*

[3.82] Et ita idem modus probandi qui prius—quod scilicet reflexio et descensus sunt in eadem superficie ortogonali super superficiem loci reflexionis, et eiusdem sunt declinationis, et quod descensus per medium efficit reflexionem per ipsum. Et declinato capite regule, erit reflexio super perpendicularem anuli, sicut dictum est in plano.

[3.83] In speculo piramidali concavo eadem in omnibus probatio.

[3.84] In speculo sperico exteriori palam quod medius eius punctus est in superficie regule, et axis cadit in punctum illud, et erit in eo idem situs linearum. Et aliorum penitus quod in plano, et eadem demonstratio.

[3.85] In speculo sperico concavo iam determinatum est quod axis foraminis descendit ad punctum eius medium, et acumen tabule enee transit per foramen in speculo iam factum usque dum sit in eadem superficie cum puncto illo medio. Et linea a puncto illo ad acumen protracta est equidistans medie linee longitudinis regule, et ita descensus et reflexio sunt in superficie ortogonali super superficiem contingentem speculum in illo puncto medio et equidistantem superficiei regule. Et eadem probatio penitus ut in aliis.

[3.86] Palam ergo quod omnis lux in quodcumque speculum eorum cadit reflexio et descensus sunt in eadem superficie ortogonali. Hic autem modus reflexionis non accidit ex proprietate axis, vel puncti in quod cadit, vel foraminis per quod intrat, vel proprietates speculi. Accidit enim in quodlibet foramen quecumque sit lux, et per quamcumque lineam descendat, et in quodcumque speculi punctum cadat. Quoniam quocumque puncto speculi sumpto, si lux in ipsum descendat, cum idem sit ei situs respectu longitudinis speculi, et cuicumque alii erunt similiter idem respectu linearum ab eo

105 *post* probandi *add.* hic *C1* 106 sunt: fit *O*; sint *R*/ortogonali: ortogonaliter *E*/ortogonali . . . superficiem *om. O*/super *om. P1* 107 *ante* eiusdem *add.* quod *R*/sunt: sint *R*/descensus *corr. ex* descendsus *F* 108 reflexionem: rationem *O* 109 perpendicularem: perpendiculares *OC1* 110 *post* speculo *scr. et del.* sperico *S* 111 medius: medium *R*/punctus: punctum *OR* 112 erit in eo: in eo erit (in eo *inter.*) *L3* 113 quod: que *S*; qui *ER*/*ante* in *scr. et del.* in superficie regule *S* 115 speculo sperico *transp. E*/determinatum: declaratum *R* 116 descendit: descendet *E*; descendat *R* 117 transit: transeat *R*/eadem *corr. ex* edem *O* 118 superficie: specie *O*; *corr. ex* specie *L3*/puncto² *corr. ex* predicto *O* 119 medie *corr. ex* medium *F* 120 *post* in *add.* eadem *R* 121 *post* contingentem *scr. et del.* secundum *O*; *scr. et del.* per *C1*; *add.* per *L3*/illo puncto *transp. O*/medio *inter. a. m. E*/equidistantem *corr. ex* equidistans *F* 122 et *inter. a. m. C1*/ut: quod *E*; que *R* 123 palam: planum *OL3C1ER*/speculum eorum: horum speculorum *R* 124 cadit: ceciderit *R*/et *om. E* 126 intrat: intravit *O*/proprietates: proprietate *ER* 127 accidet enim *om. E*/quodlibet foramen: quolibet foramine *OR*/*post* foramen *add.* in quolibet foramine *E*/et: vel *E* 128 *post* quacumque *scr. et del.* planum ergo quod omnis lux in quodcumque speculum eorum cadit reflexio et descensus sunt in eadem superficie ortogonali *S* 129 puncto speculi *transp. C1*/si . . . descendat (130) *om. P1* 130 ei *corr. ex* eis *C1*/*post* speculi *add.* qui *O* 131 cuicumque *corr. ex* quicumque *L3*; quicumque *E*

protractarum, que eiusdem sunt declinationis cum lineis a puncto priori intellectis, sicut et puncto priori, vel cuicumque alii.

[3.87] Et generaliter idem est situs cuiuslibet puncto in quod cadit lux qui et in priori sumpto et respectu axis, et respectu acuminis tabule enee. Et eadem in omnibus probatio, et similis demonstratio, unde certum non esse hoc ex proprietate lucis vel figure alicuius speculi sed ex quadam proprietate communi omni rei polite et cuilibet luci. Si autem per diversa in quodcumque punctum descendit lux foramina, videbitur reflexio diversa et angulorum diversitas suo descensui consona, et sic in omnibus.

[3.88] Manifestum ex superioribus quod, si corpus politum opponatur corpori luminoso, cadit in quodlibet punctum eius lux a quolibet luminosi puncto, unde super quodlibet politi punctum cadit piramis cuius acumen in eo, et superficies luminosi basis. Et a quolibet puncto luminosi procedit piramis cuius acumen in eo et basis superficies politi.

[3.89] Si autem inter luminosum et politum intelligatur punctum aliquod, veniet quidem ad illud punctum lux luminosi in modum piramidis cuius acumen in puncto, et latera huius piramidis procedentia usque dum cadant in superficiem politi piramidem efficiunt. Unde in puncto intellecto erunt acumina duarum piramidum quarum bases sunt superficies luminosi et superficies politi, et si ad punctum quodcumque intermedium intelligatur piramis cuius basis superficies politi, et procedant huiusmodi piramidis linee, illud quod occupabunt ex superficie luminosi hoc est a quo procedebat lux ad politum secundum duas piramides quarum acumina in puncto intellecto.

[3.90] Et quod procedit lucis in hiis duabus piramidalibus procedit et includitur in duabus primis piramidalibus, et a luminoso secundum

132 protractarum *corr. ex* postrarum *a. m. E*/priori: priore *R* 133 intellectis *corr. ex* et intellectus *C1*/et: a *C1*; *om. R*/puncto *corr. ex* predicto *O* 134 cuiuslibet: cuilibet *R*/*post* cuiuslibet *add.* in *C1*/*post* quod *scr. et del.* in quod *F* 135 qui et *transp. E*/in *om. O*/priori: priore *R*/respectu *mg. a. m. C1*/*ante* axis *add.* ita *C1* 136 *post* et[1] *add.* in *FP1*; *inter.* est *a. m. E*/similis *alter. ex* sillabica *in* simillima *O*; *alter. ex* similima *in* sillogistica *a. m. C1*; similia *L3*/*post* unde *add.* est *R* 137 figure: figura *R* 138 quadam proprietate *transp. R*/omni *inter. a. m. C1*; *om. ER*/cuilibet *corr. ex* cuiuslibet *C1* 139 in *corr. ex* et *O*/descendit: descenderit *R* 140 descensui: descensu *E* 142 *post* manifestum *add.* autem *R* 143 cadit: cadet *R*/quodlibet punctum: quolibet puncto *E* 144 luminosi puncto *transp. ER*/puncto *corr. ex punctum O* 145 cuius *inter. a. m. E*/et[1] *om. S*/*post* luminosi *add.* est *R* 146 cuius *corr. ex* eius *O* 148 *post* autem *scr. et del.* inter *F*/inter: intra *L3* 149 veniet: tenet *F*; tendit *P1* 150 piramidis cuius acumen *corr. ex* cuius acumen piramidis *E*/cuius . . . piramidis[2] *om. S* 151 cadant: cadat *O*; caderent *L3*/in puncto (152) *mg. a. m. C1* 153 superficies[2] *om. R*/superficies politi *transp. E*/quodcumque: quodlibet *P1* 154 intelligatur: intelligitur *L3C1*/cuius *corr. ex* eius *O*/et *om. FP1SOL3* 155 huiusmodi: huius *L3R* 156 procedebat *corr. ex* procedat *C1*/*post* politum *add.* erit *R* 157 *post* acumina *add.* sunt *R* 158 quod: quicquid *R*/piramidalibus *corr. ex* pyramidibus *P1*; pyramidibus *R* 159 primis *om. O*/*post* primis *scr. et del.* politis *L3*; *add.* pollitis *E*/piramidalibus: pyramidibus *L3R*/*post* secundum *scr. et del.* lineis *P1*

lineas equidistantes procedit lux ad speculum, sed hee linee includuntur in duabus primis piramidibus. Et per quascumque lineas moveatur lux ad speculum, observant linee reflexionis eundem penitus situm quem habebant linee motus lucis; unde si moveatur lux per equidistantes, reflectitur per equidistantes, et lux cadens in modum politi ad modum piramidalis reflectitur observans modum eiusdem piramidis.

[3.91] Cum descendit lux a corpore luminoso per foramen aliquod ad corpus politum, si in superficie foraminis ex parte illuminosi intelligatur punctus a quo puncto intelligantur due piramides basis unius in luminosa alterius in polito, a sola base piramidis cuius luminosum basis venit lux ad politum super illud punctum. Similiter, si in superficie foraminis ex parte politi intelligatur punctum in quo acumina duarum piramidum unius ad speculum alterius ad luminosum, a sola base piramidis que basis est in luminoso accedit lux ad speculum super hoc punctum.

[3.92] Et a parte luminosi hiis duabus piramidalibus communi accidit lux ad partem speculi commune duabus piramidalibus. Venit etiam lux a luminoso ad speculum per lineas equidistantes, sed per quascumque accedat, fit reflexio modo predicto. Et quelibet linee reflexionis conservant situm linearum descensus lucis eas respicientium, et in omni reflexione observatur ydemptitas forme lucis que fuerit in polito corpore, et hoc deinceps explanabimus explanatione evidenti.

[3.93] Amplius, patuit quod lux quanto plus ab ortu suo elongatur plus debilitatur. Patuit etiam quod lux continua fortior disgregata. Cum igitur ab aliquo puncto luminosi procedit lux ad superficiem speculi in modum piramidis, quanto magis elongatur a puncto illo tanto maior est eius debilitatio duplici de causa: et propter elongationem ab ortu suo, et propter disgregationem. Cum autem ab aliquo speculi puncto

160 sed: et *L3* 161 piramidibus: pyramidalibus *P1O*/per *om. FP1* 163 quem: quam *O*/moveatur: movetur *O* 164 modum: modicum *C1*; *om. R*/politi: politum *R* 165 *post* modum[1] *add.* politi *C1*/piramidalis: piramidis *P1R* 167 *ante* cum *add.* et *R*/a *om. L3*; *inter. a. m. C1*/aliquod *inter. OE* 168 si *inter. a. m. C1*/*post* parte *add.* illius *O*/illuminosi: luminosi *OR* 169 punctus *corr. ex* pes *O*; punctum *R*/quo *inter. a. m. E* 170 luminosa: luminoso *OR*/base: basi *R*/cuius: eius *L3C1*; *corr. ex* eius *OE* 171 *post* luminosum *add.* est *a. m. C1* 172 si *inter. a. m. E*/*post* quo *inter.* sunt *O* 173 duarum: duorum *L3C1* 174 base: basi *R*/que *corr. ex* qui *E*/in luminoso *corr. ex* illuminoso *O*/accedit: accidit *OL3* 176 piramidalibus communi: pyramidibus communis *R* 177 accidit: accedit *R*/speculi: speculum *FP1O*/commune *corr. ex* tot? *O*; communem *R*/piramidalibus: pyramidibus *R* 178 a *corr. ex* ad *S*/*ante* speculum *scr. et del.* punctum *S* 179 *post* quelibet *scr. et del.* in *C1* 180 conservant: observant *ER*/respicientium *corr. ex* respiciendum *O* 181 lucis *mg. a. m. E* 182 hoc: hec *C1R* 183 quod . . . etiam (184) *mg. a. m. S* (plus ab ortu suo: ab ortu suo plus)/quanto *corr. ex* quam *a. m. E*/*post* elongatur *add.* tanto *ER* 184 *post* fortior *inter.* it *F*; *add.* est *P1R* 185 *post* ad *scr. et del.* speculum *P1*/speculi *inter. a. m. S* 186 a puncto illo: ab illo puncto *R*/illo *inter. a. m. E* 187 est: erit *ER*/eius *corr. ex* enim *O*/debilitatio: debilitas *R* 188 aliquo *corr. ex* liquo *O*

reflectitur lux, ista fit debilior tripliciter: et propter reflexionem que debilitat, et propter elongationem a loco reflexionis, et propter disgregationem.

[3.94] Si vero lux reflexa a speculo agregetur in punctum aliquod, fiet quidem fortior propter agregationem, sed debilitatur per reflexionem et elongationem. Si igitur agregatio lucis tantum redit ei fortitudinis quantum subtrahunt reflexio et elongatio, erit lux reflexa agregata eiusdem fortitudinis cuius est in superficie speculi. Si vero agregatio minus addat fortitudinis quam diminuant illa duo, erit debilior, et si plus addat, erit fortior.

[3.95] Similiter, si a superficie luminosi procedat piramis ad aliquod punctum speculi, erit lux procedens secundum hanc piramidalitatem debilior propter elongationem sed fortior propter agregationem. Si autem agregatio potest super elongationem, erit lux in puncto speculi agregata fortior luce unica a luminoso veniente per lineam unam. Unica dico, quia ad quodlibet punctum linee ex illis sumpte venit etiam piramis a luminoso, que quidem piramis cum similibus excluditur in hac consideratione.

[3.96] Si vero elongatio ponderet super agregationem, erit lux puncti politi minor luce sola unius linee sumpta, et si agregatio plus ponderet elongatione, erit fortior. Luces autem que a luminoso ad speculum accedunt super lineas equidistantes erunt debiliores quam modo alio accedentes, quoniam debilitate propter elongationem non agregentur in speculum, et in reflexione per lineas equidistantes moventur. Unde per reflexionem et elongationem debilitantur. Et si agregentur in reflexione, conferetur eis fortitudo comparata ad fortitudinem quam habuerint in speculo secundum posse agregans super reflexionem et elongationem.

[3.97] Amplius, omnis linea per quam movetur lux a corpore luminoso ad corpus oppositum est linea sensualis, non sine latitudine,

189 tripliciter *corr. ex* multipliciter *C1* 193 *post* propter *scr. et del.* e *P1*; *add.* per *O*/debilitatur per: debilitabitur propter *R* 194 reddit: impendit *O* 195 quantum: quam *E*/et *inter. O*/ *post* reflexa *add.* et *L3C1* 196 *post* vero *scr. et del.* in *P1* 197 minus: unius *FSL3C1*; *alter. in* unius *O*/addat: addit *O*/fortitudinis: fortitudinem *FP1S*/diminuant: diminuunt *R* 199 aliquod *corr. ex* aliqud *O* 200 hanc *inter. a. m. S* 201 *post* agregationem *scr. et del.* si autem aggregatio potest super elongationem et propter fortior agregationem *S* 202 agregatio: aggregationem *O* 203 agregata: aggregati *O*/unica: unicam *O*; *corr. ex* unicam *L3C1* 204 quia *corr. ex* quod *a. m. E*/etiam: et *E* 205 quidem *inter. O* 207 *post* vero *scr. et del.* aggregatio *O*/ponderet: penderet *O* 208 politi: polita *O*; *corr. ex* polita *L3*; *corr. ex* positi *a. m. E*/*post* minor *scr. et del.* line *F*/*post* sola *scr. et del.* luce *L3*/unius: minus *P1*/ponderet: penderet *O* 211 agregentur *corr. ex* agregeretur *L3E*; aggregantur *R* 212 et: sed *FP1S*/reflexione: flexione *F*; inreflexione *E* 213 agregentur: agregerentur *E* 214 comparata *corr. ex* comparatata *O* 215 habuerint: habuerunt *R*/agregans *alter. in* agregationis *C1* 217 *post* omnis *add.* autem *O* 218 sensualis *corr. ex* sensuasensualis *F*

lux enim non procedit nisi a corpore, quoniam non est nisi in corpore. Sed in minori luce que sumi possit est latitudo, et in linea processus eius est latitudo. Et in medio illius linee sensualis est linea intellectualis, et alie eius linee sunt equidistantes illic. Et si dividatur minor ex lucibus, neutra eius pars erit lux, sed utraque extinguetur nec apparebit. Si autem lux minima duplicetur, aut amplius multiplicetur per equalia, et compacta dividatur, erit lux utraque eius pars. Si vero per inequalia fiat divisio, erit altera pars eius lux, altera minime.

[3.98] Lux autem minima procedit ad minimam corporis partem quam lux occupare possit, et processus eius est secundum lineam intellectualem linee sensualis mediam, et extremitates ei equidistantes. Et cadit lux minima non in punctum corporis intelligibilem, sed sensibilem, et refertur per lineam sensibilem cuius latitudo est equalis latitudini linee sensibilis venientis. Et si intelligatur in linea sensibili linea reflexa intellectualis media, eundem habet situm super reflexionis locum quem habet linea intelligibilis media linee sensibilis venientis, et quelibet linea intellectualis in linea reflexa sensibili eundem penitus observat situm cum linea intelligibili alterius sensibilis ipsam respiciente. Observatur ergo in omni luce reflexio linearum et punctorum intellectorum, licet ab eis aut per ipsas non procedat lux, et in hunc modum erit reflexio lucis.

[3.99] Amplius, quare ex politis corporibus non ex asperis fiat reflexio est quoniam lux, ut diximus, non accedit ad corpus nisi per motum citissimum, et cum pervenit ad politum, eicit eum politum a se. Corpus vero asperum nec potest eam eicere, quoniam in corpore aspero sunt pori quos lux subintrat; in politis autem poros non invenit. Nec accidit hec eiectio propter corporis fortitudinem vel duriciem, quia videmus in aqua reflexionem; sed est hec repulsio propria politure,

219 in *inter. L3* 220 minori: minore *R*/que: qui *P1*/possit: potest *ER*/est *inter. a. m. E* 222 illic *alter. in* illi *F*; illi *R*/si *corr. ex* non *O* 223 sed: si *P1* 224 multiplicetur *inter. E*/per . . . compacta (225): et compacta per equalia *L3ER*/equalia: inequalia *inter. a. m. E*/equalia et compacta (225): et compacta equalia *OC1* 225 eius pars *transp. ER* (pars *inter. a. m. E*)/ inequalia *corr. ex* inqualia *O*; equalia *E* 227 ad *corr. ex* in *a. m. E*; in *R* 230 intelligibilem: intelligible *R*/sed sensibilem (231) *mg. a. m. F* 231 sensibilem: sensibile *R*/et . . . sensibilem[2] *om. O*/refertur: reflectitur *R*/latitudo *corr. ex* altitudo *L3*; altitudo *E*/est *om. SL3C1E*; *inter. O*/*post* equalis *scr. et del.* a *L3* 232 si *inter. F*/sensibili: sinsibili *R* 233 habet *inter. O*; habebit *R* 234 quem: quoniam *FP1O*/*post* linea *scr. et del.* intellu *E*/linee: linea *P1*; *corr. ex* linae *F* 235 intellectualis in linea *om. S* 236 observat *corr. ex* observant *O*/*post* observat *scr. et del.* eundem *L3*/intelligibili *corr. ex* intelligibilis *L3*/*post* alterius *add.* linee *R*/ipsam *corr. ex* illam *a. m. E* 237 ergo *om. E*/reflexio: ratio *R*/*post* linearum *add.* intellectuarum *O* 238 intellectorum: intellectuum *O*; *corr. ex* intellectum *L3*/licet *inter. a. m. E* 239 in *inter. O* 240 non *corr. ex* nec *a. m. C1* 241 accedit: accedet *OL3*; *alter. ex* accidet *in* accedet *E* 242 pervenit *corr. ex* pervenerit *S*; *corr. ex* venit *E*; pervenerit *R*/eum *alter. in* eam *C1*; eam *R* 243 nec: non *C1ER* 244 poros non invenit: non invenit poros *ER* 245 hec eiectio *transp. ER*/eiectio *corr. ex* ectio *O* 246 aqua *corr. ex* qua *S*/repulsio: repulsi *FP1S*/politure *corr. ex* poture *F*; poture *S*

sicut de natura accidit quod aliquod honerosum cadens ab alto super lapidem durum revertitur in altum, et quanto minor fuerit duricies lapidis in quam ceciderit, regressio cadentis debilior erit. Et semper regredietur cadens versus partem a qua processit. Verum in arena, propter eius mollitiem, non fit regressio que quidem accidit in corpore duro.

[3.100] Si autem in poris asperi corporis sit politio, tamen lux intrans per poros non reflectitur, et si eam reflecti accidit, dispergitur, et propter dispersionem a visu non percipitur. Pari modo, si in aspero corpore partes elatiores fuerint polite, fiet reflexa dispersio, et ob hoc occultabitur visui. Si vero eminentia partium adeo sit modica, ut sit eius quasi idem situs cum depressis, tunc comprehendetur eius reflexio tamquam in polito non aspero, licet minus perfecte.

[3.101] Quare autem fiet reflexio lucis secundum lineam eiusdem situs cum linea per quam accedet ad speculum ipsa lux est quoniam lux motu citissimo movetur, et quando cadet in speculum, non recipitur; sed ei fixio in corpore illo negatur. Et cum in ea perseveret adhuc prioris motus vis et natura, reflectitur ad partem a qua processit, et secundum lineas eundem situm cum prioribus habentes.

[3.102] Huius autem rei simile in naturalibus motibus videre possumus et etiam accidentalibus. Si corpus spericum ponderosum ab aliqua altitudine descendere permittamus perpendiculariter super politum corpus, videbimus ipsum super perpendicularem reflecti per quam descenderat. In accidentali motu, si elevetur speculum secundum aliquam altitudinem hominis, et firmiter in pariete figuratur, et in acumine sagitte consolidetur corpus spericum, et proiciatur sagitta per arcum in speculum hoc modo ut elevatio sagitte sit equalis elevationi speculi (et sit sagitta equidistans orizonti), planum quod super perpendicularem accedit sagitta ad speculum, et videbis super eandem

247 sicut *corr. ex* sitis *O* / aliquod: aliquid *C1ER* / honerosum: porosum *R* 248 quanto: quando *L3C1* 249 quam: quem *ER* / ceciderit *corr. ex* cecideret *O* / *post* ceciderit *add.* tanto *R* / cadentis . . . regressio (251) *mg. a. m. S* 250 regredietur: regreditur *ER* / processit: procedit *L3C1* 251 fit *inter. a. m. E* / quidem *om. R* 253 sit: fit *FP1SOL3C1* 254 per *om. OER* / accidit: acciderit *ER* 255 in *inter. E* 256 fiet *corr. ex* fiat *E* 257 sit eius *transp. E* / sit eius quasi: eius quasi sit *R* 258 depressis: depressivi *C1* / tunc: et *FP1SOL3E*; *om. R* 259 *post* non *inter.* in *a. m. L3* 260 fiet: fit *C1*; fiat *R* 261 accedet: accidit *SOL3*; *corr. ex* accidit *a. m. C1*; accedit *R* 262 motu citissimo *transp. L3C1* / cadet: cadit *R* 263 *post* ei *scr. et del.* eo *P1* / et *om. O* / ea: eo *P1*; *corr. ex* eo *F* 264 et[1] *inter. F* / ad: in *O* 267 *post* etiam *add.* in *R* / accidentalibus *corr. ex* accidentatalibus *O*; *corr. ex* accidenta *E* / *post* si *inter.* enim *C1* 269 perpendicularem: perpendiculari *FP1* 270 *post* elevetur *scr. et del.* ad *L3*; *add.* ad *E*; *add.* aliquod *R* 271 *post* altitudinem *add.* velut altitudinem *OL3*; veluti altitudinem *mg. a. m. C1* / figuratur: figatur *OL3ER* 272 *post* spericum *scr. et del.* et proiciatur corpus spericum *S* / et *inter. a. m. E* 273 ut *mg. a. m. FC1*; *om. P1SL3E* / *ante* elevatio *mg.* scilicet *a. m. F* 275 accedit: accidit *O* / *post* et *add.* modo *L3*; *scr. et del.* modo *C1*; videbis: videbit *O*; videbitur *L3R*; *corr. ex* videbitur *E*

perpendicularem eius regressum. Si vero motus sagitte ad speculum fuerit super lineam declinatam in ipsum, videbitur reflecti non per lineam per quam venerat sed per aliam non equidistantem orizonti, sicut et alia erat, et eiusdem situs respectu speculi cum ea et respectu perpendicularis in speculo. Quod autem ex prohibitione politi corporis accidat luci motus reflexionis palam quia, cum fortior fuerit repulsio vel prohibitio, fortior erit lucis reflexio.

[3.103] Quare autem accidit idem motus reflexionis et eius accessus hec est ratio. Cum descendit corpus ponderosum super perpendicularem, reflexio corporis politi et motus descendentis ponderosi directe sibi sunt oppositi, nec est ibi motus nisi perpendicularis. Et prohibitio fit per perpendicularem, quare repellitur corpus secundum perpendicularem, unde perpendiculariter regreditur. Cum vero descendat corpus super lineam declinatam, cadit quidem linea descensus inter perpendicularem superficiei politi per ipsum politum transeuntem et lineam superficiei eius ortogonalem super hanc perpendicularem.

[3.104] Et si penetraret motus ultra punctum in quem cadit, ut liberum inveniret transitum, caderet quidem hec linea inter perpendicularem transeuntem et lineam superficiei ortogonalem super perpendicularem. Et observaret mensuram situs respectu perpendicularis transeuntis et respectu linee alterius que ortogonalis est super illam perpendicularem. Compacta enim est mensura situs huius motus ex situ ad perpendicularem et situ ad ortogonalem.

[3.105] Repulsio vero per perpendicularem incedens, cum non possit repellere motum secundum mensuram quam habet ad perpendicularem transeuntem, quia nec modicum intrat, repellit ergo secundum mensuram situs ad perpendicularem quam habet ad ortogonalem. Et quando motus regressio eadem fuerit mensura situs ad ortogonalem

276 regressus: regressum *R*/ad speculum *om. R* 278 aliam: lineam *ER*/non *inter. a. m. L3*; *om. OER* 279 situs *inter. a. m. C1* 280 politi corporis (281) *transp. R* 281 quia *corr. ex* quod *a. m. O*/fuerit: fuit *FP1SE* 282 *post* lucis *scr. et del.* erit *E* 283 accidit: accidat *OR*/motus *alter. in* modus *L3*; modus *C1*; *scr. et del. E*/*post* reflexionis *add.* motus *OC1E*; *scr. et del.* motus *L3* 285 reflexio: repulsio *OR*; *corr. ex* repulsio *a. m. C1* 286 *post* directe *scr. et del.* sibi *E*/est *rep. F*/ ibi: in *scr. et del. O*/ibi *om. O*; in *L3* 287 per *inter. SE*; *om. L3*/quare: quia *FP1* 288 vero *corr. ex* quo *L3* 289 descendat *corr. ex* descendet *E*; descenderit *R* 293 *post* penetraret *scr. et del.* locus *S*/quem *corr. ex* quod *a. m. E*; quod *R*/*post* ut *scr. et del.* punctum *E* 294 inveniret *corr. ex* inveniet *C1E*/caderet: cadet *P1*/inter *inter. a. m. S* 295 *post* transeuntem *add.* per politum *R*/et: etiam *O* 296 *post* super *mg.* hanc *a. m. C1*/observaret: observare *L3* 298 illam perpendicularem *transp. R*/compacta . . . perpendicularem (299) *mg. a. m. S*/enim est *transp. C1ER*/ est *inter. OL3C1E* (*a. m. C1*); *om. S* 299 situ: situs *OE*; *corr. ex* situs *C1* 300 per *om. SC1*; *inter. O* 1 quam: quem *P1* 2 *post* transeuntem *add.* per politum *R*/modicum *corr. ex* modum *S*/secundum (3) *om. L3*; *mg. a. m. C1* 3 *post* habet *scr. et del.* quam *F*/ad *inter. a. m. F*; quam *S* 4 ad *om. L3*

que fuit prius ad eandem ex alia parte, erit similiter ei eadem mensura situs ad perpendicularem transeuntem que fuit prius.

[3.106] Sed ponderosum corpus in regressu, cum finitur repulsionis motus, ex natura sua descendit et ad centrum tendit. Lux autem eandem habens reflectendi naturam, cum ei naturale non sit ascendere aut descendere, movetur in reflexione secundum lineam inceptam usque ad obstaculum quod sistere faciat motum, et hec est causa reflexionis.

[3.107] Patet etiam ex superioribus quod colores simul moventur cum lucibus, unde erit reflexio coloris sicut et lucis. Et si probationem eius videre volueris secundum modum in parte secunda assignatum, poteris iterum per instrumentum. Ad hanc denotandam reflexionem non plene videbis propter debilitatem coloris, debilitatur enim color per elongationem, per reflexionem, per foramen in quod intrat. Quod autem foramen debilitat planum per hoc quod lux apparet maior post foramen magnum quam parvum. Pari modo, cum foramina stricta sint, color post reflexionem aut nullus apparebit aut valde modicus. Tamen, si in predicto instrumento videre volueris, facias speculum argenteum, in ferreo enim speculo color apparet debilior, quoniam in reflexione misceretur cum luce reflexa mixta ex luce descendente et luce speculi ferrei modica, et color ferreus colori reflexo mixtus debilitaret ipsum.

[3.108] Iterum in domo unici foraminis tantum habeatur instrumentum predictum cui domui paries albus opponatur. Et instrumentum foramini domus aptetur cuius foraminis latitudo sit ut duo instrumenti foramina occupare possit per quorum alterum inspiciatur paries albus domui oppositus. Et parti comprehense parietis opponatur corpus coloris fortis, et per aliud instrumenti foramini

5 *post* que *scr. et del.* fiat *F* / erit *inter. a. m. C1* / *post* erit *scr. et del.* ei *O* / *post* similiter *scr. et del.* erit *E* / ei *om. E* 7 ponderosum *corr. ex* pondero *F* / regressu *corr. ex* essu *O*; *corr. ex* gressu *L3* 9 *post* habens *scr. et del.* reflexionem *S* 10 movetur *corr. ex* moveatur *E* / secundum *corr. ex* super *L3*; super *C1* 12 etiam *om. O* 13 unde *corr. ex* ut *O* 14 eius *inter. a. m. E* / assignatum: signatum *L3C1* 15 iterum: verum *OL3R*; *corr. ex* verum *C1*; scire *E* / reflexionem: reflexionis *O* / *post* reflexionem *add.* factum *OR*; *mg.* factum *a. m. C1* 16 non *inter. L3*; *mg. a. m. C1* / propter *inter. a. m. E* 17 in: per *ER* / intrat: intravit *O* / *post* intrat *add.* vel intravit *E* / quod autem (18) *corr. ex* autem quod *L3* 18 debilitat: debilitet *SOC1ER* / quod: quia *R* / post: per *E* 19 *post* quam *inter.* per *a. m. E*; post *R* 20 sint: sunt *R* / valde *corr. ex* vade *a. m. E* / modicus . . . speculum (21) *mg. S* 22 ferreo *corr. ex* eo *O* 23 *post* reflexione *mg.* coloris *a. m. C1* / misceretur: miscetur *R* / ex *corr. ex* cum *a. m. C1* 24 *post* et *scr. et del.* fo *C1* / color *corr. ex* calor *P1* / *post* colori *inter.* in *O* / reflexo: reflexione *O* 25 debilitaret: debilitat *FP1R* / ipsum: ipsam *O* 26 unici: unicum *O* / foraminis: foramine *L3* / *post* tantum *scr. et del.* enim *C1* 27 instrumentum predictum *scr. et del. O* / predictum . . . instrumentum (28) *mg. a. m. E* / domui *alter. in* domus *E* / albus opponatur *transp. ER* / opponatur: apponatur *O* 28 foramini *corr. ex* foraminis *L3* 29 quorum alterum: quarem altereum *O* 30 parietis *corr. ex* parieti *C1* 31 opponatur: apponatur *C1ER* / aliud *corr. ex* illud *a. m. E* / instrumenti *corr. ex* instrumentum *P1* / *post* instrumenti *scr. et del.* videbitur color reflecti *S* / foramini: foraminum *E*; foramen *R*

videatur pars parietis. Cum ergo lux intraverit per foramina instrumenti, videbitur color reflecti per foramen illud respiciens, quod est oppositum corpori colorato, per aliud minime. Et ita accidet quocumque opposito corpori foramine, et que dicta sunt in reflexione lucis considerari poterunt in reflexione coloris. Occupavit autem latitudo foraminis parietis duo instrumenti foramina ei adhibita ut maior descendat in speculum lux et melior apparet color reflexus. Et quoniam color debilitatur per foramen directus, et similiter reflexus, cum in corpus ceciderit visui oppositus percipietur secundus, unde, si post reflexionem cadat in corpus album foraminis colorationis adhibitum, forsan propter debilitatem non comprehendet eum visus. Adhibito autem secundo visu foramini colorationis, forsan comprehendetur, quoniam primus non secundus videbitur.

[CAPITULUM 4]

Pars quarta: quod comprehensio forme in corporibus fit per reflexionem

[4.1] Super modum comprehensionis forme in politis corporibus dissentiunt plurimi. Unde quidam eorum radios a visu exire ad speculum, et a speculo redire, et formam rei in reditu comprehendere. Alii affirmant formam corporis speculo ei opposito imprimi, et proinde in eo videri sicut in corporibus fit comprehensio formarum naturalium eius.

[4.2] Verum quod aliter sit palam per hoc: quoniam si quis se viderit in aliqua speculi parte motum in partem aliam, non videbit se in parte prima, sed in secunda, quod non accideret si in parte prima infixa esset eius forma. Pari modo, si ad tertiam mutetur partem, mutabitur locus apparentie forme, nec apparebit in prima vel in secunda parte.

32 ergo: igitur *FP1* 33 *post* videbitur *scr. et del.* color *C1* / respiciens: inspiciens *E* / est oppositum (34) *transp. ER* 34 corpori colorato *transp. OC1ER* / colorato *inter. L3E* (*a. m. E*) 35 corpori: corpore *L3* 36 poterunt: poterit *SL3C1*; *corr. ex* poterit *O* / coloris: corporis *FP1SOL3E* 37 foraminis: foramine *O*; *corr. ex* foramine *L3* 38 melior apparet: melius appareat *R* / apparet *alter. in* appareat *a. m. E* / *post* reflexus *scr. et del.* cum in *S* 40 oppositus: oppositum *R* / *post* percipietur *scr. et del.* sus *P1* / secundus *corr. ex* secunde *O* / si *inter. a. m. C1* 41 album *inter. a. m. E* / *post* album *scr. et del.* co *F* / foramini *alter. ex* color *in* ?? *F*; foramini *R* / colorationis: reflexionis *C1*; *corr. ex* reflexionis *L3*; *alter. ex* remotionis *in* reflexionis *a. m. E* 42 comprehendet: comprehendit *C1*; *corr. ex* comprehendat *E* 43 visu foramini *transp. E* / colorationis: reflexionis *L3C1*; *alter. ex* remotionis *in* reflexionis *E* / comprehendetur: comprehenditur *C1* 45 pars . . . reflexionem (46) *om. FP1S* / quod *corr. ex* ut *a. m. E* 46 per *om. E* 48 unde: vident *O* 49 formam *corr. ex* foramina *F*; foramina *P1* / *post* comprehendere *add.* existimant *R* 50 formam *inter. O* / *post* corporis *add.* in *C1* / imprimi *corr. ex* et primi *O* 54 motum: motus *R* / non *inter. O* 55 sed . . . prima^2 *mg. a. m. E* / parte prima *transp. C1*

[4.3] Amplius, viso corpore aliquo, et vidente ab eo situ remoto, poterit accidere quod non videat corpus illud in speculo illo, licet videat totam speculi superficiem, quod quidem non esset si imprimeretur forma in speculo, cum videatur speculum et non mutetur locum, et corpus similiter sit immotum, et forma eius inficiat speculum, sicut et prius.

[4.4] Ut plane appareat non accidere hoc ex comprehensione forme, obturetur medietas foraminum instrumenti, et in aliquo obturatorum sit scriptura aliqua. Si inspiciatur speculum regule per foramen scripturam respiciens, comprehendetur in speculo scriptura, per quodcumque aliud minime. Quod si scripture forma speculo esset impressa, per quodcumque foramen instrumenti posset percipi. Simili modo, in speculis columpnaribus per foramen respiciens tantum comprehendetur scripture situs. Verum in speculis piramidalibus et spericis situs et magnitudo scripture mutabitur.

[4.5] Amplius, speculo columpnari extracto, regula super bases suas directe sita apparebit facies hominis in eo directa. Si vero erigatur regula aut multum declinetur, videbitur distorta. Palam ergo quod non accidet comprehensio ex forma fixa in speculo, cum non comprehendatur res visa in speculis nisi fuerit visus in situ reflexionis. Palam etiam quod distortio faciei apparentis non est ex forma rei sed dispositione speculi.

[4.6] Amplius, viso corpore in speculo et post elongato, comprehendetur corpus magis intra speculum quam prius, quod non erit si forma corporis in superficie speculi sit et ibi comprehendatur. Comprehensionem igitur forme in speculo efficit reflexio.

59 quod: ut *R*/videat[1] *corr. ex* accidat *O*; viderat *L3*; *corr. ex* vidat *C1*/videat[2] *corr. ex* videa *F*; viderat *OL3*; *alter. ex* viderit *in* viderat *C1*   60 quidem: autem *L3C1*   61 in *om. OC1*/*post* in *scr. et del.* etiam *C1*; *add.* etiam *E*/et[1] *inter. P1*/mutetur *alter. in* mutet *F*; mutet *R*   63 prius *corr. ex* primus *S*   64 *ante* ut *add.* et *P1ER*; et *mg. a. m. F*/hoc: huiusmodi *E*/forme *inter. O*   66 aliqua *corr. ex* alia *a. m. E*/si *inter. a. m. C1*   67 *ante* respiciens *scr. et del.* in *O*/respiciens: inrespiciens *E*/*post* scriptura *add.* prima *E*   70 respiciens *alter. ex* inspicione *in* inspiciens *O*/*post* tantum *add.* foramen obturatum in quo est scriptura *R*   71 verum: et erit *OL3*/et *corr. ex* in *O*   75 declinetur: inclinetur *R*/accidet: accidit *R*   76 ex: et *O*/comprehendatur *corr. ex* comprehendantur *P1*   77 fuerit visus *transp. FP1*/etiam: ergo *L3*; *corr. ex* ergo *a. m. C1*   78 *post* est *add.* lux *P1*   79 *post* in *scr. et del.* cor *P1*/*post* post *scr. et del.* elongationem *P1*   80 erit: esset *R*   81 sit: esset *R*/ibi: hic *E*/comprehendatur *corr. ex* comprehendantur *P1*; comprehenderetur *R*   82 reflexio *om. P1*

[CAPITULUM 5]

Pars quinta: in modo comprehensionis formarum in corporibus politis

[5.1] Iam patuit in parte superiori quod, si opponatur speculo corpus coloratum lucidum, a quolibet eius puncto procedit lux cum colore ad totam speculi superficiem, et reflectitur per lineas reflexionis proprias. Igitur a puncto sumpto in corpore opposito speculo procedit lux cum colore ad speculum in modum piramidis continue, cuius basis est superficies speculi, et forma illa reflectitur per lineas eiusdem situs cum lineis accessus, et erit post reflexionem continuitas sicut in accessu. Et si lineis reflexis occurrat superficies corporis, propter continuitatem earum tota occupabitur ut nichil intersit vacuum. Si ergo forma illius corporis moveatur ad speculum per lineas illas (scilicet per reflexas) et ad basem piramidis pervenerit, quoniam linee piramidis eiusdem sunt situs cum lineis reflexis, reflectitur forma per lineas piramidis, et agregabitur tota in puncto sumpto.

[5.2] Quotiens ergo forma alicuius corporis per lineas aliquas ad speculum venerit, si linee ille eiusdem sunt situs cum lineis piramidis a puncto sumpto ad speculum intellecte, cum eas respicientibus, movebitur forma per piramidem illam ad punctum sumptum. Et si in puncto sumpto fuerit visus, videbit corpus cuius est forma illa. Et superius determinatum est quod in situ determinato fit adquisitio forme in speculo. Situs igitur proprius et naturalis adquisitio visus per reflexionem hic est ut linee accessus forme ad speculum eundem habeant situm cum lineis piramidis a centro visus ad capita illarum linearum intellecte unaqueque cum ea respiciente, nec accidit forme reflexe comprehensio nisi in situ isto.

83 pars . . . politis (84) *om. FP1S* 85 *post* iam *add.* autem *L3C1*/*post* superiori *add.* libri *C1*/ opponatur *corr. ex* ponatur *O* 86 eius puncto *transp. C1* 88 proprias: prias *E* 91 continuitas *corr. ex* continuitatis *P1* 92 *post* accessu *add.* fuit *C1* 93 *post* continuitatem *scr. et del.* sicut in accessu et si *O*/occupabitur *corr. ex* occupatur *mg. a. m. E*/ut *inter. O*/ergo: modo *O* 94 scilicet *om. FOL3*; *inter. a. m. S*/scilicet per (95) *om. P1ER* 95 per *inter. L3*/ad *om. O*/basem: basim *R*/*post* linee *scr. et del.* perve *F*/piramidis: piramidum *O* 96 reflectitur: reflectetur *R* 98 quotiens *corr. ex* quosiens *O*; quoties *R*/ergo: igitur *FP1*/*post* corporis *add.* ad speculum venerit *R*/lineas aliquas *transp. R*/ad . . . venerit (99) *om. R* 99 ille: iste *ER*/sunt: sint *ER*/piramidis *corr. ex* piramidalibus *a. m. E* 100 intellecte: intellige *OL3ER*/cum *om. O*; *corr. ex* tamen *L3*; tamen *ER* 101 illam: illa *O* 102 *post* illa *scr. et del.* in precedenti capitulo *E* 103 determinatum: declaratum *R*/fit: fiat *R* 104 acquisitio: acquisitionis *R* 105 *post* est *add.* dicendus *E*/*post* ut *scr. et del.* sit *E* 106 habeant: habent *P1C1*; habuerint *E*/illarum: earum *FP1* 107 intellecte: intellige *OE*; scilicet *R*/unaqueque: unaquaque *OL3C1E*/ea: eam *C1*; sua *R*/accidit: accit *P1* 108 situ isto *transp. R*

[5.3] Palam ergo quod secundum hanc dispositionem linearum tantum fit comprehensio formarum. Et palam quod ex corpore colorato luminoso procedit lux cum colore ad speculum et reflectitur, nec procedit aliquid ex corpore preter colorem et lucem. Patet igitur quod ex luce et colore tantum huiusmodi forma comprehenditur, et cum moveatur forma ex colore et luce compacta secundum predictam situs observationem, superfluum est dicere quod ab oculo exeant radii ad speculum et reflectantur super situm predictum, sicut a pluribus dictum est. Hic igitur est reflexionis modus geometrarum doctrine non adversus sed consonus, cum in eo geometrice radiorum exeuntium opinione observetur situs, et hic modus michi soli usque nunc patuit.

[5.4] Verum cum a corpore luminoso procedat forma ad speculum secundum varietatem situum propter lineas a quolibet puncto corporis ad totam speculi superficiem intellectas, erit forme eiusdem reflexio per diversas piramides quarum capita sunt diversa puncta et bases speculi superficies situm linearum motus forme observantes. Ob hoc accidit quod eadem hora speculo fixo eadem percipitur corporis forma a diversis super quorum intuitis cadunt capita piramidum reflexarum. Similiter, si idem visus moveatur super illa piramidum capita, apparebit ei speculo immoto a locis diversis eadem forma. Sed diversis in speculo eandem formam comprehendentibus in diversa speculi loca cadunt eorum intuitus, quoniam ab eodem speculi puncto diversorum punctorum corporis formas comprehendere eandem non possunt.

[5.5] Iam dictum est quod a quolibet corporis puncto procedit lux ad quodlibet punctum speculi, unde super quodlibet punctum corporis est acumen piramidis cuius superficies speculi basis. Et quodlibet superficiei speculi punctum est acumen piramidis cuius basis superficies corporis. Tota ergo forma corporis erit in quolibet speculi puncto per lineas procedentes in partes diversas, nec concurrere possibiles. Et

109 ergo: igitur *FP1* 110 tantum fit *transp. C1*/fit: fiat *R* 111 procedit: procedat *R*/ reflectitur: reflectatur *R* 112 procedit: procedat *R*/colorem et lucem: lucem et colorem *R*/ igitur: ergo *R* 113 tantum: *scr. et del.* tamen *O*/huiusmodi *alter. ex* hominis *in* omnis *O*/*post* et[2] *scr. et del.* con *F* 114 et luce *om. O*; *inter. a. m. E*/secundum *inter. O* 116 super: secundum *ER* 117 igitur est *transp. ER*/modus: modum *L3* 118 in *inter. L3*/*post* geometrice *add.* situm *O* 120 cum: non *L3* 123 per *alter. in* ad *a. m. E*/sunt *inter. O*; *om. L3C1E* 124 situm: situum *E*/observantes *inter. a. m. E* 125 quod: ut *R*/speculo fixo *transp. R*/ percipitur: percipietur *R* 126 intuitis: intuitus *SOL3C1E* 128 ei: se *C1* 129 formam *om. C1* 131 corporis *inter. L3*; *om. R*/comprehendere *corr. ex* comprehendunt *L3*; comprehendent *E*/*post* comprehendere *scr. et del.* ut *O*; *add.* vel comprehenditur et *C1*/eandem: eadem *SOE*; *corr. ex* et *L3*; easdem *R* 132 *ante* iam *add.* et *R*/iam *corr. ex* item *a. m. E*/est *inter. a. m. E*/corporis puncto *transp. R* 133 unde *corr. ex* unum *S*/punctum corporis *transp. OL3C1ER* 134 *post* superficies *add.* est *C1R* 135 *post* basis *inter.* est *a. m. S*; *add.* est *C1* 136 tota ergo forma *inter. a. m. E*/quolibet speculi *transp. FP1*/speculi *om. S*/speculi puncto *transp. R* 137 lineas: linea *FP1SO*/et *inter. O*

forma a corpore ad quodcumque speculi punctum accedens per piramidem reflectetur per piramidem. Et licet in speculi superficie super numerum multiplicetur eadem iteratio forme, cum concurrat forma totalis cum qualibet parte et in quolibet puncto. Et non sit in formis illis discretio, sed continuitas inseparabilis in reflexione. Tamen, quia forma totalis non cadit in diversas speculi partes, secundum ydemptitatem situs dirigitur ad loca diversa in quibus eam comprehendit visus.

[5.6] Cum ergo similis sibi fuerit forma speculi figure corporis, erit in speculo complementum forme corporis et figure. Quoniam in speculo eiusdem figure cum corpore forma puncti primi dirigitur ad primum punctum speculi, secundi ad secundum, et sic in omnibus se respicientibus. Et ita erit in speculi superficie figura totalis figure, quod non accidit in speculo alterius figure. Similiter, sumpta quacumque speculi parte cui eadem cum corpore figura, erit complementum figure corporis in ea. Et cum infinite sint tales speculi partes, infinite erunt forme corporis reflexionis sed ad puncta diversa procedentes ex quibus formam comprehendit visus.

[5.7] Cum igitur secundum hanc linearum dispositionem fiat forme comprehensio, non erit forme procedentis a corpore in speculi superficie fixio. Et in hunc modum accidit in omnibus speculis, sed in planis certius; in aliis autem accidit quedam diversitas ex errore visus secundum modum predictum. Et quilibet visus secundum modum predictum ab uno speculi puncto non percipit nisi unum corporis punctum, nec a duobus percipitur in eodem speculi puncto idem corporis punctus.

[5.8] Amplius, si opponatur speculum visui, et intelligatur a centro visus ad speculi superficiem piramis, et basis illius piramidis si sumatur punctum, et intelligatur linea piramidis a centro visus ad illud punctum, cum a puncto illo infinite possunt produci linee, si aliqua earum cum latere piramidis eundem habeat situm et equalem cum perpendiculari teneat angulum, et ita accidat quolibet puncto speculi

138 speculi punctum *transp. OL3C1*/*post* punctum *scr. et del.* punctum *L3* 141 totalis: talis *C1*/parte *inter. L3*/non *inter. a. m. E* 144 comprehendit: comprehenderit *L3* 145 ergo: igitur *FP1E*/sibi *om. P1R*/fuerit: fuerint *OE*/*post* speculi *scr. et del.* specu *F*; *add.* et *O*/figure: forme *R.*/*post* figure *scr. et del.* cum corpore forma *O* 146 forme corporis *transp. L3* 147 forma *mg. a. m. C1*/puncti primi *ER* 148 punctum speculi *transp. C1* (punctum *mg.*)/secundi *om. O*/ se *inter. a. m. C1* 149 *post* figure *mg.* res *F*; res *inter. a. m. S*/quod . . . figure (150) *inter. L3* 151 *post* speculi *scr. et del.* figura *C1*/eadem: eidem *L3C1* 153 ex: a *R* 157 fixio *corr. ex* fixo *O* 158 certius: circulis *FP1* 159 et *om. O*; *inter. L3*/et . . . predictum (160) *scr. et del. S*/ quilibet: quibuslibet *O*/secundum . . . predictum (160) *om. OC1* 160 uno: uni *FP1L3* 161 *post* a *add.* visibus *ER* 162 punctus: punctum *R* 163 et *om. L3*; *inter. C1*/*post* centro *scr. et del.* c *F* 164 speculi superficiem *transp. ER* (speculi *inter. a. m. E*)/si: et *R* 166 illo *corr. ex* illius *L3*; illius *C1*/possunt: possint *R* 168 *post* accidat *add.* quod *alter. in* in *S*; *add.* quod a *C1E*(a *inter. a. m. C1*; a *scr. et del. E*)/quolibet . . . planum (169) *mg. L3*

sumpto, planum quod a quolibet puncto speculi potest fieri reflexio. Dico ergo quod inter lineas a puncto sumpto productas est linea eundem habens situm cum latere piramidis, et equalem tenet angulum cum perpendiculari super illud punctum. Et est illa latus piramidis intellecte a puncto illo superficiei rei occurrentis, et quod super terminum illius linee ceciderit, cum per eam ad punctum sumptum venerit, reflectetur ad visum per latus piramidis eius iam dictum. Et hoc piramidis latus cum linea a puncto illo producta erit in eadem superficie ortogonali super superficiem speculum in illo puncto tangentem. Et hoc dico, cum lateris piramidis super punctum sumptum fuerit declinatio. Si enim ortogonaliter cadat super superficiem speculum in puncto sumpto tangentem latus piramidis productum a centro visus, reflectetur in se et redibit in visum ad originem sui motus.

[5.9] In speculo plano planum est quod diximus, quoniam in quodcumque punctum superficiei plane cadat radius a puncto illo potest erigi linea ortogonalis super superficiem illam, et a centro visus potest intelligi linea perpendiculariter cadens in superficiem planam predicte continuam, aut in eandem. Et hee due perpendiculares erunt in superficie eadem, quoniam sunt equidistantes, et linea a termino unius usque ad terminum alterius protracta in superficie plana tenebit angulum acutum cum utraque, et erit in eadem superficie cum utraque. Et radius qui a linea illa elevatur tenebit acutum angulum cum perpendiculari speculi, et similiter cum perpendiculari visus. Et intelligatur linea in alteram partem superficiei plane transiens ortogonaliter per terminos perpendicularium. Tenebit ex parte alia cum perpendiculari speculi angulum rectum, unde ex illo recto poterit

169 sumpto . . . speculi *mg. a. m. C1E*/*post* sumpto *scr. et del.* planum *S*/a *inter. L3* 170 ergo: igitur *FR*/quod *om. E*/*post* linea *add.* que *R*/eundem (171) *corr. ex* eadem *O* 171 habens: habet *R* 172 *post* punctum *scr. et del.* est *L3*/est illa latus: illa linea est latus *C1ER*; est *mg. a. m. E* 173 intellecte *corr. ex* intellige *O*/*post* illo *inter.* ad *OL3*; *add.* ad *C1*/superficiei: superficiem *L3C1*/occurentis *corr. ex* currentis *O* 175 per: et *O*/*post* latus *scr. et del.* u *F*/piramidis: piramidum *FP1SC1*/eius *om. R*/*post* dictum *scr. et del.* est *O* 176 hoc: huius *R* 177 super *om. F*; *inter. P1O*/*post* superficiem *add.* tangentem *R*/illo puncto *transp. P1*/puncto: loco *O*; *om. L3C1* 178 tangentem: contingentem *FP1*; *om. R*/sumptum: positum *FP1* 179 *post* superficiem *add.* tangentem *R* 180 tangentem *om. R* 182 quoniam in quodcumque (183): in quodcumque quoniam *C1* 183 superficiei *corr. ex* superficie *O*/cadat: ceciderit *R* 184 visus: unius *FP1* 185 linea *inter. a. m. S* 186 aut *mg. a. m. C1*/eandem: eadem *FP1C1* 187 superficie eadem *transp. R* 189 acutum *om. OC1ER*; *inter. a. m. L3*/*post* utraque *add.* ectum *alter. in* rectum *O*; *scr. et del.* et super *E*/superficie *inter. O*; *inter. a. m. C1*; *om. L3E* 190 linea illa: aliena linea *FP1*; *corr. ex* aliena linea *S*/cum *inter. a. m. L3* 191 *post* speculi *scr. et del.* et similiter cum perpendiculari speculi *S*/similiter *om. FP1*; simpliciter *L3C1*/perpendiculari: particulari *FP1*/*post* perpendiculari *scr. et del.* est *O*/*post* et[2] *inter.* si *O*; *add.* si *R* 192 intelligatur: intelligitur *O*/linea *om. OL3C1ER*/alterem partem *transp. ER*/*post* partem *add.* linea *OL3C1E*; *add.* produci linea *R*/plane *alter. in* planum *L3* 193 per: super *R*/perpendicularium: perpendicularem *O*; *corr. ex* perpendiculariter *L3E* (*a. m. E*)/*post* perpendicularium *inter.* que *a. m. C1*/alia: altera *ER*; *scr. et del.* altera *C1* 194 unde *inter. E*

abscindi angulus acutus equalis angulo acuto quem cum eadem perpendiculari tenet radius. Et hii duo anguli in eadem superficie, quare radius exiens et reflexus in eadem superficie et in superficie perpendicularium dictarum. Inspecto autem alio puncto, idem situs accidet radiorum cum perpendicularibus quarum una linea a puncto viso, alia a centro visus.

[5.10] In omni ergo superficie reflexionis accidit quatuor punctorum concursus—scilicet centrum visus, et punctus comprehensus, et terminus perpendicularis a centro visus, et punctus reflexus. Et omnes reflexionis superficies secant se in perpendiculari a puncto reflexionis intellecta, et ipsa est communis omnibus superficiebus reflexionis. Et cum idem accidat quolibet puncto superficiei plane inspecto, erit ex omnibus punctis similis reflexio, et eodem modo.

[5.11] In speculis autem spericis palam erit quod diximus. Opposito visui speculo sperico—et est oppositio ut visus non sit in superficie illius sperici aut in superficie continua et sperica—et inspecto hoc speculo, pars eius comprehensa erit pars spere circulo inclusa quam efficit motu suo radius tangens superficiem spere, si per girum moveatur contingendo speram donec redeat ad punctum primum a quo sumpsit motus principium. Et si intelligantur superficies se secantes super dyametrum spere a polo circuli predicti intellectum, quilibet arcuum superficiei spere et hiis superficiebus communium a polo circuli ad ipsum circulum intellectorum erit minor quarta circuli magni, quoniam linea a centro spere ad terminum radii speram contingentis protracta—et est ad circulum predictum—tenet cum radio angulum rectum ratione contingentie. Tenet ergo angulum acutum cum semidyametro a polo circuli producto, et hunc angulum respicit arcus interiacens polum

195 quem: quam *S*; quoniam *L3* 196 *post* radius *scr. et del.* exiens *S*/*post* anguli *inter.* sunt *a. m. O*; *add.* sunt *R*/*post* superficie *add.* sunt *C1* 197 *post* eadem *add.* sunt *R*/*post* superficie *add.* sunt *C1* 198 autem: aliquo *E*/*post* idem *scr. et del.* punctus *P1* 199 una *inter. a. m. L3*/linea *om. O*/linea . . . alia (200) *om. R* 200 *post* visus *add.* alia a puncto viso *R* 202 scilicet: que sunt *R*/punctus comprehensus: punctum apprehensum *R*/comprehensus . . . punctus (203) *mg. a. m. S*/terminus (203): tertius *FP1SL3C1* 203 *post* perpendicularis *inter.* ducte *S*/a centro visus *om. C1*; a centro visus ducte *inter. L3*/*post* visus *add.* ducte *ER*/punctus reflexus: punctum reflexionis *R* 204 se *inter. E*/*post* reflexionis[2] *scr. et del.* et cum idem accidat quolibet *S* 205 ipsa est *transp. ER*; est *inter. a. m. E*/*post* est *add.* sectio *C1* 206 idem *rep. C1*/plane *alter. ex* plan *in* plani *O*/*post* erit *add.* et *FS* 207 similis *corr. ex* simul *C1E* (*a. m. E*)/reflexio: reflexo *E* 208 spericis *corr. ex* speris *O*/*post* diximus *add.* hoc modo *C1* 209 speculo *corr. ex* specula *O* 210 sperici: speculi *R*/*post* superficie *add.* ei *R*/et sperica *om. R*/hoc *corr. ex* hic *E* 211 *post* eius *add.* a visu *R*/*post* circulo *add.* minore *R*/quam: quem *R* 212 *post* efficit *add.* in *C1*/*post* superficiem *scr. et del.* speculi *P1* 213 redeat: redderat *O*; *corr. ex* rederat *L3* 214 motus: motum *FP1*/et: quia *R*/intelligantur *corr. ex* intelligeantur *O*/*post* superficies *scr. et del.* se similiter *O* 215 intellectum *alter. in* intellecto *O*; intellectam *R*/arcuum *corr. ex* acumen *O* 216 *post* et *scr. et del.* in *O*/*post* communium scr. et del. terminum *O*/*post* circuli *scr. et del.* ad ipsum circulo *C1* 218 a *mg. L3* 219 et *inter. a. m. C1*; que *R*/est *inter. a. m. E* 221 producto: producta *R*

circuli et circulum, quare quilibet horum arcuum erit minor quarta circuli.

[5.12] Dico ergo quod a quolibet huius portionis puncto poterit fieri reflexio, quoniam, sumpto aliquo eius puncto, dyameter spere ab illo puncto intellectus erit perpendicularis super superficiem planam tangentem speram in puncto illo. Et huius rei probatio est: Intellectis duabus superficiebus speram super dyametrum a puncto sumpto intellectum secantibus, linee communes superficiei spere et hiis superficiebus sunt circuli spere transeuntes per punctum sumptum. Et intellectis duabus lineis tangentibus hos circulos in puncto sumpto, erit dyameter perpendicularis super utramque lineam, quare super superficiem in qua sunt ille linee. Et cum descenderit radius super punctum sumptum, erit in eadem superficie cum dyametro spere cuius terminus est punctus sumptus, et linea a centro visus ad centrum spere intellecta, que quidem transit per polum circuli, et est radius ortogonaliter cadens super superficiem spere. Et ex hiis tribus lineis erit triangulus, et radius super punctum sumptum incidens tenet acutum angulum cum dyametro spere ab exteriori parte, quoniam, cum elatior sit iste radius radio speram contingente, secabit speram cum productus intelligitur. Et superficies tangens speram in puncto sumpto dimissior erit hoc radio, et secabit inter speram et visum dyametrum—id est lineam a centro visus ad centrum spere intellectam per polum circuli transeuntem.

[5.13] Unde cum dyameter spere sit ortogonalis in superficie punctum tangente, tenebit angulum recto maiorem ex interiori parte cum radio in punctum descendente, unde in exteriori parte tenebit cum eo angulum minorem recto. Et productus ortogonalis erit super superficiem contingentem exterius, quare ex angulo recto quem tenebit

222 quare . . . circuli (223) *om. R* 224 ergo: igitur *R*/huius *inter. L3*/huius portionis puncto: puncto huius portionis *R* 226 intellectus: intellecta *R*/super superficiem *corr. ex* superficiem *O* 227 est *om. OL3*; *inter. a. m. C1*/intellectis *corr. ex* intellectas *a. m. E* 228 sumpto intellectum (229): sumptam intellectam *R* 231 intellectis *corr. ex* intellectus *L3* 232 perpendicularis: perpendiculariter *L3*/*ante* utramque *scr. et del.* ut *C1* 233 *post* sunt *inter.* site *L3*/et *inter. a. m. E*/*post* et *scr. et del.* de *P1*/descenderit: descenderet *L3* 235 est *inter. OL3*; *om. C1R*/est punctus *transp. E*/punctus sumptus: punctum est sumptum *R*/*post* sumptus *add.* erit *C1* 236 *post* quidem *add.* linea *O*/et *scr. et del. O* 237 *post* spere *add.* est similiter in eadem superficie *R* 238 triangulus: triangulum *R* 239 acutum angulum *transp. FP1*/*post* spere *inter.* et *a. m. E*/ab: et *OL3C1*/*post* ab *mg.* ex *a. m. C1*/cum *inter. a. m. E* 240 elatior *corr. ex* elatio *F*/speram *corr. ex* speras *O*/contingente *corr. ex* continginte *F*/cum: si *inter. OL3* (*a. m. O*); *inter. a. m. E* 241 productus: produci *OC1*; *corr. ex* productum *L3*; producta *R* 242 dimissior *corr. ex* demissior *OR*/hoc *corr. ex* ob *O*/*post* radio *scr. et del.* et eius cui centro *OL3E*/visum *rep. FP1SL3E* (*mg. L3*)/*post* visum *add.* visam *R* 245 ortogonalis: ortogonaliter *O* 246 recto: rectum *E*/interiori parte *transp. R* 247 descendente: descendentem *FP1L3E*; *corr. ex* descendentem *S*/unde in exteriori *corr. ex* quoniam in maiori *a. m. E*/in^{2} *om. O*/exteriori *corr. ex* ratiori *O* 248 productus: producta *R* 249 quem: quam *S*/*post* tenebit *add.* diameter *C1*

cum superficie ex alia radii parte poterit abscindi acutus equalis ei quem includit radius cum illo dyametro. Et erunt linee tres hos duos angulos includentes in eadem superficie, quare a puncto portionis sumpto potest produci linea in eadem superficie cum radio in punctum illud cadente et linea ortogonali in superficie punctum contingente et ad paritatem angulorum cum perpendiculari illa. Et illi linee occurret forma puncti mota ad superficiem speculi per radium illum. Igitur eiusdem est situs cum linea que poterit reflecti, et erit superficies in qua sunt hee linee ortogonalis super superficiem speram in puncto contingentem, et ita in quolibet puncto portionis intelligendum.

[5.14] Ergo in omni superficie reflexionis erunt centrum visus, centrum spere, punctus reflexionis, et punctus reflexus, et omnes hee superficies secabunt se super lineam a centro visus ad centrum spere protractam. Et cuilibet reflexionis superficiei et superficiei spere communis linea erit circulus spere, et omnes circuli secabunt se super punctum spere in quem cadit dyametrum visus, et est super circuli portionis polum. Cum autem radius ceciderit in speculum ortogonaliter super superficiem in punctum in quem cadit radius speram tangentem—et est radius ille dyameter visus per polum circuli portionis ad centrum spere—fiet reflexio ad visum per eundem radium ad motus radii ortum.

[5.15] In speculis autem columpnaribus patebit quod diximus. Opponatur speculum columpnare exterius politum oculo—et est oppositio ut non sit visus in superficie columpne aut superficie ei continua—et intelligemus superficiem a centro visus ad columpne superficiem secantem columpnam super circulum equidistantem basibus columpne. Et in hac superficie sumantur due linee tangentes circulum sectionis in duobus punctis oppositis. Ab utroque illorum punctorum producatur linea secundum longitudinem columpne, et intelligantur due superficies in quibus sunt hee due linee longitudinis

251 illo: illa *R*/linee tres *transp. F*/duos angulos *transp. R* 254 *post* et[1] *inter.* cum *O*; cum *mg. a. m. C1*/paritatem: parietem *P1* 255 cum: in *OE*; *corr. ex* in *L3*/occuret *corr. ex* occurat *O* 257 que *om. O*/hee linee (258) *om. O* 258 super *inter. O* 259 puncto portionis *transp. R* 261 punctus[1]: punctum *R*/punctus reflexus: punctum reflexum *R* 262 se *om. P1* 263 et superficiei *om. P1* 264 et . . . cadit (265) *mg. a. m. S* (et: ??; spere:??; quem: quam)/*post* circuli *scr. et del.* linee *O* 265 *ante* spere *add.* super punctum *FO*/*post* spere *add.* in quantum; quantum *del. F*/quem: quam *OC1*; quod *R*/dyametrum: dyameter *ER* 266 speculum *corr. ex* polum *L3E* (*a. m. E*)/*post* speculum *scr. et del.* speculum *O*/ortogonaliter *corr. ex* ortogonale *O* 267 punctum: puncto *OR*/quem: quod *R*/cadit radius *transp. R*/tangentem: contingentem *FP1* 269 fiet *corr. ex* et *O*/eundem: eum *FP1* 270 autem: et *E* 272 aut . . . columpne (273) *mg. a. m. S*/*post* aut *inter.* in *a. m. O* 273 intelligemus: intelligamus *R* 274 *post* superficiem *scr. et del.* seq *P1*/equidistantem *alter. in* equidistanter *a. m. E* 275 hac *alter. ex* alia *in* illa *O* 276 *post* circulum *scr. et del.* equidistantem *S*/oppositis *om. FP1* 277 punctorum *mg. a. m. C1*/*post* punctorum *add.* sumatur *L3*; *scr. et del.* sumatur *C1*/*ante* producatur *inter.* et *L3*/longitudinem: longum *C1* 278 longitudinis *om. P1*/longitudinis . . . linee (279) *mg. a. m. S*

et due linee a centro visus ducte contingentes circulum sectionis. Dico quod hee superficies tangent columpnam.

[5.16] Si enim dicatur quod altera secat illam, planum quod sectio erit super lineam longitudinis columpne in quam superficies cadit, et similiter erit sectio super lineam longitudinis columpne huic oppositam. Et circulus sectionis transit per has duas lineas longitudinis. Et linea contingens circulum sectionis, cum sit in superficie aliqua, secat columpnam super aliquas longitudinis lineas sibi invicem equidistantes, et si transit per unam earum, transibit per alteram, et ad paritatem angulorum. Cum ergo transeat per punctum in quo circulus sectionis secat primam longitudinis lineam, transibit etiam per punctum in quo alia longitudinis linea tangit hunc circulum. Et ita secat circulum, quare non erit contingens, quod est contra ypothesim. Palam ergo quod ille due superficies contingunt speculum, et quod inter illas cadit ex superficie speculi est quod apparet visui.

[5.17] Cum autem illarum duarum superficierum sit concursus in centro visus, secabunt se, et linea sectionis communis transibit per centrum visus, et est equidistans axi columpne, quoniam axis columpne ortogonalis est super eundem circulum sectionis. Et linee longitudinis columpne ortogonales super eundem circulum, et superficies tangentes columpnam secundum lineas has sunt ortogonales super circulum eundem. Quare super superficiem secantem columpnam in illo circulo, quare linea communis harum superficierum est ortogonalis super eandem superficiem, quare equidistans axi columpne.

[5.18] Dico ergo quod quocumque puncto in sectione speculi apparente sumpto, linea a centro visus ad punctum producta secabit speculum. Quoniam intellecta linea longitudinis columpne a puncto sumpto, transibit per circulum sectionis, et tanget ipsum in puncto ad

279 *post* linee *scr. et del.* a centro et due linee *O* 280 *post* hee *add.* due *O* 281 *post* altera *add.* illarum *O*; illarum *mg. a. m. C1*/illam: eam *O*/*post* planum *add.* est *R* 282 erit: est *ER*/in . . . columpne (283) *om. P1* 285 sectionis *om. O*/*post* sit *scr. et del.* cum sit *E*/secat: secans *O* 286 super *corr. ex* per *O*/aliquas: has duas *O*; *alter. in* alias *a. m. C1*/longitudinis *corr. ex* longitudines *P1*/longitudinis lineas *transp. O*/*post* lineas *scr. et del.* et linea contingens circulum cum sit in superficie aliqua longitudinis super lineas *O*/invicem *om. O* 287 si *om. O*/transit: transeat *O*/unam *corr. ex* unum *O*; unum *L3*/paritatem *corr. ex* parietatem *O* 288 ergo: igitur *FP1*; *om. S* 289 etiam *corr. ex* et *a. m. C1* 290 *post* linea *scr. et del.* cont *F* 291 erit: est *inter. a. m. E*/erit contingens: contingit *R*/contra: circa *L3*/ypothesim *corr. ex* ypothasim *O*/ergo: igitur *F*; *om. P1*/ille due (292) *transp. R* 292 et *inter. OE* (*a. m. E*) 294 illarum duarum *transp. R* 295 secabunt: secabant *L3* 296 est: erit *R*/quoniam axis columpne *inter. a. m. L3E*; quoniam: quem *L3*/*post* quoniam *add.* enim *R*/columpne *om. FP1S* 297 eundem *om. R*/*post* longitudinis *scr. et del.* or *C1* 298 eundem *om. FP1S*/circulum *om. L3*/et: etiam *R*/et . . . eundem (300) *mg. a. m.*; circulum eundem *transp. L3* 299 secundum: in *O*/lineas *alter. in* lineis *O*/has: illis *O*/sunt *om. R*/sunt ortogonales *transp. E*/*post* ortogonales *add.* erunt *R*/eundem (300) *om. O* 300 *ante* quare *add.* circulum *L3*/quare *inter. E*; ergo et *R*/super *om. P1* 2 *post* quare *add.* est *OC1* 5 a *inter. E* 6 *post* ipsum *add.* aut sectionis *S*

quem punctum, si ducatur linea a centro visus, secabit speculum quod cadit inter lineas contingentes hunc circulum. Et superficies a centro procedens in qua fuerit hec linea secabit speculum. Cum ergo in eadem superficie fuerit linea hec et linea a centro ad punctum sumptum ducta, secabit linea illa speculum, et ita quelibet linea a centro visus ad portionem speculi intellecta secat speculum. Eodem modo quelibet linea a linea communi per centrum visus intellecta ad hanc portionem ducta secat speculum, unde quelibet superficies tangens speculum in aliqua portionis apparentis linea secat superficies que contingunt portionis extremitates. Et nulla omnium superficierum portionem tangentium pervenit ad visus centra; sed inter visum extendetur et speculum.

[5.19] Dico ergo quod a quolibet puncto portionis huius potest fieri reflexio lucis. Dato enim puncto, fiat super ipsum circulus equidistans columpne basibus. Si ergo superficies a centro visus procedens et columpne superficiem equidistantem basibus secans, secet eam super hunc circulum, et linea a centro visus ad circuli centrum ducta transeat per punctum datum. Fiet reflexio forme illius puncti per eandem lineam ad linee ortum, quia linea illa est axis visus super axem columpne perpendicularis. Sumpto autem quocumque per quem transeat axis perpendiculariter super axem columpne, fiet reflexio illius puncti per eundem axim.

[5.20] Si vero pretereat axim punctus sumptus, quecumque sit linea a centro circuli super ipsum equidistantis basibus punctum ducti, ad superficiem in linea longitudinis columpne per punctum illud transeuntis contingentem erit ortogonalis super axem, quare super lineam longitudinis per punctum illud transeuntem. Et quoniam visus est altior superficie punctum contingente, linea a centro visus ad punctum sumptum ducta tenebit acutum angulum cum perpendiculari illa

7 quem: quod *R*/quod: quia *R* 8 *post* circulum *add.* ergo *R*/*post* centro *scr. et del.* visus secabit *O*; *add.* visus *C1R* 9 ergo: igitur *FP1* 10 fuerit: sit *R*/linea hec *transp. O*/hec et linea *om. R*/*post* centro *add.* visus *R* 12 secat: secabit *C1*/quelibet: queque *FP1* 13 a linea *corr. ex* alia *C1* 15 *post* aliqua *add.* linea *C1*/apparentis *om. O*/linea *scr. et del. C1* 16 omnium: omni *P1* 17 centra: centrum *R*/extendetur: extenditur *R* 20 dato *mg. O*/enim puncto *transp. C1* 21 *post* columpne *scr. et del.* et *O*/ergo *om. OL3* 22 equidistantem: equidistanter *OR*; *alter. ex* equidistante *in* equidistanter *C1*/basibus: visui *FP1SL3E*; basi *R* 23 hunc *om. FP1*/linea *corr. ex* lenea *C1*/centrum: centra *FP1SL3E*; *corr. ex* centra *O*/*post* ducta *add.* et *OL3*; et *scr. et del. C1* 25 ad *inter. a. m. E*/linea illa *transp. E*/illa est *corr. ex* est illa *O* 26 *post* autem *add.* puncto *R*/quem: quod *R* 27 perpendiculariter: perpendicularis *R*/axem: axim *L3*/illius *rep. L3*/axim: axem *ER* 29 axim: axem *C1ER*/punctus sumptus: punctum sumptum *R* 30 super ipsum *scr. et del. O*; *om. C1R*/super ipsum equidistantis: equidistantis super ipsum *S*/equidistantis: equidistante *O*; equidistantem *L3E*/*post* basibus *add.* super ipsum *C1*; *add.* per ipsum *R*/punctum *om. SO*; puncti *L3E*/ducti *corr. ex* perdicti *O*; *om. L3E* 31 *post* superficiem *add.* punctum *S*/in *inter. L3* 32 transeuntis: transeunte *O*/ortogonalis super axem: super axem ortogonalis *ER*; ortogonalis *corr. ex* ortogonaliter *a.m. E* 33 quoniam *alter. ex* qui *in* quia *O*

a puncto ad centra circuli ducta. Et hoc ex parte exteriori, quia obtusum ex interiori. Et ex angulo recto quem illa perpendicularis tenet cum linea superficiei contingente circulum poterit abscidi acutus huic equalis. Et perpendicularis illa cum centro visus in eadem superficie, quare cum linea a centro ad punctum ducta. Et erit linea reflexa in eadem superficie, et erit hec superficies ortogonalis super superficiem contingentem speculum in puncto illo, quoniam perpendicularis ortogonaliter cadit super hanc superficiem. Et huiusmodi erit reflexionis superficies.

[5.21] Est autem diversitas inter lineas superficiebus reflexionis et superficiei columpne communes, cum enim reflexio erit per eundem radium, cadet idem radius ille ortogonaliter super axem. Et linea communis superficiei columpne et superficiei reflexionis erit linea recta—scilicet latus columpne—cum in superficie reflexionis sit dyameter columpne. Et hoc planum, quoniam columpne compositio est ex motu superficiei equidistantium laterum super unum latus immotum. Unde superficiei columpnam secanti in qua sit axis—id est latus immotum—communis linea ei et superficiei columpne erit latus motum. Et dico quod ex omnibus reflexionis superficiebus una sola est cui et columpne superficiei sit linea communis recta, quoniam unica potest intelligi superficies in qua sit axis columpne et centrum visus, et non plures.

[5.22] Si vero superficies reflexionis sit equidistans basibus columpne, erit linea communis circulus, et hec sola est superficies que cum columpne superficie lineam communem habeant circularem, quoniam in omni reflexione perpendicularis super superficiem contingentem punctum reflexionis est dyameter circuli basibus columpne equidistantis. Et non potest esse in columpne superficie nisi unus circulus equidistans basibus qui cum centro visus sit in eadem

36 centra: centrum *ER*/hoc: hic *R*/*post* hoc *add.* est *ER*/exteriori: exteriore *R*/*post* obtusum *add.* habet *R* 37 ex interiori *corr. ex* exteriori *a. m. E*/interiori: interiore *R*/*post* recto *scr. et del.* videtur *P1*/*post* quem *add.* linea *C1* 38 linea: illa *C1*/contingente: contingentis *R*/abscidi: abscindi *SOL3C1ER*/huic equalis (39) *transp. C1* 39 *post* visus *add.* est *C1R*/superficie: quare *E* 40 quare . . . superficie (41) *om. OL3C1E*/*post* quare *add.* etiam *R*/et . . . superficie (41) *om. P1* 41 superficie *corr. ex* superficum *S*/*post* superficie *add.* quare cum linea a centro ad punctum ducta *R*/*post* hec *scr. et del.* li *P1* 43 huiusmodi *corr. ex* huius *S* 47 *post* radium *scr. et del.* non *P1*/cadet: cadit *P1*/idem *om. O* 49 in *inter. L3*/dyameter: dyametrum *FP1SO* 50 et . . . columpne *mg. a. m. S*/hoc *om. E*/hoc planum *transp. R*/*ante* quoniam *add.* est *R* 51 unde: unum *S* 53 communis . . . columpne: et superficiei columpne communis linea *R* 54 *post* reflexionis *add.* scilicet *E*/cui *inter. OL3* 55 communis . . . plures (56) *inter. O*/superficies (56) *corr. ex* ipsi *L3* 56 in *om. E*/qua: que *E* 58 *post* est *add.* reflexionis *OC1*; scilicet reflexionis *inter. a. m. E* 59 communem *om. FP1*/habeant *alter. in* habeat *L3*; habeat *C1* 60 *post* reflexione *scr. et del.* que *E*/super superficiem *corr. ex* superficiem *O* 62 equidistantis *corr. ex* equidistantie *a. m. E*/et *inter. OL3*/et non *mg. a. m. E*/*post* esse *scr. et del.* co *P1* 63 unus: unius *FP1SOL3C1*

superficie. Omnes alie reflexionis superficies secant columpnam et axem columpne, quoniam perpendicularis ducta a puncto reflexionis secat axem columpne, et linee communes hiis superficiebus et superficiei columpne sunt sectiones quas in columpnis et piramidalibus assignant geometre.

[5.23] Cum superficiebus columpne et reflexionis linea recta fuerit communis, quodcumque punctum illius linee intueatur visus, fit reflexio in superficie eadem in qua scilicet est axis, quoniam est superficies unica contingens columpnam in linea illa longitudinis. Et quocumque puncto huius linee sumpto, perpendicularis ab eo ad axem ducta erit in eadem superficie cum axe, et hec longitudinis linea ortogonalis est super superficiem contingentem superficiem columpne. Sed centrum visus est in superficie ortogonali, ut super eandem, et sit in ea axis columpne et linea communis, et una sola est superficies ortogonalis super illam superficiem in eadem, quare omnes reflexiones a punctis huius linee facte sunt in eadem reflexionis superficie.

[5.24] Verum cum linea communis superficiei reflexionis et columpne fuerit circulus, quocumque puncto illius circuli viso, fiet in una et eadem superficie reflexio, quoniam quecumque perpendicularis a puncto viso ducta erit dyameter huius circuli, quare in superficie huius circuli, et punctum visus similiter. Et superficies huius ortogonalis est super superficiem quodcumque punctum huius circuli sumptum contingentem, quare in hac sola superficie erit cuiuslibet puncti predicti circuli reflexio. Quacumque vero alia linea communi sumpta, non fiet in eadem reflexionis superficie reflexio nisi ex uno tantum huius linee puncto, quoniam perpendicularis ducta a puncto reflexionis ortogonalis est super lineam longitudinis columpne per punctum illud transeuntis, quare et super axem. Et perpendicularis illa est dyameter circuli equidistantis basibus columpne. Et superficies reflexionis et circulus ille secant se, et linea eis communis est dyameter illius circuli, et est illa

64 *post* omnes *add.* autem *R*/reflexionis *corr. ex* reflexiones *E*/reflexionis superficies *transp. R* 67 superficiei *corr. ex* superficiebus *a. m. E*; superficiebus *R*/et *om. E*/piramidalibus: pyramidibus *R* 69 et *inter. O* 70 fit: fiet *R* 71 eadem *corr. ex* eaidem *F*; earumdem *P1*/scilicet *om. C1R*/est axis *transp. E*/unica *om. P1* 73 *post* ducta *scr. et del.* erit *S* 74 longitudinis *om. R*/*post* linea *add.* erit *R*/est *om. OL3C1ER* 75 *post* contingentem *scr. et del.* super *L3E* 76 ut *om. R*/eandem: eadem *FP1*/*post* eandem *add.* superficiem quia in una superficie est centrum visus et *R*/et . . . et[1] (77) *om. R* 77 *post* communis *add.* et axis columpne *R*/superficies *inter. a. m. O* 78 in eadem *om. R* 79 facte *corr. ex* factis *a. m. E*/sunt *om. P1* 80 communis: omnis *E* 81 *post* columpne *scr. et del.* et *L3E*/fuerit: fuit *O* 82 quecumque: quacumque *FP1* 83 viso: reflexionis *R*/dyameter: dyametrum *FP1SOL3C1*/huius[2]: huiusmodi *L3* 84 *post* circuli *add.* est *R*/et[1] *corr. ex* est *SO* (*a. m. S*); est *C1E*/huius *inter. O*; hec *R* 85 *ante* super *scr. et del.* supp *P1*/super superficiem *corr. ex* superficiem *O* 87 vero: non *FP1*/linea communi *transp. O*/non: nec *E* 88 huius *om. FP1S* 90 *post* super *scr. et del.* superficiem *P1*/lineam: lineas *L3*; *corr. ex* lineas *C1* 91 illa *inter. a. m. C1E*/illa est *transp. FP1*; *corr. ex* est illa *L3* 93 secant se *transp. P1*/se *om. F*; *inter. a. m. E*/*post* linea *add.* est *C1*/eis: iis *R*/est[1] *inter. a. m. C1*

perpendicularis, et superficies reflexionis secans est, et est declinata super ipsum. Et in superficie super lineam aliquam declinatam non potest intelligi nisi una linea ortogonaliter cadens in illam. Si a duobus reflexionis superficiei punctis fieret reflexio in eadem superficie, essent due linee illius superficiei ortogonales super axem, quod esse non potest, cum ipsa sit delinata super eum.

[5.25] Amplius, perpendicularis a puncto reflexionis cadit in circulum equidistantem basibus columpne et in puncto axis communis circuli et superficiei reflexionis. Si ergo ab alio linee communis puncto in eadem superficie fieret reflexio, alia perpendicularis ab alio puncto ducta esset dyameter alterius circuli columpne huic equidistantis, et caderet in punctum axis in quod non cadit superficies reflexionis. Et ita in omnibus reflexionis superficiebus est intelligendum quod ab uno tantum puncto linee communis fiat reflexio in eadem superficie respectu eiusdem visus. Quoniam respectu duorum visuum potest fieri a duobus punctis circuli dyametri terminus, id est perpendicularis. Respectu vero unius visus non accidit, quoniam illa duo puncta non simul ab eodem visu possunt comprehendi, semper enim necesse est partem columpne medietate minorem videri.

[5.26] Palam ex predictis perpendicularem super punctum reflexionis intellectam exterius intus transeuntem dyametrum circuli efficere, quia, si non, cum constet dyametrum circuli super punctum illud transeuntem perpendicularem esse super superficiem contingentem columpnam in puncto illo, et perpendicularem exterius similiter, erit continuitas inter has perpendiculares, et unam efficient

94 perpendicularis . . . superficie (95): diameter perpendicularis super superficiem columnam in illo puncto contingentem et superficies reflexionis secat illam lineam longitudinis columne super quam fit contingentia et est declinata super ipsam ergo et super axem erit illa superficies reflexionis declinata et in superficie plana *R* 96 illam: illa *P1*/*post* illam *add.* sed *ER*/si: scilicet *O* 97 reflexionis superficiei *transp. R*/punctis *corr. ex* punctus *O*/fieret . . . superficie *rep. L3* 98 illius superficiei *transp. C1* 99 *post* cum *add.* superficies *R*/ipsa: illa *R*/eum: eam *L3* 100 amplius: nam *R*/*post* perpendicularis *scr. et del.* a *S* 101 circulum equidistantem: circulo equidistante *O*/puncto: punctum *R*/*post* axis *add.* et est sectio *R* 102 *ante* circuli *scr. et del.* axis *OL3*; *add.* superficiei *R*/*post* et *scr. et del.* in *S*/ab *inter. O*; *inter. a. m. L3E*/*post* ab *scr. et del.* illo *FP1* 103 superficie *corr. ex* superficiei *L3*/fieret: fiet *FSOL3*; fiat *E*/reflexio *rep. P1* 104 *post* ducta *scr. et del.* ut *C1*/alterius: alius *L3*/*post* circuli *scr. et del.* licet *E*/columpne *corr. ex* columpna *O*/huic *alter. in* basibus *O* 105 punctum *corr. ex* puncto *P1*/cadit *corr. ex* caderet *C1* 106 reflexionis superficiebus *transp. R* 107 tantum puncto *transp. ER*; puncto *inter. a. m. E* 108 *post* duorum *scr. et del.* visus *P1* 109 *post* potest *add.* reflexio *R*/duobus *corr. ex* duabus *O*/*post* punctis *add.* superficiei speculi ut *R*/circuli dyametri *transp. O*/dyametri: dyameter *FP1C1E*/terminus id: terminis que *R* 110 *post* perpendicularis *add.* super ipsam sectionem *R* 112 medietate *corr. ex* mediate *O*; mediate *R* 114 exterius intus transeuntem: extra et intra produci *R*/*post* exterius *inter.* et *a. m. C1*/transeuntem *corr. ex* transeundtem *F* 115 si *om. FP1*/cum *om. FP1SL3*/circuli *om. O* 116 super superficiem *corr. ex* superficiem *OL3* 117 puncto illo *transp. R*/exterius: extra *R* 118 *post* has *inter.* duos *L3*; *add.* et *C1*/efficient: efficiet *O*; *corr. ex* efficit *a. m. E*

lineam. Quia, si non est quod dyametrum extra productum perpendiculare sit super illam superficiem, accidet ex eodem superficiei puncto duas erigi perpendiculares. In omni ergo superficie reflexionum patet quatuor punctorum concursus: centri visus, puncti axis in quem cadit perpendicularis, puncti visi in speculo, puncti a quo forma corporis procedit.

[5.27] In speculis piramidalibus super bases suas ortogonalibus politis exterius est oppositio visus ut non sit visus in superficie speculi aut in ei continua, et secundum visus situm respectu speculi piramidalis erit quantitas comprehense in eo partis.

[5.28] Igitur, si radius ab oculi centro ad terminum axis piramidis—id est ad acumen intellectus—faciat cum axe angulum acutum ex parte piramidis, intelligemus a centro visus superficiem secantem piramidem super circulum equidistantem basi piramidis. Et intelligemus duas lineas a centro quidem visus tangentes illum circulum in punctis oppositis, a quibus punctis protrahemus lineas secundum longitudinem piramidis. Superficies ergo ex una harum linearum longitudinis et altera contingentium circulum continget piramidem, si enim secaverit, continget aliud punctum quam punctum contingentie circuli. Super illud punctum producatur linea longitudinis piramidis, et illud punctum et acumen piramidis sunt sicut in hac superficie, quare illa linea erit in hac superficie et transibit per aliquod punctum circuli. Illud ergo punctum est in hac superficie et in circulo, quare est in linea communi circulo et superficiei. Sed illa est contingens circulum, quare contingens transit per duo puncta circuli quem contingit, quod est impossibile. Restat ergo quod superficies illa tangat piramidem.

[5.29] Et generaliter omnis superficies reflexionis in qua concurrunt linea tangens aliquod punctum piramidis et linea longitudinis per illum

119 *post* non *scr. et del.* esset *C1*/quod *om. O*/dyametrum: diameter *R*/productum: producta *R*/perpendiculare (120) *corr. ex* perpendiculariter *P1*; perpendicularis *OR*; *corr. ex* perpendicularis *L3* 121 perpendiculares *corr. ex* perpendicularis *F*/superficie: superficiei *FP1SL3*/*post* superficie *scr. et del.* puncto duos erigi *S*/reflexionum: reflexionis *R* 122 quatuor: quod *OC1*; *corr. ex* quod *L3*/*post* quatuor *inter.* est *O*/quem: quod *R* 123 puncti . . . speculo *mg. a. m. S*/visi: reflexionis *R* 126 politis: positis *FP1SO*/est: et *FP1SL3* 127 ei *corr. ex* eo *a. m. E*/ei continua *transp. R*/piramidalis: piramidis *OC1* 130 ad *om. FP1S*/faciat *corr. ex* fiat *a. m. C1E* 131 secantem *corr. ex* seguntem *O* 132 super . . . piramidis *mg. a. m. C1* 133 quidem *om. O*/*post* illum *scr. et del.* punctum *L3* 134 punctis *om. R* 135 ergo: igitur *O*/*post* longitudinis *scr. et del.* et *P1* 136 secaverit: secuerit *R* 137 continget *corr. ex* contingens *L3*; contingens *C1E* 138 piramidis *om. R* 139 *post* piramidis *add.* simul *R*/sunt *corr. ex* fiunt *a. m. E*/*post* sunt *add.* simul *P1*/sicut *om. R*/quare *inter. O* 141 ergo: igitur *R*/est in hac superficie: in hac superficie est *ER*/hac superficie *transp. L3C1*/in^3 *inter. P1* 142 circulo *inter. a. m. E*/est contingens: contingit *R*/circulum . . . contingens (143) *om. O* 143 *post* contingens *mg.* et *O*; *add.* circulum tangit; *del.* tangit *C1*/quem *mg. a. m. C1* 144 ergo: igitur *FP1R*; *om. E*/quod: ut *R*/superficies illa *transp. R*/tangat *corr. ex* tangant *S* 145 omnis: omnes *O*/reflexionis *inter. a. m. E*; *om. R* 146 linea tangens: linee tangentes *E*/*post* et *add.* illa *E*/linea longitudinis *transp. ER*/illum: illud *OER*/illum punctum (147) *transp. R*

punctum transiens tangit piramidem super lineam longitudinis. Habemus ergo duas superficies ab oculi centro procedentes piramidem contingentes inter quas est portio piramidis apparens visui in hoc situ, et est minor medietatum piramidis, quoniam linee contingentes circulum includunt eius partem medietatum minorem.

[5.30] Si vero linea a centro ad acumen piramidis ducta tenet angulum rectum cum axe, intelligatur circulus secans piramidem equidistans basi. Linea communis huic circulo et superficiei in qua sunt axis piramidis et centrum visus erit ortogonalis super axem piramidis, quoniam axis est ortogonalis super superficiem circuli. Et super lineam communem protrahatur per centrum circuli dyameter ortogonaliter super hanc lineam, et a terminis huius dyametri protrahantur due contingentes circulum, et etiam due linee usque ad acumen piramidis. Due superficies in quibus erunt hee due linee cum contingentibus contingunt piramidem secundum modum predictum. Et quoniam linea communis circulo et superficiei in qua sunt centrum visus et axis piramidis est equidistans linee a centro visus ad terminum axis producte, et huic linee communi sunt equidistantes linee circulum in predictis punctis contingentes, erunt ille contingentes equidistantes linee a centro visus ad terminum axis ducte, quare erunt in eadem superficie cum illa. Igitur utraque superficierum circulum contingentium transit per centra visus, et communis illarum superficierum sectio est linea a centro visus ad terminum axis ducta. Et quod inter illas superficies cadit ex piramide apparet visui, et est medietas piramidis, quoniam lineas has contingentes circulum interiacet medietas circuli. Et ita palam quod in hoc situ apparet medietas piramidalis speculi.

[5.31] Verum si linea a centro visus ducta ad terminum axis piramidis teneat cum axe angulum obtusum ex parte superiori apparenter, et fiat circulus secans piramidem equidistantem basi, linea communis huic

148 duas superficies *transp. C1*/procedentes: cedentes *L3E* 149 apparens: apparentis *ER* 150 medietatum *corr. ex* medietum *O*; medietate *L3C1R*/piramidis: pyramidum *L3*/contingentes: tangentes *FP1R*; contingens *E* 151 includunt . . . tenet (152) *mg. O*/medietatum *corr. ex* miedietatum *F*; medietate *C1R*/minorem *corr. ex* minorum *F*; minore *P1* 152 *post* centro *add.* visus *ER*/tenet: teneat *L3C1ER* 153 *ante* angulum *scr. et del.* in *O*/cum *corr. ex* cur *L3*/*post* axe *add.* et *R*/secans *inter. E* 154 equidistans: equidistanter *R*/communis *corr. ex* longitudini *O*/et *inter. L3* 155 centrum: centra *FP1*; *corr. ex* centra *L3*/erit *om. L3*; *mg. a. m. C1* 157 et . . . circuli *mg. a. m. E*/centrum: centra *FP1SL3*; *corr. ex* centra *C1* 158 ortogonaliter: ortogonalis *R*/et *inter. L3*/*post* dyametri *add.* ortogonaliter *E*; *add.* ortogonalis *R* 160 hee *mg. a. m. C1*/cum *inter. L3* 161 contingunt: contingent *R*/piramidem *corr. ex* piramides *E* 163 *post* centro *add.* illius *ER* 164 producte *om. O*/et *om. S* 165 *post* predictis *scr. et del.* d *S*/*post* ille *add.* linee *R*/contingentes *inter. a. m. E*; *om. R* 166 axis *inter. a. m. E* 169 linea *inter. a. m. C1*/et *om. O*/*post* inter *add.* illa *C1*/illas *corr. ex* illa *E* 170 piramide *corr. ex* piramidis *E*/medietas . . . interiacet (171) *mg. a. m. S* 171 has: habens *S*; *om. O* 172 medietas *alter. ex* radius *in* radiis *O*/speculi *inter. a. m. S* 174 teneat *corr. ex* tenet *C1*/apparenter: apparente *R* 175 piramidem *corr. ex* piramides *C1*/equidistantem *alter. in* equidistanter *L3*; equidistanter *C1ER*

circulo et superficiei in qua est centrum visus et axis est perpendicularis super axem piramidis. Et hec linea communis extra producta concurret cum linea a centro visus ad terminum axis ducta propter angulum acutum quem facit hec linea cum axe ex inferiori parte. A puncto concursus linearum protrahantur due linee contingentes circulum in duobus punctis oppositis, et producantur linee ab hiis punctis ad acumen piramidis. Superficies in quibus sunt linee contingentes cum hiis longitudinis lineis contingunt piramidem, et in utraque harum superficierum sunt duo puncta linee a centro visus ad terminum axis ducte—scilicet terminus axis et terminus perpendicularis in quo scilicet concurrunt linea illa et perpendicularis—quare linea illa est in utraque superficie. Igitur utraque superficies transit per centrum visus, et includunt hee superficies ex interiori in inferiori parte minorem partem piramidis, quia linee contingentes circulum includunt partem eius minorem medietatem. Unde ex parte superiori interiacet superficies piramidem contingentes pars medietatum maior, et illa est que apparet visui, quare in hoc situ comprehendit visus piramidis partem medietatum maiorem.

[5.32] Si autem linea a centro visus ad terminum axis producta cadat super latus piramidis, ut ex ea et latere unum efficiatur continuum, dico quod non latebit visum ex hac piramide preter lineam quandam intellectualem, quoniam omnis superficies in qua est linea a centro visus ad terminum axis ducta et secundum lateris longitudinem prolongata secat piramidem una tantum excepta que contingit piramidem in latere quod est pars linee. Et hoc solum latus intellectuale in tota piramidis superficie super hoc situ visum preterit.

[5.33] Et huius rei veritas patet ex hoc quod, quocumque superficiei piramidis puncto sumpto, si ad ipsum ducatur linea a centro visus et

176 et[1] *inter. P1*/*post* axis *scr. et del.* perpendicularis *O* 177 concurret *corr. ex* concurrat *OE*; concurrat *L3* 179 *post* puncto *add.* igitur *R* 180 *post* linee *scr. et del.* contingentes *O*/*post* circulum *scr. et del.* in duobus *L3* 181 oppositis *corr. ex* oppositionis *a. m. E* 182 sunt *om. FSOL3E*/sunt linee *inter. a. m. C1* 184 duo puncta *transp. C1* 186 et . . . illa[2] *inter. L3*/illa[2] *inter. a. m. E*/*post* illa[2] *add.* que ducitur a centro visus per terminum axis *R* 188 includunt: includuntur *FP1*/*post* hee *inter.* scilicet *a. m. E*/superficies *inter. a. m. E*/interiori in *om. R*/in *om. SOC1*; *inter. a. m. E* 189 *post* piramidis *add.* medietate *R* 190 medietatem: medietate *OC1ER* 191 medietatum: medietate *OC1R* 192 piramidis: piramidum *P1*/piramidis partem *transp. R* 193 medietatum: medietate *OC1R* 194 cadat: cadit *P1ER* 195 piramidis: piramidum *P1*/efficiatur *corr. ex* ficiatur *O*/*post* continuum *add.* latus *R* 196 non *inter. a. m. S*/*post* preter *scr. et del.* lumen *P1* 197 omnis: omnes *F*; *corr. ex* omnes *P1OL3* 198 *post* lateris *add.* venientis ad centrum visus *P1* 199 contingit: contingerit *P1*/piramidem[2]: piramidis *O*/*post* piramidem[2] *add.* superficiem *O* 200 *post* pars *add.* illius contingentis *C1*/et: in *P1*/latus *om. O*/intellectuale *corr. ex* intellectualem *S* 201 super: sub *L3C1ER*/situ *corr. ex* situm *P1*; sui *S* 202 ex hoc *mg. a. m. E*/quocumque *corr. ex* quodcumque *L3*/*post* quocumque *inter.* in *O*/superficiei *om. R* 203 piramidis puncto *transp. L3C1E*/puncto: puncti *FP1*/*post* sumpto *add.* extra latus intellectuale *R*

ab eo linea longitudinis piramidis ad terminum axis, efficient hee due linee triangulum cum linea lateri applicata. Et est triangulus in superficie a centro intellecta piramidem secante, et ex lineis huius superficiei non nisi due cadunt in superficie piramidis—scilicet linea lateris et linea longitudinis a puncto sumpto ad acumen piramidis. Et linea a centro ad punctum sumptum ducta secat lineam longitudinis reflexionis in puncto sumpto et lineam lateris in centro, quare huic linee non accidet concursus de centro cum aliqua harum linearum. Cum igitur non posset sumi punctus alius ad quem linea a centro accedat et in hoc punctum transeat, non occultatur punctus iste ab alio puncto. Et ita apparet visui, cum ei et visui non intercidat corporis solidi obiectio. Et eadem probatio in quolibet superficiei piramidis puncto.

[5.34] Et si linea a centro visus in terminum axis cadens intret piramidem, dico quod nullus occultatur visui punctus in tota piramidis superficie. Sumpto enim quocumque puncto in piramidis superficie, intelligatur ad ipsum linea a centro et alia ab eo usque ad acumen piramidis. Hee due includunt superficiem triangularem cum linea a centro visus ad terminum axis ducta piramidem intrante, et est iste triangulus in superficie piramidem secante. Cum omnis superficies in qua fuerit linea intrans piramidem secet eam, linea a centro ad punctum sumptum ducta secat in illo puncto lineam longitudinis ab eo ad acumen piramidis ductam. Et ex lineis superficiei in qua sunt hee due linee non sunt nisi due linee in superficie piramidis—scilicet hec linea longitudinis a puncto ad acumen ducta et alia opposita secans angulum quem includit hec cum linea piramidis intrante. Igitur linea illa opposita extra piramidem producta secat lineam a centro ad punctum sumptum

204 eo: ea *FP1S*/*post* piramidis *add.* ducatur *C1*/*post* ad *scr. et del.* centrum *E*/*post* terminum *mg.* B *a. m. E*/axis *mg. a. m. E*/hee *inter. a. m. E* 205 et *corr. ex* quod *L3*/*post* et *scr. et del.* quod *S*/ est triangulus: erit triangulum *R* 206 *post* centro *add.* visus *R*/*post* ex *add.* hiis *E* 207 due cadunt *transp. E*/*ante* in *add.* linee *E*/superficie: superficiem *R*/linea lateris et (208) *om. R* 208 lateris: lateri *FP1*; *corr. ex* latitudinis *L3*; latitudinis *inter. a. m. E*/longitudinis *corr. ex* longitudins *P1*/*post* piramidis *add.* et linea opposita huic ex altera parte *R* 209 *post* centro *add.* visus *R* 210 reflexionis: ei eius *O*; *om. R*/reflexionis in puncto: in puncto reflexionis *C1*/lateris: lateri *P1*; *corr. ex* latitudinis *OL3*/*post* lateris *add.* continuati cum visu *R*/*post* centro *add.* visus *R*/*post* quare *scr. et del.* hunc *P1*/*post* linee *add.* a centro visus *R* 211 accidet: accidit *OL3C1*/de centro *scr. et del. O*/de . . . posset (212) *om. R*/*post* centro *inter.* inter ipsum et centrum *a. m. O*; *add.* visus *C1* 212 posset: possit *C1*; *alter. in* possit *E*/*post* posset *scr. et del.* po *F*/punctus alius ad quem: punctum aliud ad quod *R*/*post* centro *add.* visus *R* 213 occultatur *corr. ex* occultatum *O*/punctus iste *alter. in* punctum hoc *a. m. E*; punctum istud *R* 214 et[1] . . . ei: quod non perveniat ad centrum visus quare apparet visui cum inter ipsum *R*/et[2] *inter. a. m. C1*/visui: visum *R*/obiectio: abiectio *FP1* 215 *post* eadem *add.* est *OC1*/*post* probatio *add.* est *R*/in *inter. L3*; de *R* 216 et: quod *O* 217 nullus: nullum *R*/punctus: punctum *R* 218 *post* superficie[2] *add.* et *C1* 219 *post* centro *add.* visus *C1R* 220 *post* due *add.* linee *R* 221 *post* centro *scr. et del.* videtur *P1*/visus *om. O*/intrante: intrantis *O*/iste: istud *R* 222 triangulus: triangulum *R*/*post* in *scr. et del.* parte *P1*/secante *corr. ex* secantem *L3* 223 secet *corr. ex* secent *O*/*post* linea[2] *add.* vero *R*/*post* centro *add.* visus *R* 225 due *om. O* 226 sunt *inter. O* 227 alia: illa *P1* 228 piramidis: piramidum *P1*; pyramidem *R* 229 secat *corr. ex* secans *O*

ductam; quare linea hec secat duas lineas que sole ex lineis huius superficiei sunt in piramidis superficie, unam extra piramidem aliam in puncto sumpto, quare producta in infinitum non concurret cum altera illarum linearum. Unde non occultatur visui punctum sumptum secundum modum supradictum.

[5.35] In hoc ergo situ nulla superficierum piramidis tangentium transibit per centrum visus, sed quelibet secabit lineam visus super terminum axis piramidem intrantis inter visum et piramidem, et est in termino axis. Cum vero linea visus linee longitudinis piramidis applicatur, nulla superficierum piramidis tangentium perveniet ad centrum preter illam que in predicta linea contingit piramidem. Et omnes superficies contingentes secabunt lineam illam inter visum et piramidem.

[5.36] Similiter, in situ in quo superficies due contingentes piramidem per centrum transeunt, quelibet superficies tangens piramidem in portione piramidis apparente que duas contingentes interiacet a centro visus divertit. Super quodcumque punctum illius portionis cadat linea visualis, secabit piramidem, cum intercidat duas contingentes visuales. Et superficies in qua fuerit hec visualis et linea longitudinis piramidis secabit piramidem, et erit hec visualis superficies cuicumque superficiei piramidis in hac portione contingat, quare et visus.

[5.37] Dico ergo quod in quolibet situ a quolibet puncto potest fieri reflexio. Sumatur enim punctus, et intelligatur circulus per punctum transiens basi piramidis equidistans. Dyameter huius circuli ab hoc puncto incipiens erit perpendicularis super axem, cum axis sit perpendicularis super circuli superficiem, quare linea longitudinis a puncto ad acumen piramidis ducta tenet angulum acutum cum dyametro et acutum cum axis termino in eadem superficie. Sit linea visualis super punctum cadens in superficie in qua est linea longitudinis

232 in[2] *om. C1*; *inter. E*/concurret *corr. ex* concurrit *F*/altera: aliqua *R* 233 illarum: aliarum *C1*/unde *corr. ex* unum *S*/punctum sumptum *transp. R* 235 ergo *om. FP1*; *inter. L3*; *mg. a. m. E*/ergo situ *transp. R*/piramidis: piramidum *P1*; piramidem *ER*/*post* tangentium *scr. et del.* et *O* 236 visus: a visu *R* 237 piramidem[1]: piramidis *FP1*/intrantis: intrantem *OC1*; *corr. ex* intrantem *L3* 238 vero: ergo *L3C1*/piramidis: piramidum *P1* 239 piramidis: piramidem *C1ER*/perveniet: pertinet *R* 240 *ante* preter *add.* visus *R* 242 piramidem: verticem pyramidis *R* 243 situ: visu *FP1*; *corr. ex* visu *OL3*; *alter. ex* usu *in* visu *S*/superficies due *transp. R* 244 centrum: centra *FP1SOL3*/*post* centrum *add.* visus *R* 245 in portione piramidis *om. FP1*/apparente: apparentem *P1* 246 divertit *scr. et del. O*/*post* divertit *inter.* et *L3*; et *inter. a. m. C1E*; *add.* et *R* 247 *post* intercidat *add.* inter *R* 248 *post* fuerit *add.* linea *R*/*post* visualis *add.* linea *C1*/et[2] . . . visualis (249) *mg. a. m. E* 250 contingat *corr. ex* contingant *OL3*; continua *R* 252 *post* quod *scr. et del.* visus *F* 253 sumatur: similiter *S*/punctus: punctum *R* 254 piramidis: pyramidi *R*/*post* dyameter *add.* igitur *R* 256 circuli superficiem *transp. L3C1* 259 in[1] . . . qua *om. O*/est: et *inter. O*

et axis, in qua superficie ducatur perpendicularis super lineam longitudinis in puncto illo. Concurret hec quidem perpendicularis cum axe, et ex ea, et axe, et linea longitudinis efficietur triangulus. Super punctum illud intelligatur linea contingens, et super dyametrum circuli quem fecimus intelligatur dyameter alius ortogonalis super ipsum, qui erit ortogonalis super ipsum axem, et ita super superficiem in qua axis et dyameter primus. Et hic dyameter secundus est equidistans contingenti, quoniam contingens est perpendicularis super dyametrum primum. Et ita linea contingens ortogonalis est super superficiem in qua axis et dyameter primus, quare erit ortogonalis super perpendicularem quem primo fecimus. Et ita illa perpendicularis ortogonaliter cadit super superficiem contingentem piramidem in qua punctus sumptus.

[5.38] Igitur, si linea visualis cadens in punctum sumptum transeat secundum processum perpendicularis, erit quidem ortogonalis super superficiem piramidis illam in puncto contingentem, et fiet reflexio forme per eandem lineam. Si autem deviet a processu perpendicularis, faciet quidem angulum cum perpendiculari acutum in puncto sumpto. Et poterit produci in superficie huius linee visualis alia linea a puncto illo que equalem angulum huic teneat cum perpendiculari, cum perpendicularis ortogonalis sit super superficiem contingentem. Linea autem quacumque super superficiem contingentem in puncto sumpto ortogonaliter cadente, transit ad axem. Et si ab axe ducatur ortogonalis ad hanc superficiem, efficient perpendicularis interior et exterior lineam unam. Quod si non cum perpendicularis interior extra producta sit etiam perpendicularis super superficiem, accidet ab eodem puncto super illam superficiem erigi duas perpendiculares in eandem partem.

[5.39] Palam igitur quod quocumque puncto superficiei piramidis viso potest fieri reflexio ad paritatem angulorum. Et cum linea

260 ducatur: deducatur *ER* 261 concurret: concurrit *FP1*/quidem: quod *O* 262 et^2 *inter. a. m. C1*/triangulus: triangulum *R* 264 intelligatur: intelligigatur *S*/dyameter *corr. ex* diametrum *O*/alius: alia *R*/ipsum qui (265): ipsam que *R* 265 axem *corr. ex* ?? *O*/ita *om. R*/super^2 *inter. L3*/*post* superficiem *add.* est *C1*/*post* qua *add.* est *R* 266 primus: prima *R*/hic: hec *R*/secundus: secundum *L3*; secunda *R* 267 est *inter. O*; *om. L3*/est perpendicularis *ER* 268 primum: primam *R*/*post* primum *scr. et del.* quare erit ortogonalis super perpendicularem *C1*/*post* ita *add.* etiam *E*/super *om. O* 269 *post* qua *add.* sunt *R*/primus: prima *R*/ortogonalis: perpendicularis *R* 270 quem: quam *R*/*post* illa *add.* prima *R* 271 cadit *om. O*/punctus sumptus (272): punctum est sumptum *R*/*post* punctus *add.* est *E* 272 *post* sumptus *add.* est *C1*; est *inter. a. m. S* 275 piramidis: pyramidum *L3*; piramidem *ER*/fiet *corr. ex* fiat *O* 276 *post* a *scr. et del.* perp *P1* 277 faciet *corr. ex* fiet *O*/*post* in *scr. et del.* punctum *L3* 278 huius: eius *R* 279 cum perpendiculari *inter. a. m. E*/cum perpendicularis (280) *om. L3*/*post* cum^2 *add.* illa *C1* 281 quacumque: quecumque *E*/super *inter. a. m. E* 282 cadente: cadens *R*/transit: transeat *L3*; transibit *C1*/si *corr. ex* sic *L3* 283 perpendicularis *alter. in* perpendiculares *E*; perpendiculares *R* 285 etiam: et *P1O*; *om. L3*; *inter. a. m. E*/accidet *corr. ex* accidit *E* 286 illam: aliam *L3C1*; *alter. ex* aliam *in* aliquam *a. m. E*; aliquam *R*/erigi duas *transp. O* 287 *post* quod *add.* a *R* 288 paritatem: partem *FP1*

reflexionis occurrerit forma, veniet ad speculum super lineam hanc et reflectetur ad visum super aliam, et sunt hee due linee in eadem superficie ortogonali super superficiem contingentem piramidem in puncto reflexionis. Et hec est superficies reflexionis in qua semper fit comprehensio quatuor punctorum: scilicet centri visus, puncti visi, puncti reflexionis, terminus perpendicularis.

[5.40] Diversificantur autem linee communes superficiebus reflexionis et superficiei piramidis. Cum enim radius visualis continuus fuerit axi piramidis—scilicet cum in qualibet superficie reflexionis sit totus axis et perpendicularis ad axem transiens—erit cuilibet superficiei reflexionis et superficiei piramidis communis linea linea longitudinis in hoc situ. Quoniam quelibet superficies in qua est totus axis hanc habet lineam communem cum superficie piramidis.

[5.41] Et in omni alio situ unica longitudinis piramidis linea erit communis illa—scilicet que fuerit in superficie centrum visus et axem continente. Et quia centrum visus non erit in directo axis, una tantum erit superficies talis, et omnis alia communis linea erit sectio piramidalis non circulus. Si enim fuerit circulus, erit superficies illius circuli in superficie reflexionis, et quia axis est ortogonalis super illum circulum (cum quilibet circulus piramidis sit equidistans basi), erunt latera piramidis declinata super circulum, et ita super superficiem reflexionis. Quare in superficie illa non potest duci perpendicularis super lineam longitudinis piramidis. Sed perpendicularis ducta super superficiem contingentem locum reflexionis est in superficie reflexionis, et perpendicularis super lineam longitudinis, cum quelibet superficies tangens piramidem tangat in linea longitudinis.

[5.42] Accidit igitur impossibile, quare restat omnes alias communes reflexionis lineas sectiones piramidales esse, et cum fuerit linea com-

289 reflexionis: declinata *R* / occurrerit: occurit *E* / *post* occurrerit *add.* alicui *C1* / lineam hanc *transp. O* 290 reflectetur: reflectet *O* / due *om. OL3C1E* 291 contingentem *inter. L3* / in *om. O* 292 et . . . reflexionis *om. P1* / fit: sit *S* 293 puncti visi *mg. a. m. E* 294 terminus: terminum *O*; termini *R* / terminus perpendicularis *om. FP1*; termini perpendicularis *mg. L3* / *post* perpendicularis *add.* et puncto axis *O* 295 communes *om. L3* / *post* communes *scr. et del.* a *E* / superficiebus: superficiei *R* 296 superficiei: superficie *FP1S* 297 piramidis: piramidum *P1* / qualibet *om. R* / sit *alter. ex* licet *in* fuerit *O*; fuerit *R* 298 cuilibet *om. R* 299 *post* superficiei *scr. et del.* pima *F* / linea[2] *om. O*; *mg. a. m. C1* 300 est *inter. SO*; *om. L3E* / est totus *transp.*; est *inter. a. m. C1* 1 habet: habeat *L3* / communem *om. P1* / cum *inter. a. m. C1* 2 et *inter. O* / unica *corr. ex* iuncta *a. m. E* 3 centrum: centra *FP1SL3C1* / centrum visus *transp. ER* 4 quia: quando *R* 5 talis: taliter *C1* 6 si enim *om. P1* / fuerit circulus *transp. FS* / circulus *om. P1* 7 axis est *om. O* / est *inter. S*; *om. L3C1* / est ortogonalis *ER* 8 quilibet *alter. ex* quibus *mg. F*; quibus *P1* / piramidis *corr. ex* piramidalis *a. m. E* / *post* sit *scr. et del.* equalis *C1* / latera: lata *F* 9 super *inter. a. m. E* 10 super: per *FP1S* 11 longitudinis: longitudini *P1* / super *corr. ex* superficiem *F* 12 *post* reflexionis[1] *add.* apud locum reflexionis *SC1*; *mg. a. m. E* / reflexionis[2] *om. L3* / reflexionis et *om. O* 13 perpendicularis: perpendiculari *O* / *post* lineam *scr. et del.* longidinis *P1* 14 piramidem *om. R* 15 *post* restat *add.* ut *FP1* 16 sectiones *corr. ex* sectionis *a. m. E* / piramidales *alter. in* piramidis *O* / et *om. OL3E*

munis linea longitudinis, ex quocumque puncto illius linee fiat reflexio erit in eadem superficie cum cuiusque alterius puncti reflexione. Quoniam a quolibet huius linee puncto ducta perpendiculari continget axem; et erunt in superficie reflexionis centrum visus, et punctum reflexionis, et punctum axis; et huius reflexionis est superficies in qua sunt linea longitudinis et axis, quare in hac superficie fit reflexio a quocumque puncto.

[5.43] Si vero communis linea non fuerit linea longitudinis, dico quod vel ab uno communis linee puncto in eadem superficie fiat reflexio, vel a duobus tantum. Quoniam ducta perpendiculari a puncto reflexionis, perveniet ad axem, et cadet in aliquod punctum eius. Intellecto circulo super punctum reflexionis, ortogonaliter secabit circulus axem, et cum perpendicularis secat axem equidistans basi, erit perpendicularis declinata super circulum. Et circumquaque ducta, semper erit equalis, unde fiet piramis cuius basis circulus, acumen punctus axis in quem cadit perpendicularis. Igitur superficies reflexionis aut tanget hanc piramidem aut secabit.

[5.44] Si tangat, dico quod a puncto reflexionis sumpto possit tantum fieri in eadem superficie reflexio. Planum quod superficies reflexionis continget hanc piramidem super perpendicularem, que est linea ortogonalis in superficie reflexionis, et si ab acumine totalis piramidis ducantur linee ad sectionem communem superficiei reflexionis et piramidi magne prius, cadent in circulum qui est basis piramidis intellecte quam in sectione preter unam que in punctum reflexionis cadit. Si ergo ab alio sectionis communis puncto fieret reflexio, linea ab illo puncto ad acumen intellecte ducta erit perpendicularis super lineam longitudinis piramidis per punctum illud transeuntem. Sed linea ab acumine piramidis intellecte ad punctum circuli per quem tran-

17 ex *om. O*/quocumque: cuicumque *O*; quecunque *L3*/fiat: fiet *L3C1* 18 in *inter. L3*/cum *inter. L3*; *inter. a. m. E*/cuiusque *corr. ex* cuiuscumque *P1*; cuiuslibet *C1*; cuiuscumque *ER* 19 ducta: producta *C1*/perpendiculari: perpendicularis *R*/continget *corr. ex* contingit *E* 20 et punctum reflexionis (21) *mg. a. m. C1*/punctum: punctus *P1* 21 et[2] . . . axis (22) *om. R* 22 sunt *om. FP1*; *inter. SOL3*; est *mg. a. m. C1*; est *E* 23 *post* puncto *add.* eius *C1* 25 quod *inter. a. m. E*/fiat *alter. in* fiet *O*; fiet *C1* 26 quoniam *inter. O* 27 cadet *alter. ex* cadat *in* cadit *E*/eius *om. O* 28 *ante* intellecto *add.* et *R*/ortogonaliter: ortogonalis *FP1S*/secabit *corr. ex* secat *E* 29 *post* axem[1] *add.* in termino *FP1*; *scr. et del.* in terminum perpendicularis in terminum axis cum *O*/et . . . axem[2] *om. O*/cum: quia *R*/perpendicularis secat *transp. L3*/secat . . . basi: tenet angulum acutum cum axe *R*/*post* secat *inter.* scilicet axem *a. m. E*/equidistans *corr. ex* equidistantem *C1*/erit perpendicularis (30) *om. C1* 31 erit *inter. a. m. C1*/punctus: punctum *R* 32 quem: quod *R*/aut *mg. a. m. C1* 33 tanget *alter. in* continget *O* 34 a *om. FP1* 35 *post* planum *add.* enim *R* 36 linea ortogonalis (37) *transp.*; ortogonalis *mg. L3* 37 ortogonalis in superficie *om. O*; *mg. a. m. E*/in superficie *inter. L3* 38 communem: commune *FP1* 39 piramidi magne: pyramidis totalis *R*/est *inter. a. m. E* 40 *post* intellecte *add.* quam *P1*/sectione: sectionem *R* 41 ergo: igitur *O*/alio: illo *E*/sectionis communis puncto: puncto communis sectionis *R* 42 *post* intellecte *add.* pyramidis *C1R* 43 sed *corr. ex* et *a. m. C1* 44 quem: quam *P1*; quod *R*

sit illa linea longitudinis absque dubio est perpendicularis super eam, quare alia angulum tenet acutum cum hac linea, non rectum.

[5.45] Si vero superficies reflexionis secet intellectualem piramidem, secabit circulum qui est eius basis in duobus punctis. Dico quod hec sola sunt puncta in tota sectione communi a quibus fieri possit reflexio in eadem superficie, quoniam ab utroque istorum punctorum linea ducta ad acumen intellecte piramidis est perpendicularis super lineam longitudinis super punctum suum transeuntem. A quocumque enim sectionis alio puncto ducatur linea ad acumen illius piramidis, tenebit angulum acutum cum linea longitudinis per ipsum transeuntem, cum perpendicularis cum eadem longitudinis linea angulum rectum teneat in circulo. Et linee ducte ab acumine intellecte piramidis ad puncta sectionis que intercidunt speculi acumen et circulum facient angulos obtusos cum lineis longitudinis versus partem acuminis piramidis totalis. Et que ducuntur ad puncta circulum et basem speculi interiacentia faciunt cum linea longitudinis angulos acutos ex parte acuminis speculi, obtusos ex parte basis.

[5.46] In speculis spericis concavis, si fuerit intra concavitatem speculi tota speculi superficies apparebit ei. Quod si extra fuerit visus, poterit comprehendere portionem eius maiorem medietatum quam scilicet fecerit circulus spere quem contingunt duo radii a centro visus ducti.

[5.47] Visu autem in centro huius speculi existente, non fiet ab aliquo puncto speculi reflexio nisi in se, quoniam quelibet linea a centro spere ad speram ducta perpendicularis est super superficiem speram in puncto illo tangentem. Unde in hoc situ non comprehendet visus per reflexionem nisi se tantum.

[5.48] Si vero statuatur visus extra centrum spere, poterit fieri reflexio in aliud corpus a quocumque speculi puncto preter quam ab eo in quem cadit dyametrum a centro visus ad speram per centrum spere ductus,

46 acutum *corr. ex* acumen *L3*; acumen *C1*/hac *om. P1* 48 eius *om. FP1R*/hec *rep. P1* 49 tota *om. P1*/fieri possit *transp. FP1*/possit: posset *L3C1* 51 perpendicularis: perpendiculariter *L3C1E* 53 alio puncto *transp. ER*/ad . . . linea (54) *mg. a. m. S*/*post* illius *scr. et del.* puncti *P1* 54 acutum *om. FP1SL3C1E*/transeuntem: transeunte *R* 55 perpendicularis: perpendiculari *P1*; *corr. ex* perpendiculariter *E* 56 intellecte piramidis *transp. R*/piramidis: piramidum *P1* 57 *post* intercidunt *add.* inter *R*/speculi *corr. ex* speculum *P1*/circulum: circulus *FP1S*; *corr. ex* circuli *O* 59 *post* puncta *add.* inter *R*/basem: basim *R* 60 faciunt: facient *R* 61 *post* basis *add.* ergo a nullo istorum punctorum potest fieri reflexio *R* 62 *post* si *add.* visus *R* 63 si *inter. a. m. E*/extra fuerit: sit extra *O*/visus *om. R* 64 medietatum: medietate *OC1R* 65 fecerit: fecit *R*/quem: quam *P1L3C1E*; *alter. ex* quam *in* quod *a. m. S* 67 *post* in *scr. et del.* co *F* 68 *post* quoniam *add.* enim *R* 69 ducta: ductam *FP1L3*/ducta . . . speram *mg. a. m. E*/superficiem *corr. ex* superficies *L3* 70 unde: ergo *R*/non *inter. OL3*/comprehendet: comprehendit *FP1S*; *corr. ex* comprehendat *O* 71 *post* nisi *scr. et del.* in *E* 73 in aliud corpus: alterius rei visibilis *R*/preter *inter. O*/preter quam: preterquam *R*/ab *corr. ex* in *O*/quem: quod *R* 74 dyametrum: diameter *ER*/ductus: ducta *R*

quoniam dyameter cadit super superficiem contingentem speram ortogonaliter. Sumpto autem alio puncto, ducatur ad ipsum dyameter a centro spere et linea a centro visus. Ex hiis ergo lineis acutus includetur angulus, quoniam linea visualis cadit inter dyametrum et superficiem contingentem punctum, que scilicet est extra speram. Et sive sit oculus intra speculum sive extra, cadit hec visualis linea intra speculum, quia cadit inter lineas visuales contingentes circulum portionis spere cum visus fuerit extra.

[5.49] Oculo cadente intra, planum quod intra cadit linea. Cum igitur dyameter angulum rectum teneat cum contingente, secetur ex eo acutus equalis predicto in eadem superficie. Dico ergo quod linea reflexionis cadit intra speculum, quoniam linea communis superficiei speculi et superficiei reflexionis est circulus tenens cum dyametro angulum acutum maiorem omni rectilineo acuto, et in singulis punctis erit hic modus reflexionis.

[5.50] Palam ex hiis quod in omni superficie reflexionis erunt centrum visus, centrum speculi, punctus reflexionis, punctus visus, terminus dyametri a centro visus per centrum spere ad speram ducti. Et communis omnium linea cum superficie speculi est circulus, et a quolibet linee communis puncto potest fieri in eadem superficie reflexio.

[5.51] In speculis columpnaribus concavis potest totum comprehendi speculum si fuerit visus intra ipsum. Sed eo extra sito, videbitur maior medietatum speculi, portio que scilicet interiacet duas superficies a centro visus procedentes columpnam contingentes.

[5.52] Intelligemus autem superficiem a centro visus procedentem basibus columpne equidistantem. Hec superficies aut cadet in columpnam aut non. Si ceciderit, linea communis huic superficiei et columpne erit circulus, et linea visualis transiens per centrum huius circuli cadet ortogonaliter super superficiem contingentem columpnam

75 dyameter *corr. ex* dyametrum *F*/*post* dyameter *scr. et del.* su *F*/*post* superficiem *scr. et del.* diamet *O*/*post* contingentem *scr. et del.* sumpto *O*; scilicet speram *inter. a. m. E*/speram *om. O* 76 sumpto *inter. O*/dyameter: diametrum *L3C1* 77 hiis *inter. O* 78 angulus: triangulus *C1E* 79 est *om. P1*/et *mg. a. m. C1*/sit: sumpsit *S* 80 *ante* intra[1] *scr. et del.* e *C1*/sive . . . speculum *mg. a. m.*; visualis linea *transp. L3*/hec *om. L3ER*/intra: inter *E* 81 cadit *om. C1*/*post* visuales *add.* spere *L3C1E* 82 visus . . . cum (83) *om. R*/extra *om. FP1*/*post* extra *add.* et *O* 83 *post* oculo *mg.* existente *a. m. C1*/cadente: existente *O* 85 acutus *corr. ex* actus *O* 86 linea communis *transp. ER*/*post* communis *scr. et del.* spe *S*/superficiei *om. R* 91 *post* visus[1] *scr. et del.* centrum visus *E*/punctus[1]: punctum *R*/punctus visus: punctum visum *R* 92 ad speram *om. R*/ducti: ducte *R*/et: palam quod *C1*/*post* et *add.* quod *R* 93 *post* omnium *add.* superficierum reflexionis *R*/est *om. L3E*/est circulus *transp.*; est *inter. a. m. S*/*post* et *add.* quod *R* 94 *post* quolibet *scr. et del.* et *O* 95 totum *om. E*/totum comprehendi (96) *transp. R* 96 intra: inter *L3E*/*post* intra *add.* speculum *P1*/*post* sed *add.* ex *L3*; *scr. et del.* ex *C1E*/sito: scito *P1* 97 medietatum: medietate *OR* 98 *post* procedentes *scr. et del.* a centro visus *F* 99 procedentem *corr. ex* procedentes *P1* 100 cadet: cadit *OE* 101 ceciderit *corr. ex* cecideret *O*

in puncto in quem cadit linea. Et fiet reflexio per eandem lineam ad eius originem.

[5.53] Quicumque alius sumatur punctus. Linea perpendiculariter ab hoc puncto ducta cadet in axem, et linea visualis in punctum illud cadens faciet angulum acutum cum linea perpendiculari, cum sit inter perpendicularem et contingentem. Et quia hec linea cadet intra speculum, planum ex hoc quod cadit inter superficies portionem apparentem contingentes. Poterimus igitur in eadem reflexionis superficie ex angulo quem facit perpendicularis cum contingente excipere angulum acutum equalem angulo predicto acuto. Et cadet linea reflexionis hunc angulum continens intra columpnam, quoniam cadet inter perpendicularem et lineam longitudinis per terminum perpendiculariter transeuntem. Erunt igitur in superficie reflexionis centrum visus, punctum reflexionis, punctum visum, punctum axis in quem cadit perpendicularis.

[5.54] Et si hoc modo statuatur visus ut communis linea superficiei reflexionis et superficie columpne sit linea longitudinis, a quocumque puncto communis linee fiat reflexio. In una et determinata erit superficie omnibus hiis reflexionibus communi—ea scilicet in qua centrum visus et axis columpne totus—sicut dictum est superius in columpnari speculo non concavo.

[5.55] Similiter, si linea communis fuerit circulus, omnes reflexiones a punctis illius circuli facte procedent in eadem superficie, sicut in aliis circulis patuit.

[5.56] Et si sectio columpnaris fuerit linea communis, a duobus quidem eius punctis tantum fiet reflexio in eadem superficie, licet in superioribus columpnis circulus tantum ab uno puncto in unica superficie fieret reflexio, unico visu adhibito, quoniam supra latebant visum puncta sectionis se respicientia per que scilicet transit circulus columpne equidistans basibus. Viso enim uno latebat alius propter

104 quem: quod *R*/fiet *corr. ex* fiat *a. m. E* 106 quicumque alius: quodcumque aliud *R*/ punctus: punctum *R*/perpendiculariter: perpendicularis *S* 108 faciet *corr. ex* faciens *P1* 109 quia: quod *C1ER*/cadet: cadat *R*/*post* cadet *add.* in punctum *P1*/intra: inter *OL3E* 110 *ante* planum *add.* est *C1*/*post* planum *add.* est *R*/apparentem *om. R* 111 poterimus: perimus *O*/ex *mg. a. m. C1* 113 angulo *om. O*/predicto acuto *transp. R* 114 cadet *corr. ex* cadit *E*/inter: intra *E* 115 perpendiculariter *alter. in* perpendicularis *a. m. S*; perpendicularis *R* 117 quem: quam *O*; quod *R*/perpendicularis: perpendiculariter *L3* 119 superficie: superficiei *C1ER*/sit: fit *FP1S* 120 *post* una *scr. et del.* in una *F*/et *inter. E*; *om. R*/superficie *corr. ex* superficiei *L3* 121 *post* communi *inter.* in *O*/*post* qua *inter.* est *a. m. S*; *inter. L3* 122 superius *om. P1* 123 non *mg. a. m. L3*; *inter. a. m. E* 124 si *inter. L3*; *inter. a. m. E* 125 *post* circuli *scr. et del.* facte *F* 126 circulis *alter. in* speculis *OL3* 128 punctis *corr. ex* punctus *O* 129 circulus *om. R* 130 fieret *corr. ex* fierit *E*/reflexio *corr. ex* reflecto *O*/*post* adhibito *add.* quod *C1*/quoniam: quem *FL3*; qui *S*; *corr. ex* quod *E*/supra: illic *R*/*post* supra *add.* quem supra *FP1O*/latebant *corr. ex* latebunt *E* 131 puncta sectionis *corr. ex* punctionis *O*/ respicientia *corr. ex* respicienda *O*/scilicet *om. FP1C1* 132 equidistans basibus *transp. R*/*post* uno *add.* illorum punctorum *R*/latebat *corr. ex* latebit *E*/alius *om. P1*; *alter. ex* visus *E*; aliud *R*

minoris columpne portionis apparentiam, sed in hiis apparet maior columpne portio, unde ab unico visu percipiuntur puncta circuli equidistantis basibus et sectionis communis.

[5.57] In speculis piramidalibus concavis, si fuerit visus intra speculum, videbit ipsum totum. Si vero extra, et linea a centro visus ad acumen piramidis ducta intret piramidem aut applicetur linee longitudinis piramidis, nichil videbitur ex speculo. Quoniam quecumque alia linea ab oculo ad piramidem ducta cadet in piramidis superficiem exteriorem, unde occultabitur interior superficies.

[5.58] Si autem auferatur portio a piramide, poterit videri pars piramidis cadens inter contingentes superficies a centro ductas, scilicet maior, et si linea a centro visus sit perpendicularis super superficiem contingentem piramidem et continuetur axi, erunt linee communes, sicut dictum est in aliis piramidalibus, aut linee longitudinis piramidis aut sectiones. Et in hiis a duobus punctis sectionis poterit reflexio in eadem superficie respectu eiusdem visus; et in superficie reflexionis erunt centrum visus, punctus visus, punctus reflexionis, punctus axis.

[5.59] Sed speculum piramidale integrum, si apponatur visui, et sit visus ex parte basis, non percipiet nisi hoc quod fuerit intra speculum, quoniam perpendicularis tenet angulum acutum cum linea ab oculo ad ipsam ducta ex parte basis. Unde fit reflexio ex parte acuminis, et cadent omnes linee reflexe intra piramidem, et videri poterit quod intra piramidem positum sit.

[5.60] Si autem auferatur ex eo portio secundum longitudinem, poterunt quidem comprehendi exteriora, cum pateat exitus lineis reflexionis. Similiter, si secetur piramis ad modum anuli ut auferatur conus, liberum habebunt linee gressum, et exteriora apparebunt. Et si fuerit visus ex parte concavi, plura poterit comprehendere exteriora quam ex parte basis, quia latior reflexis lineis datur ad egrediendum via.

134 unico: uno *ER*/percipiuntur *alter. ex* percipiantur *in* percipientur *O*/*post* puncta *add.* terminantia diametrum *R*/circuli equidistantis (135) *corr. ex* circulo equidistantibus *O*; circulo equidistanti *L3C1E* 135 basibus: ba *P1*/*post* basibus *add.* columpne *R*/sectionis communis: sectioni communi *L3C1E*; *om. R* 139 *post* ex *inter.* hoc *L3* 140 ad *corr. ex* et *a. m. C1* 144 sit *om. O* 146 est *om. FS*/piramidalibus: piramidibus *SOL3*/piramidis: piramidum *P1R* 147 sectiones *corr. ex* sectionis *E*/*post* sectiones *add.* pyramidales *C1*/a *scr. et del. E*/*post* poterit *add.* fieri *R*/*post* reflexio *add.* fieri *O* 148 reflexionis *om. E* 149 erunt . . . punctus[2] *mg. a. m. E*/centrum: centra *FP1SOL3C1*/punctus visus punctus: punctum visum punctum *R*/*post* visus[2] *scr. et del.* et in superficie reflexionis erunt centra visus punctus visus *S*; *add.* et *E*/punctus[3]: punctum *R*/*post* axis *add.* in quod cadit perpendicularis *R* 150 integrum si *transp.*; integrum *mg. a. m. E*/apponatur *alter. in* opponatur *E*; opponatur *R* 151 hoc *om. O*/fuerit: sit *L3*/*post* speculum *scr. et del.* quoniam perpendicularis nisi quod fuerit intra speculum *O* 155 sit: est *R* 157 quidem *om. O* 159 conus *alter. in* capud *O*; vertex *R*/linee *om. P1*/gressum: ingressum *R* 160 concavi *corr. ex* coni *L3C1* (*a. m. C1*); superficiei concavitatis speculi *R* 161 *post* latior *scr. et del.* est *L3*; *add.* est *C1*/reflexis: reflexionis *O*; incidentibus *R*/lineis datur *transp. ER*/ad egrediendum *om. R*

[5.61] Amplius, sumpto uniuscuiusque speculi puncto, non est possibile in eo percipi formam nisi formam unius puncti ab eodem visu. Quoniam super perpendicularem et centrum visus unica transit superficies, et una sola est linea a centro visus ad punctum, et unicus angulus ex linea et perpendicularis acutus, et unicus angulus in eadem superficie acutus equalis huic, unde unica linea angulum equalem huic cum perpendiculari faciens. Et cum linea pervenit ad punctum corporis, non potest forma alterius puncti per ipsam vehi, cum punctum precedens occultet postpositum. Sed duobus visibus possunt in eodem speculi puncto comprehendi due punctales forme, quoniam infinite possunt sumi superficies super perpendicularem secantes in quarum qualibet circa perpendicularem sumi poterunt duo anguli equales acuti.

[5.62] Iam ergo proprietatem reflexionis declaravimus, et similiter cuiuslibet speculi proprium. Visus, cum per reflexionem formas comprehendit, non advertit quod hec adquisitio per reflexionem sit. Non enim accidit ex proprietate visus reflexio, quoniam, visu remoto, procedit non minus forma a corpore ad speculum et reflectetur secundum modum predictum. Et si accidit visum esse in loco in quem linearum reflexarum fit agregatio, comprehendet visus formam illam in capitibus harum linearum, et est in speculo tamquam non adveniens sed naturalis esset forma in speculo. Amplius, aliquando adquirit visus formas in speculis in sola superficie, aliquando intra speculum, aliquando ultra. Et erit apparens locus forme secundum figuram speculi et secundum situm rei vise, et semper comprehenditur forma in loco proprio, mutato situ visus et speculi. Et erit diversitas elongationis loci forme ad speculi superficiem secundum diversitatem figure speculi. Et locus forme dicitur locus ymaginis, et forma dicitur ymago. Visus autem comprehendit rem visam in loco ymaginis, et nos dicemus locum illum et eius proprium in quolibet speculorum, que numerabimus et

162 uniuscuiusque: cuiuscumque *O*/speculi *inter. E* 163 formam[1] *om. C1*/nisi formam *om. L3* 164 super: enim per *R* 165 ad punctum *inter. L3*; *inter. a. m. C1* 166 *post* angulus[1] *scr. et del.* in eadem superficie *S*/et[1] *inter. P1*; *om. R*/perpendicularis: perpendiculari *R*/*post* acutus *add.* est *C1*/et[2] *inter. P1*; *corr. ex* est *L3* 167 unde *inter. a. m. E*; ergo est *R*/*post* linea *add.* que *R*/*post* huic *scr. et del.* per *S* 168 faciens: facit *R*/pervenit *alter. ex* venit *C1*; perveneritt *R*/punctum: partem *R* 170 precedens: procedens *FP1SO* 171 due *mg. a. m. C1*/punctales: pictales *F*; punctuales *R* 172 perpendicularem: superficiem *P1*/*post* perpendicularem *add.* se *R*/*post* secantes *add.* se *O* 173 qualibet: quelibet *E* 174 ergo: igitur *FP1*/*post* reflexionis *scr. et del.* ut *L3C1* 175 *post* speculi *inter.* est *a. m. O*/*post* proprium *add.* est *C1*; *inter.* est *a. m. E*/*post* visus *inter.* est *L3*; est *mg. a. m. S*; *add.* autem *R*/cum: tamen *P1L3*/cum . . . visus (177) *om.O* 176 comprehendit: comprehenderit *E*/advertit: animadvertit *R* 177 remoto *corr. ex* remotu *a. m. C1* 178 reflectetur: reflectitur *R* 179 accidit: accidet *R*/in loco in quem: aliquam *O* 180 *post* linearum *scr. et del.* reflarum *P1*/fit: sit *L3C1* 182 in *om. OC1ER*; *inter. L3* 185 secundum *om. R*/comprehenditur: comprehendetur *R* 186 *post* proprio *add.* et immutabili et *O*/mutato: mutatu *FS*; immutato *O*; *corr. ex* mutatio *L3*; *alter. in* immutato *a. m. C1* 187 loci *corr. ex* locum *O*/superficiem *corr. ex* superficiei *O* 189 dicemus *corr. ex* dicamus *O*; dicamus *L3E*/locum illum (190) *transp. R* 190 numerabimus: enumerabimus *E*; enumeravimus *R*

assignabimus causas comprehendi res visas in loco illo, et hoc in sequenti libro, si deus voluerit.

191 *post* causas *add.* propter quas *R*/comprehendi *alter. in* comprehendendi *a. m. E*; comprehendantur *R*/visas: vise *R*/hoc: hee *FS* 192 sequenti: sequente *R*/*post* sequenti *scr. et del.* loco *F*

[QUINTUS TRACTATUS]

Liber iste in duas partes partitus est. Prima pars est prohemium libri; secunda in ymaginibus.

[CAPITULUM 1]

Prima pars

[1.1] Liquet ex libro quarto quod forme rerum visarum reflectuntur ex corporibus politis, et visus adquirit eas in corporibus politis propter reflexionem. Et patuit quomodo fiat adquisitio rerum ex reflexione formarum. Et visus comprehendit rem visam in loco reflexionis determinato et primo cum non fuerit situs rei vise ad visum mutatio. Et forma in corpore polito comprehensa nominatur ymago. Et nos explanabimus in hoc libro loca ymaginum ex corporibus politis, et dicemus quomodo adquiratur horum locorum scientia, et quomodo inveniantur sillogistice, et demonstratur.

[CAPITULUM 2]

Pars secunda: loqui in ymaginibus.

[2.1] Ymaginis cuiuscumque puncti locus est punctus in quo linea reflexionis secat perpendicularem a puncto rei vise intellectam super lineam contingentem lineam communem superficiei speculi et

1 liber . . . ymaginibus (2) *om. FP1*/*post* prima *scr. et del.* est *L3* 3 prima pars *om. FP1O* 4 libro quarto *transp. ER* 5 adquirit: comprehendit *R*/*post* politis[2] *scr. et del.* et visus *S* 6 et *corr. ex* quod *O*/fiat *inter. a. m. F*; fieret *R* 7 reflexionis *om. R* 9 in . . . comprehensa: comprehensa in corpore polito *R* 12 inveniantur: inveniuntur *C1*/sillogistice: ille *O*; sillogismo *L3*; *corr. ex* sillogismo *a. m. S*; *corr. ex* simile *a. m. E*/*post* sillogistice *add.* vel sillogismo *C1*/et demonstratur *om. R*/demonstratur: demonstrantur *F*; *alter. ex* demonstrative *in* demonstrationis *O* 13 pars . . . ymaginibus *om. OR* 14 punctus: punctum *R* 15 a *om. S*/a puncto *mg. F* 16 et . . . reflexionis (17) *om. ER*

superficiei reflexionis aut superficiei speculo continue et superficiei reflexionis. Et nos declarabimus.

[2.2] Sumatur speculum planum, et statuatur equidistans orizonti, et lignum directum et politum ortogonaliter erigatur supra speculum. Et sit speculi quantitas ut totum possit videri lignum, nisi enim totum apparuerit, error inerit. Et signetur in ligno punctum aliquod nigrum. Apparebit quidem visui lignum huic equale ultra speculum huic ligno continuum et ortogonale supra speculum, et in ligno apparenti apparebit punctus signatus tantum distans a superficie speculi quantum ab eadem distat in ligno superiori. Et si declinetur lignum supra speculum, apparebit apparens eadem declinatione declinatum, et punctus signatus in apparenti signato eque remotus a superficie speculi. Et si a puncto signato lignum aliquod erigatur ortogonaliter supra speculum, videbitur hoc etiam lignum a puncto apparenti ortogonaliter supra speculum et huic ortogonali continuum. Idem accidit pluribus punctis in ligno signatis. Idem penitus accidet elevato aut depresso speculo.

[2.3] Planum ergo per hoc quod ymago puncti visi apparet in perpendiculari ducta a puncto viso ad superficiem speculi, et in hoc speculo que perpendicularis est super superficiem speculi est perpendicularis super lineam communem superficiei speculi et reflexionis.

[2.4] Idem patere poterit in piramide super basim ortogonali, cuius basis plana speculo plano ortogonaliter sit adhibita, apparebit enim huic piramis alia continua quarum eadem basis et harum piramidum acumina equaliter a speculo distantia. Et planum quod, si ab acumine

17 aut . . . continue *mg. a. m. E*; aut: et *E*; vel *R*/*post* aut *scr. et del.* sp *F*/et *inter. O*/et superficie *om. FP1* 18 *post* nos *add.* hec *R*/*post* declarabimus *add.* hoc *C1* 19 sumatur *om. O*/statuatur *alter. in* statuantur *P1* 20 ortogonaliter *om. R*/supra: super *FP1R* 21 possit: posset *O*/nisi *inter. L3* 22 apparuerit: appareat *ER*/in *inter. L3C1E* (*a. m. C1*) 23 huic equale *transp. R* 24 ortogonale: ortogonalis *E*/apparenti: apparente *R*/*post* apparenti *scr. et del.* apparenti *E* 25 punctus signatus: punctum signatum *R* 26 ligno *corr. ex* loco *a. m. F*/superiori: superiore *R* 28 punctus signatus: punctum signatum *R*/in *om. O*; *inter. L3E*/*ante* apparenti *inter.* erit *O*/apparenti: apparente *R*/apparenti signato: apparien scilicet signatus *O*/signato *om. R*/*ante* eque *add.* apparebit *ER*/remotus: remotum *R*/*post* remotus *add.* vel speculum apparens signatus eque remotus *C1* 29 supra: super *FP1* 30 videbitur . . . speculum (31) *om. FP1*/hoc etiam *transp. ER*/etiam lignum: in ligno *C1*/lignum *corr. ex* ligno *OL3*/apparenti: apparente *R* 31 accidit: accidet *R* 32 idem: ? *C1*; idemque *R*/penitus: punctus *S*; *corr. ex* punctus *OL3* 33 speculo: speculi *FSO* 34 *post* ergo *scr. et del.* quod *F*/per *om. P1*/per hoc *transp. F*/apparet *om. FP1*/in *inter. L3E*/*post* in *inter.* nisi *a. m. O* 35 a *corr. ex* in *O*/ad *corr. ex* a *O*/in *om. FP1* 36 *post* speculo *add.* linea *C1*/est[1] *om. FP1* 37 *post* et *inter.* superficiei *a. m. E*; *add.* superficiei *R* 39 poterit: potest *R*/in *om. O*/piramide *corr. ex* piramidis *O*/ortogonali *corr. ex* ortogonalis *L3* 40 ortogonaliter sit *transp. R*/sit *inter. SO*; *om. L3*; est *inter. a. m. C1*/sit adhibita *transp. E*/*post* enim *inter.* in *a. m. E* 41 alia *om. O*/*post* alia *add.* linea *C1*/quarum . . . basis *om. R*/*post* quarum *scr. et del.* ut *E*/*post* et *scr. et del.* et *O*/harum *alter. in* earum *O*/*post* piramidum *add.* basis eadem et *R* 42 acumina *mg. a. m. S*/*post* acumina *add.* ipsarum *R*

ad acumen ducatur linea recta, erit perpendicularis super basim, et ita super speculum, cum eadem sit superficies speculi et basis, quare conus piramidis in perpendiculari videbitur ab eo ad speculum ducta. Similiter a quocumque puncto piramidis ducatur linea ad punctum respiciens ipsum in apparenti piramide. Erit linea ortogonalis super basim et super speculi superficiem, quia ymago cuiuscumque puncti piramidis cadit in perpendiculari intellecta a puncto illo in speculi superficiem.

[2.5] Sed quicumque corporis punctus opponatur speculo plano est intelligere piramidem cuius punctus ille sit conus, que quidem piramis super basim ortogonalis, et etiam super speculi superficiem aut ei continuam. Et est intelligere aliam huic piramidi oppositam quarum basis eadem et super speculum ortogonalis, et perpendicularis a cono ad conum ortogonalis erit supra speculum, quare ymago cuiusque puncti speculo quidem oppositi cadit in perpendiculari a puncto ad speculi superficiem aut ei continuam. Sed planum quod in speculis non accedet comprehensio formarum nisi per lineas reflexionum, quare ymago puncti visi cadit in lineam reflexionis, et quelibet talis linea est recta, quare ymago cuiuscumque puncti cadit in punctum sectionis perpendicularis ab illo puncto in superficiem speculi et linee reflexionis. Et in speculis planis cadit, et unica est linea communis superficiei speculi et superficiei reflexionis cum linea contingente locum reflexionis, quare planum quod in speculis planis proprius ymaginis est locus punctus sectionis perpendicularis a puncto visi super lineam contingentem communem lineam superficiei speculi et superficiei reflexionis et linee reflexionis.

44 speculi *om. FP1SL3*; *inter. E*/quare: quoniam *P1*/conus (45): vertex *R* 45 in . . . videbitur: videbitur in perpendiculari *O*/ducta . . . quocumque (46) *rep. F* 46 *post* quocumque *scr. et del.* ducta *F*; *add.* ducta *P1*/piramidis: piramis *O*/*post* linea *add.* equidistans axi cadet *R* 47 ipsum . . . piramidis (49) *mg. a. m. E*/apparenti: apparente *R*/piramide . . . puncti (48) *mg. a. m. L3*/*post* piramide *add.* et *R*/*post* linea *add.* illa *OL3C1ER*/ortogonalis: perpendicularis *ER* 48 basim: basem *O*/quia: quare *OL3C1ER*/cuiuscumque: cuiusque *SE*; cuiuslibet *L3C1* 49 *ante* cadit *scr. et del.* maxime *E*/perpendiculari intellecta: perpendicularem intellectam *OR* 51 quicumque: quodcumque *R*/corporis: corpus *P1*; *om. R*/punctus: punctum *R*/opponatur: apponatur *SL3E* 52 *post* intelligere *add.* illam *C1*/punctus ille: punctum illud *R*/sit *om. R*/conus: vertex *R* 53 basim *corr. ex* bases *O*/*ante* ortogonalis *add.* est *O*/*post* ortogonalis *add.* est *R* 54 quarum: quare *E*; *corr. ex* quare *L3C1* 55 basis: basisis *F*/et[1] . . . ortogonalis *om. R*/ortogonalis: ortogonale *FP1*; ortogonalem *E*/cono: vertice *R* 56 conum: verticem *R*/ortogonalis *corr. ex* ortogonale *P1*/supra: super *R*/cuiusque: cuiuscumque *SL3ER*/cuiusque . . . ymago (60) *om. O* 57 quidem *om. R*/perpendiculari: perpendicularem *R*/*ante* a *add.* ductam *R* 58 *post* planum *add.* est *R*/speculis: speculo *FS* 59 accedet: accidet *S*; accidit *L3C1R* 61 *post* ymago *scr. et del.* puncti visi *S*/cuiuscumque puncti *transp. P1* 62 *post* perpendicularis *add.* ductae *R*/illo puncto *transp. P1*/in: ad *R* 63 cadit *scr. et del. OL3E*; *om. C1*/cadit . . . est *om. R*/*post* communis *scr. et del.* in *C1*/superficiei *corr. ex* superficie *C1* 64 *post* reflexionis[1] *add.* est una linea *R* 65 *post* speculis *scr. et del.* sperus *O*/planis *om. P1*/proprius: propriis *L3*; *om. R*/est locus *transp. R*/punctus: punctum *R* 66 visi: viso *L3C1ER* 67 et[2] . . . reflexionis (68) *inter. L3*; *mg. a. m. C1E*

[2.6] In speculis spericis et extra politis patebit quod diximus. Queratur superficies speculi talis magna in qua appareat forma baculi gracilis perpendiculariter erecta super ipsum. Apparebit quidem forma baculi baculo continua, et apparebit in forma baculi punctus signatus distans a superficie speculi secundum distantiam eius ab eodem in baculo. Et si fuerit baculus gracilior ex parte unius capitis quam ex parte alterius, apparebit quidem in hoc speculo forma eius piramidalis, et est error visus quem postea assignabimus.

[2.7] Amplius, fiat piramis cum eo ortogonalis super basim circularem circulatione perfecta, et applicetur etiam huic speculo. Videbitur quidem piramis huic continua super eandem basim erecta, sed minor ista. Quod appareat piramis planum per hoc quod omnes linee ab apparenti ymagine coni ad circulum basis videantur equales, et si declinetur piramis modicum supra speculum a situ in quo tota videtur, ut scilicet aliquid ex eo abscondatur, dum tamen locus reflexionis in speculo visui exponatur, apparebit inde ymago piramidis. Et si elongetur visus a speculo aut accedat dum tamen super lineam a loco reflexionis ad ipsum protractatum cedat, comprehendetur ymago piramidis, sed et accessus vel recessus secundum hanc lineam erit ut notetur locus reflexionis. Et a nota ad locum visus ducatur linea secundum quam processus fiat.

[2.8] Verum quoniam ymago piramidis ortogonalis super basim piramidis, et basis est circulus ex circulis in spera, erit linea a cono piramidis ad conum ymaginis ducta ortogonalis super circulum illum, et transibit per centrum eius. Et erit ortogonalis super speram et transibit per centrum spere, et erit ortogonalis super superficiem speram contingentem in puncto per quem transit hec linea. Et erit similiter

69 et *om. R* 70 *post* forma *scr. et del.* basis *P1* 71 erecta: erecti *R* 72 et *inter. a. m. E* / punctus signatus: punctum signatum *R* 74 si *om. L3E*; *inter. a. m. C1* / fuerit: fuit *L3* 75 *post* alterius *scr. et del.* sed non *E* 76 et *inter. a. m. C1* / est error: ex toto *O* / est error visus *corr. ex* ex toto visu *L3* 77 cum eo *om. R* / basim *corr. ex* basum *O* 78 *post* perfecta *add.* et concavum *O* / etiam *om. O* 79 basim: basem *OE* / erecta: recta *SC1*; *corr. ex* recta *O* 80 *post* quod[1] *add.* autem *R* / *post* planum *add.* est *R* / omnes *corr. ex* omnis *P1* 81 apparenti: apparentis *P1*; *alter. in* apparentis *C1*; apparente *R* / ymagine *alter. in* ymaginis *OC1* / coni *alter. ex* conu *in* cono *O*; verticis *R* / basis *corr. ex* bases *O* 83 eo: ea *P1R* 84 visui: visu *F* / *post* apparebit *add.* etiam *R* / piramidis *corr. ex* paramidis *O* 85 tamen *inter. E* / a loco (86) *om. P1* 86 reflexionis *om. R* / protractatum: protractum *S*; pertractatam *C1* / cedat: incedat *O*; cadat *ER* 87 et *om. FP1R* / erit: exit *P1S*; erat *C1* 88 notetur: vocetur *FP1SL3* 89 processus fiat *transp. OL3C1ER* 90 quoniam: quando *O* / *post* piramidis *inter.* est *O* / ortogonalis *corr. ex* ortogono *L3* / *post* ortogonalis *add.* est *P1R* 91 basis: basi *F* / *post* est *scr. et del.* ? *O* / in *om. O* / spera *alter. in* spere *O* / a: in *FSL3E*; *mg. a. m. C1* / cono: vertice *R* 92 conum: verticem *R* / ducta ortogonalis *transp. O* / ortogonalis: ortogonaliter *E* 93 et[2] . . . spere (94) *mg. S* / super . . . ortogonalis (94) *mg. a. m. L3* / speram . . . super (94) *mg. a. m. E* / et[3] . . . speram (94) *om. P1* 94 spere: eius *F*; *corr. ex* speres *C1* / superficiem *om. F* 95 quem: quod *R*

ortogonalis super lineam contingentem circulum spere per punctum illum transeuntem, et hec contingens est linea communis superficiei reflexionis et superficiei contingentis speram in puncto illo, et hec linea est contingens circulo spere communi superficiei spere et superficiei reflexionis. Linea ergo a cono piramidis ad conum ymaginis ducta est perpendicularis super lineam contingentem lineam communem superficiei reflexionis et superficiei speculi, que quidem est circulus.

[2.9] In hac igitur perpendiculari videtur ymago coni, et planum quod ymago coni est in linea reflexionis, quare comprehendetur ymago coni in concursu linee reflexionis et perpendicularis a cono ad speram ducte sive ad contingentem circulum communem superficiei spere et reflexionis. Sumpto autem quocumque puncto huic speculo opposito est intelligere piramidem super superficiem speculi ortogonalem aut super continuam, cuius conus sit punctus sumptus. Et linea ab illo puncto ad ymaginem puncti illius erit in superficie reflexionis et perpendicularis super superficiem speculi vel ei continuam modo predicto, quoniam punctus visus et ymago eius semper sunt in superficie reflexionis, quare et linea a puncto viso ad eius ymaginem ducta.

[2.10] In speculis columpnaribus exterius non apparent que in ligno et piramide diximus, quoniam recta in hiis speculis videtur non recta, et est error visus cuius postea causam assignabimus. Accidit tamen in solo corporis puncto videre locum ymaginis predictum.

[2.11] Verbi gratia, adhibito precedentis libri instrumento, immittatur regula cui sit infixum columpnare speculum ut media portionis speculi linea sit in superficie regule. Et non transeat hec regula tabulam eneam sed super ipsum cadat ortogonaliter ita quod altitudo regule sit super lineam dividentem triangulum tabule enee. Erectio facta in hac tabula,

97 illum: illud *R*/*post* hec *add.* linea *C1* 98 contingentis: contingenti *SOL3C1ER*/*ante* speram *scr. et del.* et *L3* 99 est contingens: contingit *R*/*post* contingens *add.* a *P1*/circulo: circulum *ER*/ communi: communem *R*/spere et superficiei *om. FP1* 100 *post* reflexionis *scr. et del.* et *L3C1*/ergo *inter. a. m. E*/*ante* a *add.* erit *E*/cono: verticem *R* 102 quidem: quasi *L3* 103 in *inter. OL3*; *om. E*/perpendiculari *corr. ex* perpendicularis *L3*/coni: verticis *R* 104 ymago[1] *om. R*/coni: verticis *R*/ est *rep. P1*/*post* quare *scr. et del.* consistit *P1*/comprehendetur: comprehenditur *OC1E* 105 coni: verticis *R*/cono: vertice *R* 106 et *inter. L3E* (*a. m. E*) 107 *ante* reflexionis *add.* superficiei *R*/ autem: *scr. et del.* a *L3* 109 super *om. OL3E*/*post* continuam *scr. et del.* erit *O*; *add.* ei *R*/conus: vertex *R*/punctus sumptus: punctum sumptum *R* 110 puncti illius *transp. L3C1* 111 vel ei *corr. ex* in *a. m. S*/ei *om. O*; *inter. L3*/ei continuam *transp. C1* 112 punctus visus: punctum visum *R*/eius *om. R*/*post* semper *scr. et del.* habuerit fuit *P1*/*post* sunt *inter.* sicut *a. m. E*; *add.* simul *R* 113 et *inter. P1*/eius *om. O* 114 *post* ducta *add.* erit in superficie reflexionis *C1* 115 exterius: extremis *L3*/*post* exterius *add.* politis *C1R* 117 *post* visus *add.* non communis *FP1*; *add.* communis *alter. in* non communis *S*; *scr. et del.* communis *L3C1*; *add.* communis *ER*; *scr. et del.* communium *O*; *inter. c*ausam *O*/causam *om. O* 118 predictum *inter. a. m. E* 119 verbi gratia: hoc modo *R* 120 ut: in *F*/portionis: proportionis *FP1* 122 ipsum: ipsam *L3ER*/cadat: cadit *L3E*; *corr. ex* cadit *C1*/quod: ut *R* 123 erectio *alter. in* reflexio *L3E*; erectione *R*/*post* erectio *add.* vel reflexio *C1*

impleatur cera, et inducatur ei planities ut sit in eadem superficie cum tabula, et est ut certior fiat ortogonalis regule directio super tabulam.

[2.12] Deinde queratur regula acuta, et acuatur extremitas, et applicetur huius regule acuitas medie superficiei anuli linee. Et descendat super hanc lineam, et ubi ceciderit super regulam fiat signum. Postea acus descendat super hanc lineam in qua sit infixum modicum corpus album, et hoc in termino, nec descendat acus usque ad regulam.

[2.13] Adhibeatur autem visus ut sit in superficie regule, et claudatur unus visuum. Videbitur quidem ymago corporis super lineam a puncto signato ad acumen acus protractam, que quidem linea perpendicularis est super superficiem regule, que superficies tangit columpnam in linea longitudinis; et est perpendicularis super lineam longitudinis columpne, que est in superficie regule et est linea communis superficiei columpne et superficiei reflexionis. Et in superficie reflexionis sunt linea longitudinis et linea perpendicularis.

[2.14] Et si situs visus mutetur, et circa anuli superficiem visus volvatur, apparebunt sicut prius, et in eadem linea corpus et ymago corporis et acus. Et est linea illa perpendicularis super mediam longitudinis columpne lineam, et est hec perpendicularis in superficie reflexionis, quoniam superficies anuli secat columpnam super circulum equidistantem basi columpne, et in hac superficie est visus. Et nos probabimus postea quod, quando visus et visum corpus fuerint in superficie equidistanti basi columpne, illa est superficies reflexionis. In hoc autem situ linea communis superficiei columpne et superficiei reflexionis est circulus, et perpendicularis in qua videntur ymago et corpus ortogonaliter cadit super lineam super circulum contingentem.

[2.15] Hiis peractis, auferatur acus a loco suo, et ponatur regula acuta super lineam mediam ita quod cadat super mediam longitudinis regule lineam, et adhibeatur regula acuta superficiei anuli cera firmiter.

124 *post* ut *inter.* non *L3* 127 huius: huiusmodi *E* 128 super[1]: secundum *SOL3C1R*/hanc *inter. L3E* (*a. m. E*)/hanc lineam *transp. ER*/super[2]: secundum *L3C1* 129 super: secundum *F*; *scr. et del. C1*/super . . . lineam *om. R*/*post* lineam *add.* secundum *C1*/sit infixum *transp. R* 130 nec: ne *R*/*post* nec *add.* ut *L3C1E* (*inter. L3*)/*post* ad *scr. et del.* terminum *P1* 131 claudatur: clauda *FP1* 132 quidem: quid *O*; *corr. ex* quod *L3*; *alter. in* quod *E*/*ante* ymago *scr. et del.* h *O* 133 perpendicularis est (134) *transp. O* 134 est *inter. OL3C1* (*a. m. OC1*); *om. E* 135 et . . . longitudinis *om. S* 136 superficiei: superficie *FP1*; *corr. ex* superficie *O*/columpne *corr. ex* reflexionis *a. m. E*; regulae *R* 137 superficiei: superficie *FP1*/et[2] . . . sunt *mg. a. m. E* 139 si *om. FP1* 140 et[2]: ht *O* 141 *ante* acus *scr. et del.* est *E*/illa *inter. a. m. E* 142 est hec *transp. ER* 143 *post* super *scr. et del.* super *C1* 144 *post* nos *scr. et del.* cerpu *P1* 145 probabimus: probamus *FP1*/postea *om. C1*/*post* et *scr. et del.* et *C1* 146 equidistanti: equidistante *R*/est *corr. ex* et *O*/*post* reflexionis *scr. et del.* est circulus *E* 147 in . . . reflexionis (148) *mg. a. m. L3*/superficiei[1] . . . reflexionis (148): superficiei reflexionis et superficiei columpne *O* 148 *post* reflexionis *scr. et del.* et superficies reflexionis *C1*/videntur: videtur *R*/et[2] *inter. OC1E* 149 cadit: cadunt *R*/super: hunc *OER* 151 *post* lineam *add.* annuli *R*/quod: ut *R*/*post* quod *add.* non *FP1*/mediam[2] *corr. ex* lineam *a. m. E* 152 regula *inter. a. m. E*

Postea auferatur regula in qua est speculum, et accipiatur regula acuta, et applicetur eius acuitas medie longitudinis regule linee, et secundum processum acuitatis fiat cum incausto super speculum protractio. Post sumatur triangulus cereus modicus cuius unum latus sit equale altitudini regule in qua est speculum, et sit altitudo huius trianguli moderata, et superficies huius trianguli sit plane pro posse. Et adhibeatur columpne regule tabule ceree triangulus firmiter sub base regule, et latus eius equale altitudini regule ponatur super latus basis regule. Cum ita fuerit, erit huius trianguli altitudo super basem columpnalis equalem tabule regule, et ut efficiatur superficies plana ad modum superficiei regule, includatur triangulus inter regulam et superficiem planam, et comprimatur donec sit bene planitus. Et super superficiem huius trianguli ponatur regula acuta, et secetur finis huius trianguli cum acuitate regule, et erit finis eius linea recta. Et erit linea hec basis regule in qua est speculum.

[2.16] Postea ponatur regula super superficiem tabule que est in instrumento, et ponatur finis basis eius, que est in longitudine que est latus trianguli cerei, super lineam que est in longitudine eris, sicut factum est prius. Et erit superficies regule in qua est speculum ortogonalis super tabulam eneam, et hec superficies secat tabulam eneam super lineam que est in longitudine eris, et hec superficies tangit superficiem speculi super lineam que est in superficie speculi. Et hec superficies est superficies regule in qua est speculum, et erit angulus regule acute adherentis in media linea superficiei anuli in qua superficie erit speculum declinatum in partem in qua est caput trianguli, quia regula exaltavit unam partem eius cum corpore trianguli, et alia pars que est

153 acuta *corr. ex* in qua *L3* 154 medie: media *L3*; *corr. ex* media *O*/regule: recte *L3*; *corr. ex* recte *a. m. C1* 156 triangulus cereus modicus: triangulum cereum modicum *R* 157 est *inter. OC1* (*a. m. C1*); *om. SL3E*/altitudo: spissitudo *R*/trianguli: anguli *F* 158 sit: sint *OC1ER*; *alter. ex* situm *in* sint *L3*/pro *corr. ex* post *C1* 159 tabule ceree *om. R*/tabule . . . triangulus *inter. a. m. L3*/triangulus: triangulum *R*/*post* triangulus *add.* regule *O*/base: basi *R* 160 *post* regule[1] *add.* tabule enee triangulus regule firmiter sub base regule *L3*; *scr. et del. C1*/altitudini *om. FP1*/ponatur . . . regule (161) *mg. a. m. L3* 161 basem: basim *R* 162 columpnalis: columpnas *S*; columpne *R*/tabule *om. R*/tabule regule *mg. a. m. C1* 163 includatur: inclinatur *FP1*/ triangulus: triangulum *R*/et *inter. O* 164 planitus: complanatum *R*/et[2] *inter. L3*/super *inter. a. m. OE* 165 *post* finis *scr. et del.* huius *F* 166 trianguli: anguli *O*; *corr. ex* anguli *L3*; *corr. ex* anguli *a. m. E*/finis *om. C1*/linea hec (167) *transp. O* 168 in *om. S*/in . . . est[1] (169) *om. L3* 169 basis eius *transp. ER*/in . . . est[2] *om. O* 170 *post* latus *scr. et del.* eius *O*/*post* super *scr. et del.* basim *P1*/*post* longitudine *scr. et del.* erit *F*/eris *om. R*/*ante* sicut *add.* tabule *R* 172 tabulam[1] *corr. ex* lineam *SL3C1* (*a. m. SC1*)/eneam[1]: eream *FSL3*; *alter. in* eream *P1*/et . . . eneam[2] *om. P1*/ eneam[2]: eream *F* 173 longitudine: longitudinem *L3*/eris: eius *R*/tangit: tangitur *FP1* 174 hec superficies *om. R* 175 est superficies *om. FP1*/erit: est *C1* 176 *post* superficie *scr. et del.* et *OL3* 177 declinatum: declinatus *R*/*post* in[1] *scr. et del.* qua *F*/trianguli: triangulum *O* 178 *post* trianguli *add.* quia *FP1*/*post* que *scr. et del.* cum in partem in qua est capud trianguli quia regula exaltavit *O*

post caput trianguli est superficies eris, et erit linea que est in medietate speculi declinata.

[2.17] Et quando fuerit latus trianguli cerei super lineam que est in longitudine eris, movebitur regula in qua est speculum, et latus trianguli in hoc motu, si sit super lineam longitudinis eris. Et procedat vel retrocedat donec concurrat angulus regule acute cum puncto aliquo linee superficiei speculi donec firmetur regula acuta, et auferatur linea in speculo cum incausto facta. Et fiat punctus in superficie speculi in directo capitis regule acute. Et auferatur regula acuta, et apponatur acus, et sit acus super lineam mediam superficiei anuli, et adherere cogatur cum cera. Et erit linea intellectualis ab acu in punctum signatum in superficie speculi perpendicularis super superficiem regule que tangit superficiem speculi super punctum signatum et perpendiculariter super quamlibet lineam ab illo puncto protractam in superficiem contingentem speculum. Erit igitur perpendicularis super lineam rectam contingentem lineam communem superficiei alte anuli et superficiei.

[2.18] Ponatur autem visus in superficie anuli in capite eius et videbit in speculo donec comprehendat formam corporis parvi quod est in acu, et tunc percipiet corpus illud, et punctum in speculo signatum, et ymaginem illius corporis. Et linea transiens per corpus parvum et per punctum signatum in superficie est perpendicularis super superficiem contingentem speculi superficiem super punctum signatum. Et hec superficies anuli est ex superficiebus reflexionis, et corpus parvum et centra visus sunt in hac superficie, et punctus reflexionis est in hac superficie, et hoc deinceps probabimus. Et ymago corporis parvi in hoc situ erit super lineam rectam a corpore parvo protractam rectam super superficiem contingentem superficiem speculi, et cum hec linea

179 *post* superficies *add.* tabule enee *R*/eris *om. R*/*post* est *scr. et del.* ortogonalis *P1*/medietate (180): medietatum *O* 180 declinata *alter. in* declinati *a. m. E* 182 eris *om. R*/*ante* movebitur *add.* enee tabule *R*/*post* regula *scr. et del.* est *E* 183 eris *om. R*/*post* longitudinis *add.* tabule enee *R* 184 concurrat: occurat *C1* 186 punctus: punctum *R* 187 directo: directa *FP1*/*post* auferatur *scr. et del.* linea in speculo *S*/apponatur: ponatur *FP1* 189 cera: cerea *E*; *corr. ex* cere *O*; *corr. ex* cerea *L3C1*/et *om. R* 190 super *inter. a. m. E*/regule . . . superficiem (191) *rep. C1*/*post* tangit *scr. et del.* superficiem regule que tangit *E* 191 *post* superficiem *scr. et del.* super *O*/super *inter. O*; ad *L3*/*post* punctum *add.* illud *O*/*post* signatum *scr. et del.* in superficie speculi perpendiculari super *O*/et *inter. O*/perpendiculariter: perpendicularis *OC1ER* 192 quamlibet *corr. ex* libet *a. m. C1*/protractam: pertractam *S* 193 contingentem *corr. ex* contingentis *O*; contingentis *L3E*; contingente *C1* 194 alte *corr. ex* ate *F*; *corr. ex* arte *L3C1*; *corr. ex* aree *a. m. E*/*post* et *scr. et del.* in *S* 195 *post* superficiei *add.* speculi *R* 196 autem *corr. ex* atem *S*/*post* autem *scr. et del.* ei *C1*/videbit *corr. ex* vibit *F* 197 in acu *corr. ex* natu *C1*/acu: actu *FP1E*; *corr. ex* actu *O* 198 corpus *corr. ex* corporis *C1*/illud *corr. ex* illum *F* 200 signatum *inter. E*/signatum in superficie: in superficie signatum *R*/*post* superficie *scr. et del.* e et *O*; *scr. et del.* est et *L3*; *add.* reflexionis et *C1*; *add.* et *E* 201 speculi superficiem *transp. O*/*post* hec *scr. et del.* in *E* 203 centra: centrum *C1ER*/et . . . superficie (204) *om. FP1* 204 hoc: hec *R*/probabimus *corr. ex* probamus *C1* 205 protractam: pertractam *S*/rectam[2] *scr. et del. E*; *om. R* 206 cum: est *R*/hec: hoc *FL3*; *inter. a. m. E*/hec linea *transp. E*

perpendicularis super lineam rectam contingentem lineam communem superficiei speculi et superficiei reflexionis que est superficies anuli, et superficies reflexionis est ex superficiebus declinantibus secantibus columpnam inter lineas longitudinis columpne et circulos eius equidistantes basibus, quia regula et speculum quod est in ea sunt declinata, linea ergo communis huic superficiei et superficiei speculi est ex sectionibus columpnaribus. Et ita explanabimus locum ymaginis si mutetur situs regule in qua est speculum et declinetur super superficiem eius alia declinatione minori vel maiori.

[2.19] Palam ergo ex hiis quod ymago percipitur ubi perpendicularis a viso puncto ad speculi superficiem ducta concurrit cum linea reflexionis, et hic est situs predictus. Si a puncto viso ad speculi superficiem ducantur linee ad speculi superficiem, que perpendicularis est minor qualibet alia, quoniam quelibet alia prius secat lineam communem superficiei contingenti speculum in qua ortogonaliter cadit perpendicularis et huic superficiei reflexionis quam veniat ad speculum, et quelibet linea a puncto viso in hac superficie ad hanc lineam communem ducta est maior perpendiculari, quia maiorem respicit angulum, quare propositum.

[2.20] Eadem poterit adhibi comparatio in speculo piramidali exteriori, et idem patebit sive sint ymagines rerum visarum in sectionibus piramidalibus, sive in eis que fuerint secundum lineas longitudinis.

[2.21] In speculis spericis concavis comprehenduntur ymagines quedam ultra speculum, quedam in superficie, quedam citra superficiem, et harum quedam comprehenduntur in veritate, quedam preter veritatem.

[2.22] Omnes quarum comprehenditur veritas apparent in loco sectionis perpendicularis et linee reflexionis, quod sic patebit. Fiat

207 *post* perpendicularis *add.* sit *C1* 208 que . . . reflexionis (209) *om. FP1*; *inter. a. m. E* 209 declinantibus: declinatis *O*; *alter. in* declinatis *L3* 210 circulos: circulus *O* 212 et superficiei *om. FP1* 213 est: erit *FP1* 214 si *corr. ex* ut *a. m. E*; ut *R*/super *inter. a. m. E* 215 alia: aliqua *R*/minori vel maiori: maiore vel minore *R* 216 *post* quod *scr. et del.* h *O* 217 concurrit *corr. ex* concurrat *P1* 218 est *om. C1*/predictus: productus *E*/*post* puncto *add.* a *FP1*/viso *om. C1* 219 ad . . . superficiem: a . . . superficiem *scr. et del. O*; *om. R* 220 est² *om. L3*; *inter. E*/est minor *transp. R*/qualibet *corr. ex* quamlibet *O*/*post* quoniam *scr. et del.* quem *P1*/*post* secat *scr. et del.* o *O*/lineam *om. FP1SL3E*; *corr. ex* lineis *O*/lineam communem (221) *transp. C1R* 221 *post* superficiei *inter.* linea *a. m. E*/contingenti: contingentis *R*/in qua: in quam *inter. a. m. F*/qua: quam *R* 222 huic *om. R*/*post* reflexionis *inter.* autem *a. m. E*/quam: antequam *R* 223 et *inter. O*/a *inter. OL3*; *om. E*/*post* hanc *scr. et del.* superficiem *L3* 224 maiorem: maiore *O* 225 *post* angulum *add.* scilicet rectum *O*/quare *alter. in* quod erat *O*/*post* quare *add.* patet *R* 226 eadem . . . longitudinis (228) *transp. ad* 218 *post* predictus *R*/adhibi: adhiberi *OL3C1ER*/comparatio: operatio *E*/piramidali: piramidis *P1SL3C1E*/exteriori (227): exteriore *R* 227 *post* sint *scr. et del.* exteriores *P1*/*post* visarum *add.* sive *FP1SL3*; *scr. et del.* sive sint *E* 228 eis: iis *R*/fuerint: fiunt *R* 229 spericis *corr. ex* speciricis *F* 230 citra: circa *FP1C1*; *corr. ex* circa *L3*; *corr. ex* cera *O* 231 *post* veritate *add.* et *O* 233 comprehenditur *corr. ex* comprehenduntur *L3*/apparent *corr. ex* appareant *F*

piramis, et ea ortogonalis super basem, et dyameter basis sit minor medietate dyametri spere, et linea longitudinis piramidis sit maior eadem semidyametro. Et secetur ex parte basis ad quantitatem eius, et fiat super sectionem circulus, et secetur piramis super hunc circulum. Postea in medio speculi fiat circulus ad quantitatem basis piramidis remanentis, et aptetur huic circulo piramis, et firmetur cum cera.

[2.23] Deinde statuatur visus in situ in quo ymaginem piramidis possit comprehendere, et adhibeatur lux ut certior fiat comprehensio. Non videbis quidem piramidem huic coniunctam, sed comprehendes hanc ultra speculum extensam, unde apparebit piramis quedam continua cuius basis ultra speculum et pars eius piramis cerea. Et si in hac piramide signetur linea longitudinis cum incausto, videbitur hec linea protendi super superficiem piramidis apparentis, et quoniam conus piramidis est centrum spere, linea a cono secundum longitudinem piramidis ducta erit perpendicularis super contingentem cuiuslibet circuli spere per caput linee transeuntis.

[2.24] Quare quelibet linea longitudinis piramidis apparentis est perpendicularis super lineam contingentem lineam communem superficiei reflexionis et superficiei spere, que quidem linea est communis, et est circulus. Et quilibet punctus piramidis in hac videtur perpendiculari, et quelibet perpendicularis est in superficie reflexionis, quoniam punctus visus et ymago eius sunt in perpendiculari et in hac superficie reflexionis. Et omnis ymago comprehenditur in linea reflexionis, quare ymago cuiuscumque puncti piramidis erit in puncto sectionis perpendicularis et linee reflexionis.

[2.25] Puncta autem quorum ymagines citra speculum comprehenduntur, hoc est inter visum et speculum sunt, cum a quolibet eorum linea ducta ad centrum speculi secet latitudinem vie visum et speculum interiacentis. Et ut videatur hoc, auferatur piramis a medio speculi,

235 et ea: cum *OL3*; cerea *mg. a. m. C1*; *corr. ex* cum *SE*/ea: eius *R*/*ante* ortogonalis *add.* axis sit *R*/ basem: basim *R*/sit *inter. a. m. F* 236 medietate *corr. ex* mediante *O*/eadem (237) *inter. L3* 237 *post* eius *add.* scilicet semidyametri *R* 238 et . . . circulus (239) *mg. a. m. E*/piramis: piramidis *O* 242 ut *corr. ex* et *a. m. E* 243 quidem: quod *FP1* 244 unde: inde *FP1* 245 *post* speculum *add.* est *R*/*post* eius *add.* est *C1*/cerea: cera *FP1* 246 incausto *corr. ex* tantum *a. m. C1* 247 *post* protendi *add.* secundum *L3*/super: secundum *C1*/quoniam: quia *S*/conus: vertex *R* 248 piramidis *corr. ex* piramis *P1*/centrum: centra *O*/cono: vertice *R*/longitudinem *corr. ex* longitudinis *O* 249 *post* super *add.* lineam *R*/cuiuslibet circuli (250): quemlibet circulum *R* 250 *post* caput *add.* cuiuslibet *C1*/transeuntis: transeuntem *R* 253 reflexionis et superficiei *rep. F*/est communis (254) *transp. R* 254 et est *om. R*/quilibet punctus: quodlibet punctum *R*/piramidis *om. P1* 255 perpendiculari: perpendicularis *FC1*; ? *OL3*/perpendiculari et quelibet *om. P1*/et . . . perpendicularis *mg. a. m. C1*/*post* reflexionis *add.* que quelibet perpendicularis *C1* 256 *post* quoniam *scr. et del.* a *S*/punctus visus: punctum visum *R* 257 reflexionis *inter. a. m. S*; *om. OL3C1ER*/*post* ymago *scr. et del.* non *O* 260 citra: circa *S*; *corr. ex* cita *O* 261 cum *inter. E*/eorum *inter. L3*; horum *C1* 262 secet: secat *C1R*/vie *inter. O*/ *post* vie *add.* inter *OR* 263 videatur *corr. ex* videtur *O*/speculi *corr. ex* speculo *E*

et collocetur in parte. Erit conus centrum speculi, et remotio visus a speculi superficie sit maior semidyametro spere. Deinde sumatur lignum gracile album, et statuatur in speculo ut sit centrum speculi directe medium inter caput ligni et centrum visus, et dirigatur intuitus in punctum speculi a quo linea ad conum piramidis ducta sit inter caput ligni et visum. Et inspiciatur speculum donec non appareat caput ligni et lignum, et apparebit forma capitis ligni citra speculum et propinquior visui cono piramidis. Et erunt in eadem linea recta conus piramidis, et caput ligni, et ymago capitis, et hec linea est perpendicularis super lineam contingentem lineam communem superficiei speculi et superficiei reflexionis. Quoniam superficies reflexionis transit per centrum et punctum visus, et linea transiens per hec duo puncta est in superficie reflexionis, et linea communis est circulus. Et hec linea huic circulo erit dyametrum, quoniam centrum illius circuli est centrum spere, quare erit hec linea perpendicularis super lineam contingentem circulum in capite huius linee, et hec linea transit per punctum visum et per eius ymaginem. Et ita quodlibet punctum citra speculum visum comprehenditur in eadem linea cum centro et cum ymagine eius, et quodlibet punctum videtur in linea reflexionis, quare in loco sectionis perpendicularis et linee reflexionis.

[2.26] Et ea quorum veritas comprehenditur in hiis speculis sunt quorum ymagines apparent ultra speculum vel citra superficiem eius, et preter hec nulla sunt que in hoc speculo in veritate comprehendat visus, ipsa enim prohibent ymagines suas veras apparere. Ymagines que apparent in superficie speculi huius sunt ex ultima partitione, et hoc explanabimus, cum aderit sermo in erroribus visus. Quodlibet ergo punctum in veritate in hoc speculo comprehensum apparet in concursu perpendicularis et linee reflexionis, que quidem perpendicularis transit a puncto viso ad centrum spere et cadit ortogonaliter in contingentem lineam communem.

264 *post* parte *inter.* et *O*/conus: vertex *R*/a . . . superficie (265) *om. R* 265 speculi superficie *transp. L3C1*/semidyametro *corr. ex* semidiametri *P1* 268 conum: verticem *R*/*post* inter *scr. et del.* sit *F* 269 et[1] . . . ligni *om. FP1R*; *inter. S*/non *om. L3C1E* 270 lignum *alter. in* visum *a. m. E*; visum *R* 271 cono: vertice *R*/conus: vertex *R* 272 *ante* ymago *scr. et del.* conus piramidis *P1* 274 centrum: centra *O*/*post* et *scr. et del.* fluxum *C1*/punctum visus (275) *corr. ex* visus punctum *C1* 275 *post* transiens *add.* et *P1* 276 huic . . . erit: erit huic circulo *C1*/erit *om. L3*/*post* erit *scr. et del.* transiens *O* 277 dyametrum: diameter *R* 278 circulum *inter. a. m. E* 279 per *om. R* 280 citra: circa *FP1L3*; ? *E*/visum: visus *FP1* 281 cum *om. FP1* 282 *post* punctum *scr. et del.* et quilibet punctum *O*/in[2] . . . reflexionis (283) *om. L3*; *mg. a. m. C1* 284 *post* quorum *add.* est *FP1*/comprehenditur . . . speculis: in hiis speculis comprehenditur *R* 286 hoc *inter. a. m. L3* 288 speculi huius *transp. OL3C1ER*/partitione: apparitione *C1* 289 hoc: hec *R*/aderit: adherit *FP1O*; erit *R*/in: de *R*/ergo: vero *FP1* 291 perpendicularis: perpendiculariter *FP1S*; *corr. ex* perpendiculariter *L3* 292 viso *corr. ex* suo *a. m. C1*

[2.27] In speculis columpnaribus concavis diversificatur ymago, aliquando enim erit locus eius in superficie speculi, aliquando ultra, aliquando citra. Et in hiis omnibus aliquando in veritate comprehenditur, aliquando non.

[2.28] Cum volueris in hiis locum ymaginis percipere, facias sicut fecisti in columpnis exterioribus. Adhibeatur enim regula in qua sit columpna concava sicut adhibita est superius, et acus similiter, et corpus modicum in summitate acus. Et ponatur visus oppositus in medio circuli et in medio superficiei anuli, et sublevetur visus modicum a superficie anuli, et inspiciat donec ymaginem corporis videat et comprehendat formam corporis, et corpus, et punctum in speculo signatum in eadem linea perpendiculari super superficiem speculi—et hoc per sillogismum sensualem. Et erit ymago ultra speculum, et erit reflexio ex puncto linee que est in medio speculi.

[2.29] Deinde statuatur visus in superficie anuli, sed extra medium, donec videat ymaginem corporis parvi. Videbis quidem eam citra speculum, et videbis corpus, et eius ymaginem, et punctum in speculo signatum in una linea recta perpendiculari super lineam rectam contingentem circulum equidistantem basi speculi super punctum signatum in superficie speculi. Et superficies huius circuli est superficies reflexionis in hoc situ, et est superficies faciei anuli, et punctus reflexionis est punctus illius circuli.

[2.30] Postea adhibeatur cum manu alia acus in cuius summitate sit corpus modicum, et statuatur in superficiem et axem hoc modo ut hoc corpus et punctus signatus sint in eadem linea secundum sensualem sillogismum. Et sit visus in superficie anuli inter caput eius et medium. Videbit quidem ymaginem corporis, et videbit hanc ymaginem, et corpus eius, et punctus signatum in superficie speculi in eadem linea recta.

294 speculis *corr. ex* speculo *P1* / diversificatur: diversatur *FP1*; diversicatur *S* 296 aliquando: aliqua *O* / aliquando citra *mg. a. m. C1*; *om. R* / comprehenditur (297): comprehendetur *E* 297 non *corr. ex* et *a. m. C1* 298 locum ymaginis *transp. E* 299 fecisti: fecistis *C1* / columpnis: columpnaribus *R* 1 summitate: sumpmitate *S*; *corr. ex* sumpmitate *F*; *alter. ex* veritate *in* acuitate *O* 2 sublevetur . . . et[1] (3) *mg. a. m. E* 4 *ante* corpus *scr. et del.* p *L3* / in speculo *rep. E* 5 perpendiculari: perpendicularis *O* 7 *post* linee *add.* recte *SOL3C1ER* 9 videat *corr. ex* videa *F* / videbis: videbit *OL3C1ER* / citra: circa *FP1*; *corr. ex* circa *L3* 10 videbis: videbit *OL3C1ER* 11 una: eadem *O* 12 basi *corr. ex* basis *O* 13 superficie speculi *R* / circuli *om. FP1SR* / est . . . situ (14) *rep. C1* 14 *ante* reflexionis *scr. et del.* faciei anuli et punctus *S* / in . . . reflexionis (15) *rep. C1*; faciei[1,2] *om. C1* / faciei: facie *FP1* / punctus: punctum *R* 15 punctus: punctum *R* 16 in *inter. S* / summitate: sumpmitate *S* 17 superficiem: superficie *FP1* / hoc *om. C1R* 18 punctus signatus: punctum signatum *R* / *post* signatus *add.* in speculo *C1* 19 *post* sillogismum *scr. et del.* sit *F* / et[1] *mg. F* 20 *post* ymaginem[2] *scr. et del.* corporis *F* 21 et[1] . . . superficie *inter. a. m. E* / et[2]: in *C1* / punctus: punctum *C1ER*

[2.31] Si autem declinetur linea recta cum triangulo parvo quem fecimus—et visus sit in medio anuli—videbis ymaginem citra speculum, sed in eadem linea recta cum corpore et puncto signato. Et hec reflexio erit ex sectionibus columpnaribus, quoniam speculum est declinatum, et scimus quod non percipitur ymago nisi in linea reflexionis. Palam ergo quod locus ymaginis est ubi secat perpendicularis predictam lineam reflexionis, cum comprehenditur veritas, et licet non comprehendatur certitudo ymaginis, tamen erit modus harum ymaginum cum veritatis ymaginibus.

[2.32] Pari modo videre poteris in piramidibus concavis in concursu perpendicularis cum lineis reflexionis. Palam ergo quod in omnibus speculis comprehenduntur ymagines in loco predicto, qui quidem locus similiter dicitur ymago.

[2.33] Quare autem comprehendantur res vise per reflexionem in locis ymaginis et quare ymago sit super perpendicularem a re visa in speculi superficiem declarabimus causam. Visus, cum adquirit formam per reflexionem, adquirit eam statim sine certitudine, et adquirit longitudinem per estimationem. Et hanc longitudinem comprehendet forsitan in veritate per diligentiam intuitus adhibitam, forsitan non. Et istud explanavimus in libro secundo, et ibi dictum est quod visus adquirit longitudinem per sillogismum ex magnitudine corporis et angulo aliquo sub quo comprehenditur magnitudo. Et adquisitio rei site ignote manifeste est in hunc modum. Res etiam note comprehenduntur in hunc modum, conferuntur enim rebus cognitis et magnitudinibus vel longitudinibus notis. Cum visus comprehendit rem aliquam per reflexionem, non comprehendit longitudinem ymaginis nisi per estimationem; deinde adhibita diligentia, adquirit longitudinem, et verificat per sillogismum ex magnitudine rei vise et angulo piramidis super quam forma reflectitur ad visum.

23 linea *om. P1*/linea recta *transp. F*/quem: quod *R* 24 visus *om. FP1*/videbis: videbit *OL3C1ER*/ videbis ymaginem *transp. C1* 25 sed in: eorum *C1* 26 sectionibus columpnaribus *transp. R*/ columpnaribus *om. FP1SOE* 27 quod: quoniam *C1* 28 secat *corr. ex* seca *S* 30 *post* tamen *add.* fueris *O*/erit *corr. ex* eirit *L3*/modus: modum *O*; *corr. ex* modum *L3* 31 veritatis *corr. ex* veritas *O*/ymaginibus *om. O* 32 *post* poteris *add.* imaginem *R*/piramidibus: piramidalibus *R*/ *post* concursu *scr. et del.* piramis *P1* 33 lineis: linea *R* 34 comprehenduntur: comprehendantur *P1S*; *alter. ex* comprehendandantur *in* comprehendantur *F* 35 ymago: imaginis locus *R* 37 ymaginis: imaginum *R* 38 *post* visus *scr. et del.* autem *F*/cum *corr. ex* autem *a. m. L3* 40 et *corr. ex* ad *O*/comprehendet: comprehendit *FP1* 42 explanavimus: explanabimus *FP1S*; *corr. ex* explanabimus *O* 43 longitudinem *inter. L3*/ex *alter. in* et *a. m. E* 45 site *alter. in* vise *a. m. L3*; vise *R*/ignote: ceperant *O*; note *R*/res . . . modum (46) *om. P1*/note: ignote *R* 46 conferuntur *corr. ex* confirmitur *O*/*post* enim *add.* in *O*; *scr. et del.* in *L3E* 47 visus *corr. ex* visum *O*/comprehendit: comprehendat *FP1*; *corr. ex* comprehendet *E* 48 comprehendit: comprehendet *E*/ ymaginis . . . longitudinem (50) *rep. F* 50 *post* longitudinem *scr. et del.* ymaginis nisi per estimationem *P1*/verificat *corr. ex* verficat *S* 51 quam forma *corr. ex* quem forman *F*

[2.34] Cum ergo res visa ex rebus notis fuerit, visus adquirit eius longitudinem per iam notam longitudinem equalem angulum huic tenentem et huic longitudini similem. Similiter res visa, cum fuerit ignota, conferetur magnitudo magnitudinis eius alii magnitudini rerum notarum, et adquiritur longitudo huius ymaginis per sillogismum mensure anguli quem tenet ymago in centro visus in hora reflexionis. Et locus in quo est forma rei vise comprehensus per reflexionem, forma ab eo directe veniens ad angulum circa oculum, accedet super piramidem ipsam per quam forma reflectitur ad visum, et eadem piramis occupabit totam formam que fuerit in loco ymaginis. Visus ergo, cum adquirit rem visam per reflexionem, adquirit eam in loco ymaginis, quoniam forma comprehensa in loco ymaginis per reflexionem, quare similis est forme directe comprehense, occupare ab illa piramide, et hec est causa quare comprehendatur in loco ymaginis.

[2.35] Quare autem comprehendetur ymago in perpendicularem dicemus. Scimus quod punctum visui perceptibile non est intellectuale sed sensuale, et forma eius sensualis. Dico igitur in speculis planis quod ymago, cum non apparet in superficie speculi sed ultra, competentius est et rationabilius quod appareat super perpendicularem quam extra eam. Cum enim in loco perpendicularis assignata, fuerit distantia eius a puncto reflexionis speculi—que scilicet est pars linee reflexionis a loco ymaginis ad punctum reflexionis ducte—est equalis distantie puncti visi a puncto reflexionis. Quoniam superficies speculi est ortogonalis super perpendicularem, unde linea a puncto reflexionis ad perpendicularem ducta est latus duobus triangulis commune, et

53 iam . . . longitudinem: longitudinem iam notam *C1*/equalem angulum *transp. R* 54 longitudini: longum *FP1S*; longitudinem *L3C1*/similem *corr. ex* sillogismum *O*/visa: nota *FP1* 55 conferetur *corr. ex* confirmetur *O*; confertur *R*/magnitudinis *om. R*/magnitudinis . . . alii *rep. FP1* 56 *ante* notarum *add.* visarum *R*/adquiritur: adquiretur *O*/huius: eius *R* 57 in[1] *om. FP1* 58 locus: a loco *R*/comprehensus: comprehensa *R* 59 ab eo *om. R*/circa: citra *OE*; *alter. in* citra *L3*; *alter. ex* citra *in* intra *a. m. C1*/*ante* oculum *add.* circulum *O*; *inter.* ad *O*/accedet: accidet *O*; *corr. ex* accidet *E*; accedit *R*/super: supra *FP1*; *corr. ex* semper *O* 60 reflectitur *corr. ex* reflectetur *O* 61 piramis *corr. ex* piramidis *O*/totam *corr. ex* notam *O* 62 *post* reflexionem *scr. et del.* rem *O* 63 *post* ymaginis[1] *scr. et del.* visus ergo cum acquirit rem visam *O*/quoniam . . . ymaginis[2] *rep. P1*/*post* comprehensa *add.* est *ER* 64 occupare: occupatae *R*/illa *corr. ex* alia *O* 65 quare *inter. L3C1E* (*a. m. C1E*) 66 comprehendetur: comprehendatur *R*/perpendicularem: perpendiculari *R* 68 sensuale: densuale *L3* 69 ymago cum *transp. R*/cum non *inter. OL3*/non *om. FP1S*; *inter. a. m. E*/apparet: apparent *R*/*post* sed *scr. et del.* una nec *L3*; *scr. et del.* non citra nec *E* 70 *post* competentius *scr. et del.* enim *L3*; *add.* enim *E*/rationabilius: rationalius *L3E*/quod: ut *R*/super: supra *R* 71 enim: est *S*/assignata: assignato *OL3*; assignatio *C1* 73 est: erit *R* 74 quoniam: quia *ER*/quoniam . . . reflexionis (75) *om. L3*/*post* quoniam *add.* enim *R* 75 unde: et *R* 76 *post* perpendicularem *add.* in superficie speculi *C1*/ducta . . . perpendicularem (77) *om. S*/*post* latus *scr. et del.* i *F*/duobus triangulis: duabus angulis *FP1*/*post* et *add.* angulus lineae accessus est equalis angulo reflexionis quare duo anguli unius trianguli sunt equales duobus angulis alterius trianguli et unum latus commune est quare reliqua latera equalia sunt reliquis lateribus *R*

dividet perpendicularem per duo equalia. Quare duo latera unius trianguli erunt equalia duobus lateribus alterius, et angulus angulo, quia uterque est rectus, quare basis basi.

[2.36] Si ergo ymago in perpendiculari apparuerit, equaliter a puncto et a visu distabit cum corpore a quo procedit, et erit ymagini idem situs respectu puncti reflexionis qui est puncto viso respectu eiusdem, et idem est situs respectu visus, unde in hoc situ apparebit veritas et puncti visi et ymaginis. Si vero ymago fuerit extra perpendicularem, cum sit necesse eam in linea reflexionis esse, aut erit ultra perpendicularem aut citra respectu visus. Si fuerit ultra, erit quidem remotior a puncto reflexionis et a visu quam punctus visus, unde tenebit minorem angulum in oculo quam punctus. Et minorem occupat visus partem, unde, cum sit equalis, videbitur minor eo. Si autem fuerit citra perpendicularem, videbitur maior cum sit propinquior.

[2.37] In speculo sperico exteriori videtur ymago super perpendicularem, aut enim videtur ymago centri visus, aut alterius puncti. Si ymago centri, dico quod dignior est perpendicularis ab oculo ad centrum spere ducta, ut super eam appareat ymago centri quam alia, si enim forma directe procedat secundum hanc perpendicularem usque ad centrum spere, eundem semper servabit situm respectu visus, et ita cuicumque puncto spere opponatur forma, perpendicularis ad centrum mota ydemptitatem situs tenebit respectu visus. Et idem situs erit forme in una perpendiculari qui et in alia, quoniam centrum spere eundem habet situm respectu cuiuslibet puncti spere, et omnes huiusmodi perpendiculares eiusdem sunt situs.

[2.38] Si autem extra perpendicularem ymago moveatur ad quodcumque punctum spere, mutabitur eius situs respectu visus, quoniam alium habebit situm extra perpendicularem quam in perpendiculari, et extra speculum movetur perpendicularis et non intra. Et si extra

77 dividet: dividit *L3C1E*/dividet . . . basi (79) *om. R* 78 *post* trianguli *scr. et del.* e *S*/erunt *corr. ex* sunt *a. m. E*/et *inter. E* 80 puncto: speculo *R* 81 et[1] *om. FP1S*; *scr. et del. C1*/et a visu *om. R*/*post* idem *add.* est *S* 82 puncti . . . respectu (83) *om. S*/*post* puncti *add.* respectu *FP1OE*; *scr. et del.* respectu *L3*/*post* est *add.* in *ER*/*post* respectu[2] *add.* puncti *R*/et *mg. F* 83 idem: eidem *FP1* (*mg. F*)/est *inter. OE*; eis *C1*/*post* est *add.* eis *O* 84 visi *om. S*/sit: fuerit *ER* 85 esse: inesse *C1*; *corr. ex* inesse *L3* 86 aut *corr. ex* et *S*/si *corr. ex* aliquid *O* 87 reflexionis *om. FP1S*/punctus visus: punctum visum *R*/unde: unum *S* 88 punctus: punctum visum *R*/*post* punctus *scr. et del.* visus unum *S*/occupat: occupabit *R* 89 unde: unum *S*/citra: circa *FP1S* 91 exteriori: extra polite *R* 93 *post* centri *add.* visus *R*/dignior *corr. ex* digniorem *O*/ad *om. E* 94 ducta . . . spere (96) *om. FP1*/alia: aliud *E*/si enim (95): sicut *SE*; ? *C1* 95 *post* perpendicularem *scr. et del.* ai *S* 97 centrum: centra *FP1SO*; *corr. ex* centra *L3* 98 mota: noto *FP1*; nota *SO*/situs erit *transp. R* 99 qui: que *R* 100 huiusmodi: huius *O* 101 *post* perpendiculares *add.* puncti qui respectu spere *O*/eiusdem: eius *O* 102 ymago *inter. O* 103 eius situs *transp. ER* (situs *inter. a. m. E*) 104 alium: aliquando *FP1SO*/quam in perpendiculari *mg. a. m. C1* 105 movetur *alter. in* moverit *a. m. E*; movebitur *R*/non intra et *om. FP1S*/et si *inter. L3*/si *inter. a. m. E*/*post* si *add.* intra *S*

perpendicularem appareat, non servabit situm, et convenientius fuit ut servaret ymago situm quam ut mutaret, ut visus rem visam certius comprehenderet. Ob hoc ymago centri super perpendicularem apparet, et huic ymagini non possumus certum in perpendiculari assignare punctum, quoniam non invenitur dignitas in uno perpendicularis puncto plus quam in alio, ut hec ymago determinate appareat in eo. Sed scimus quod in quocumque huius perpendicularis puncto appareat semper apparet continua cum apparenti oculo, et semper in totali forma apparenti eundem tenet locum et situm.

[2.39] Cuiuscumque puncti ymago preter centrum visus ad speculum accedat, movetur declinate, quare non durat ei similitudo situs respectu puncti visus, et perpendicularis a puncto viso ad speculum ducta cadat supra centrum spere in qua quidem perpendiculari observaret ymago similitudinem situs. Non est ergo punctus in quo comprehensa ymago servet similitudinem situs nisi in perpendiculari illa, et cum oportet ipsam comprehendi in linea reflexionis, comprehendetur in concursu linee huius cum hac perpendiculari. Iam ergo assignavimus causam huius rei, verum rerum naturalium status respicit statum suorum principiorum, et principia rerum naturalium sunt occulta.

[2.40] Idem erit modus probationis in speculo sperico concavo; similiter in piramidali concavo vel exteriori. Et universaliter erit locus ymaginis in perpendiculari in quocumque speculo, quoniam non est locus extra perpendicularem in quo forma observet similitudinem et situs ydemptitatem.

[2.41] Hiis explanatis, restat demonstrative declarare locum ymaginis in qualibet speculorum specie.

106 perpendicularem: speculum *R*/appareat: apparea *S*/situm . . . servaret (107) *om. FP1S*/et *om. O* 107 ymago *corr. ex* ymabigo *F*/ymago situm *transp. R* 108 *post* centri *add.* visus *R* 109 in . . . assignare: assignare in perpendiculari *R*/assignare: assignate *FS* 110 quoniam: quod *O*/non *om. O*/*ante* invenitur *inter.* scilicet *O*/*post* puncto *scr. et del.* post *S* 111 plus: maior *R*/determinate: determinata *FP1*; determinatum *S*/appareat *corr. ex* apparet *E*; appateat *R* 112 huius . . . puncto: puncto huius perpendicularis *R*/semper: sed *FP1* 113 apparet *corr. ex* appareat *O*/apparenti: apparente *R*/oculo: oculi *O*; *alter. in* oculi *L3*/apparenti (114): apparente *R* 115 *post* cuiuscumque *add.* vero *R* 116 accedat *corr. ex* concedat *L3*; accedit *R*/movetur: cogetur *O*/declinate: declinare *P1SO*; *alter. in* declinare *E*/durat: ducat *SL3*; esset *O*/ei *corr. ex* ai *S* 117 puncti *om. R*/speculum *corr. ex* punctum *O* 118 cadat: cadet *OL3C1*; *alter. in* cadet *E*; cadit *R*/supra: super *ER* 119 observaret: observat *R*/*post* ymago *scr. et del.* et *L3*/similitudinem *corr. ex* similitudo *F*/est *om. O*/punctus: punctum *R* 120 *post* comprehensa *add.* est *E* 121 oportet: oporteat *R* 122 linee huius *transp. ER* 123 rerum *corr. ex* tunc *O* 124 statum: statim *S*; situs *ER* 126 erit *corr. ex* rerum *O*/*post* concavo *add.* et *O* 127 piramidali: piramide *FP1SL3*; *corr. ex* piramide *a. m. C1*/exteriori: extra polito *R*/universaliter: vel *FP1*/erit: dicitur *O* 128 in . . . perpendicularem (129) *om. S* 129 et situs (130) *transp. R* 131 explanatis *corr. ex* explatis *F*

[2.42] Dicimus quod quodcumque vel quodlibet punctum comprehensum a visu in speculo plano, quando egressus est a perpendiculari que a centro visus cadit in superficiem speculi plani, quod linea per quam reflectitur forma illius puncti ad visum concurret cum perpendiculari producta ab illo puncto ad superficiem speculi. Et erit punctus concursus, qui est locus ymaginis, intra speculum, et erit longitudo illius a superficie speculi equalis longitudini puncti visi a superficie speculi. Et visus non adquirit ymaginem puncti visi nisi in loco illo, et quodcumque punctum adquirit visus in hoc speculo non apparebit ex eo nisi unica ymago.

[2.43] Quodcumque autem punctum comprehendit visus in speculo [sperico] exteriori, quando egreditur forma perpendicularem ductam a centro visus ad centrum speculi, linea per quam reflectitur ymago ad oculum concurret cum linea producta a puncto illo ad centrum speculi, que linea est perpendicularis ducta a puncto illo ortogonalis super lineam contingentem lineam communem superficiei reflexionis et superficiei speculi. Et situs puncti concursus (qui est locus ymaginis) a superficiei speculi erit secundum situm visus a superficie speculi. Et forsitan erit punctus concursus ultra speculum, forsitan in superficie speculi, forsitan erit intra speculum. Et visus comprehendit ymagines omnes ultra speculum, licet diversa sint eorum loca, et non comprehendit locum cuiuslibet ymaginis sillogistice in superficie speculi. Et quodlibet punctum comprehensum a visu in hoc speculo non pretendit nisi unam ymaginem.

[2.44] In speculo columpnari exteriori, quodcumque punctum comprehendit visus (eadem in speculo piramidali exteriori), cum fuerit extra perpendicularem ductam a centro visus ortogonalem super superficiem contingentem superficiem speculi, linea per quam

133 *post* dicimus *add.* ergo *R*/quodcumque vel *om. O*/vel quodlibet *scr. et del. C1* 134 *post* quando *add.* ipsum *R*/a^{2} *om. FP1E*/perpendiculari: perpendicularis *FP1* 135 quod . . . visum (136) *om. R* 137 ad: ultra *O*; *alter. in* ultra *L3* 138 punctus: punctum *R*/qui: qua *S*/intra: ultra *C1E* 139 *post* speculi *add.* et *FP1S*; *scr. et del.* et *L3*/a^{2}: in *FP1SE* 140 visi nisi *corr. ex* nisi visi *E*/nisi *mg. a. m. C1* 144 *ante* exteriori *add.* columpnari *O*/exteriori: extra polito *R*/*post* forma *add.* super *O*; *add.* a *R*/perpendicularem: perpendicularis *FP1*; perpendiculari *R*/ductam: ducta *FP1E* 145 ad centrum *rep. L3* (centrum^{1,2}: centra *L3*)/centrum: centra *FP1SE*/ymago *om. S* 146 concurret *om. FP1S*; *inter. L3E* (*a. m. E*)/ad . . . illo (147) *om. FP1*; *rep. S*/*post* speculi *scr. et del.* linea per quam reflectitur ymago ad oculum *O* 147 ortogonalis: ortogonaliter *OR* 150 erit . . . speculi *om. FP1S* 151 punctus: punctum *R*/*post* speculum *add.* et *C1* 152 erit *om. R*/intra *om. FP1SL3*; *inter. a. m. E* 153 eorum: earum *R* 154 *post* ymaginis *add.* nisi *R*/ in *om. FP1SL3* 155 quodlibet: quolibet *S*/comprehensum *corr. ex* comprehend *F*/a visu *inter. L3*; *om. R* 157 exteriori *mg. a. m. C1*/*post* exteriori *add.* extra polito piramidali extra polito *R* 158 eadem *om. OC1*/eadem . . . exteriori *om. R*/*post* eadem *add.* et *C1E* (*inter. E*)/piramidali: piramidalis *P1S*/*post* fuerit *add.* ille punctus *C1* 159 ortogonalem: ortogonalis *FP1SOL3*/ super *om. FP1*; *inter. OC1* (*a. m. C1*)

reflectitur ad visum forma concurrit cum perpendiculari ducta a puncto illo rectam super lineam contingentem lineam communem superficiei reflexionis et superficiei speculi. Et loca ymaginum horum speculorum quedam sunt ultra superficiem speculi, quedam in superficie, quedam citra. Et visus adquirit omnes ymagines horum speculorum ultra superficiem speculi, et quodcumque punctum comprehendat visus in hiis speculis non efficit nisi unam ymaginem tantum.

[2.45] In speculo sperico concavo, linee per quas reflectuntur forme punctorum visorum quedam concurrunt cum perpendicularibus ductis a punctis illis rectis super lineas contingentes lineas communes superficiei speculi et superficiei reflexionis, et quedam sunt equidistantes hiis perpendicularibus. Et que concurrunt cum perpendicularibus, locus concursus, qui est locus ymaginis, quidam ultra speculum, quidam citra speculum. Qui citra speculum fuerit quidam inter visum et speculum, quidam super centrum ipsum visus, quidam ultra centrum visus. Et adquisitio visus formarum rerum visarum quas adquirit in hiis speculis quasdam comprehendit in loco ymaginis, qui est punctus concursus—et hee sunt quas visus certe comprehendit—quasdam comprehendit extra locum concursus—et est comprehensio sine certitudine. Et res visas quas adquirit visus in hoc speculo quedam unam prefert ymaginem tantum, quedam duas, quedam tres, quedam quatuor, nec potest esse quod plures.

[2.46] In speculo piramidali concavo et columpnari concavo, linee per quas flectuntur forme ad visum quedam concurrunt in perpendicularibus ductis a punctis visis rectis super lineas contingentes lineas communes, et quedam sunt equidistantes perpendicularibus. Que concurrunt cum perpendicularibus, in quibusdam punctis concursus est ultra speculum, in quibusdam citra. Que autem citra fuerint, quedam

161 ad . . . forma: forma ad visum *R*/concurrit: concurret *R*/*post* a *scr. et del.* tro *E*/puncto illo (162) *transp. ER* 162 *ante* rectam *add.* super *C1E*; (*inter. a. m. E*)/rectam: recte *O*/rectam super *transp. R*/super *scr. et del. C1* 163 et[1] *inter. E*/et superficiei *om. FP1SL3*/superficiei *inter. a. m. C1*; *om. ER* 164 quedam: que *FP1SL3*; *corr. ex* que *E* 166 comprehendat: comprehendit *FP1SR* 167 ymaginem: magnitudinem *FP1S* 169 ductis: ductus *L3* 170 illis: vel *L3*/rectis *om. R* 171 et[2] *om. L3ER* 172 *post* et *add.* earum *R*/*post* perpendicularibus *add.* quedam habent *R*/locus (173): locum *R* 173 concursus *corr. ex* con circulus *L3*/quidam[1]: quedam *FP1SL3*; *om. R*/quidam[2]: quedam *FP1L3R* 174 *post* speculum[1] *scr. et del.* fuerit *C1*; *add.* et *R*/qui: que *R*/citra: circa *FP1S*/speculum fuerit *inter. a. m. E*/fuerit *om. P1*; habent *R*/quidam: quedam *FP1SR* 175 quidam[1]: quedam *FP1SR*/centrum ipsum *transp. R*/ipsum *om. FP1C1*/quidam: quedam *R* 176 adquisitio *om. R*/*post* visus *add.* quasdam *R* 177 quasdam *om. R*/qui: que *FP1SL3E*/punctus: punctum *R* 178 certe: recte *L3C1*; certo *R* 179 concursus: cursus *O*/est *om. E* 180 visas: visae *R*/prefert: profert *O*; prae se ferunt *R* 181 duas . . . quedam[3] *om. FP1* 182 *post* quod *add.* una res pretendat *R* 184 flectuntur: reflectuntur *OR*/in: cum *R* 185 rectis *om. R* 186 communes: communem *FS* 187 *post* perpendicularibus *scr. et del.* que concurrat *F*/in . . . est: quedam habent concursum *R* 188 in *om. R*/quibusdam: quedam *R*/fuerint *inter. E*; *om. R*

erunt inter speculum et visum, quedam supra centrum visus, quedam ultra centrum visus. Et comprehensio rerum visarum in hoc speculo per visum, quedam sit in loco ymaginis, qui est locus concursus, quedam extra locum concursus. Et eorum que comprehenduntur aliud pretendit unam ymaginem tantum, aliud duos, aliud tres, aliud quatuor, nec aliquid est quod potest pretendere plus quam quatuor. Et nos declarabimus hec omnia demonstrative.

[2.47] **[PROPOSITIO 1]** Sit A [FIGURE 5.2.1, p. 563] punctus visus, B centrum visus, DGH speculum planum. Et sit G punctus reflexionis, DGH linea communis superficiei reflexionis et superficiei speculi. A puncto G ducatur EG perpendicularis super lineam communem. A puncto A ducatur perpendicularis super speculi superficiem, que sit AH, et producatur ultra speculum. Et AG sit linea per quam accedit forma ad speculum, BG per quam reflectitur ad visum. Igitur BG, EG, AG sunt in superficie reflexionis, et cum EG sit equidistans AH, et BG declinata sit super EG, concurret BG cum AH. Concurrat ergo in puncto Z. Dico quod ZH est equalis HA.

[2.48] Quoniam angulus BGD equalis angulo AGH, et angulus AHG equalis angulo ZHG, et latus HG commune, quare triangulus equalis triangulo; quare ZH equalis AH.

[2.49] Et si voluerimus per perpendicularem invenire locum reflexionis, secetur ex perpendiculari ultra speculum pars equalis parti eius usque ad speculum, et est ut sit ZH equalis AH. Et ducatur linea a centro visus ad punctum Z, que sit BGZ. Dico quod G est punctus reflexionis.

189 erunt *om. R*/inter: intra *FP1*/supra: super *OR*/quedam[2] . . . visus (190) *om. FP1* 190 comprehensio: comprehensionum *FP1S*/rerum *om. FP1S* 191 qui: que *FP1S*/*post* concursus *add.* et *C1* 192 quedam: quidam *FP1SE*/quedam . . . concursus *om. L3*/eorum: eo *FP1*/aliud: alium *FP1* 193 aliud[1]: alium *FP1*/duos *om. FP1S*/*post* duos *add.* et *E*/aliud[2] *inter. a. m. E* 194 aliquid est *transp. O*/potest: possit *R*/plus: plures *R*/quam *om. O* 195 demonstrative: demonstrare *FP1S* 196 punctus visus: punctum visum *R* 197 DGH: CDE *R*/speculum . . . DGH (198) *mg. a. m. E*/G: D *R* 198 DGH . . . reflexionis *om. FP1*/DGH: CDE *R* 199 G: D *R*/EG: DF *R*/perpendicularis: perpendiculariter *O*/*post* communem *add.* et *ER* 201 AH: AC *R*/AG: AD *R*/per *om. O*/accedit: accedat *E*/accedit forma (202) *transp. R* 202 BG[1]: AG *P1S*; *corr. ex* AG *a. m. F*; BD *R*/BG[2]: BD *R*/EG AG (203) *corr. ex* AG EG *S*; FD AD *R* 203 sunt in *inter. a. m. F*; *om. P1*/EG: FD *R*/AH: AC *R*/BG: DB *R* 204 sit *om. FP1*/EG: HG *O*; FD *R*/concurret: concurrit *FP1E*/BG: BD *R*/AH: AC *R* 205 Z: G *R*/ZH *corr. ex* HZ *C1*; GC *R*/*post* est *scr. et del.* perpendicularis *P1*/HA: AHA *S*; CA *R* 206 *post* quoniam *add.* enim *R*/BGD: BDE *R*/*post* equalis *add.* est *R*/AGH: ADC *R*/AHG: ACD *R* 207 *post* equalis[1] *add.* est *C1*/ZHG: GCD *R*/HG: CD *R*/triangulus equalis: triangulum equale *R* 208 triangulo: angulo *FP1S*/quare *inter. E*/ZH: GC *R*/AH: AC *R* 209 per *om. FP1SO*; *inter. a. m. E*/per perpendicularem *inter. a. m. L3*/perpendicularem: perpendicularis *FP1SO*/*ante* invenire *add.* perpendicularis *L3* 210 *post* ex *scr. et del.* pyramide *L3* 211 *post* speculum *scr. et del.* pf pars equalis parti eius usque ad speculum *F*; *scr. et del.* pars equalis parti *P1*/ZH: GC *R*/AH: AC *R* 212 *post* punctum *scr. et del.* visus *C1*/Z: G *R*/Z . . . BGZ: quod est G *O*/BGZ: BDG *R*/*ante* dico *scr. et del.* et *O*/G: D *R*/punctus: punctum *R*

[2.50] Quoniam AH et HG sunt equalia HG et HZ, et angulus angulo, quare triangulus triangulo. Igitur angulus ZGH equalis angulo HGA. Sed ZGH est equalis angulo DGB. Restat ergo ut angulus BGE sit equalis angulo EGA, et ita G punctus reflexionis. Et ita propositum.

[2.51] **[PROPOSITIO 2]** Amplius, sit A centrum visus [FIGURE 5.2.2, p. 563], et AG perpendicularis super speculum planum, et D secet hanc perpendicularem in superficie oculi. Dico quod in hac perpendiculari non est punctus qui reflectatur ab hoc speculo ad visum preter D.

[2.52] Si enim sumatur ultra visum punctus in hac perpendiculari, et sit H. Non perveniet forma eius ad speculum super perpendicularem propter solidi corporis interpositionem, et ita non reflectitur forma eius super perpendicularem.

[2.53] Et si dicatur quod ab alio puncto speculi potest reflecti, sit illud B. Movebitur quidem forma eius ad punctum B per lineam HB, et reflectitur per lineam BA. Dividatur angulus HBA per equalia per lineam TB. Igitur TB erit perpendicularis super superficiem speculi. Sed TG est perpendicularis super eandem, quare ab eodem puncto est ducere duas perpendiculares ad superficiem speculi, quod est impossibile.

[2.54] Eadem erit probatio quod forma puncti D non potest reflecti ab alio speculi puncto quam a puncto G, quare non reflectitur nisi super perpendicularem. Punctum autem in hac perpendiculari sumptum inter G et D, si dicatur formam per reflexionem ad visum mittere improbatio, quoniam aut erit corpus solidum aut rarum.

[2.55] Si solidum, procedet secundum perpendicularem forma eius ad speculum et regredietur secundum eandem usque ad ipsum, et

214 *post* quoniam *add.* enim *R*/AH: AC *R*/HG: CD *R*/sunt equalia HG *mg. a. m. E*/HG: CD *R*/HZ: CG *R* 215 quare triangulus: ergo triangulum *R*/ZGH: GDC *R*/*ante* equalis *add.* est *OER*/HGA: ADC *R* 216 ZGH: GDC *R*/DGB: DGH *S*; BDE *R*/BGE: BGC *P1*; BGD *O*; BDE *R* 217 EGA: HGA *O*; ADC *R*/*post* ita[1] *add.* est *O*/G *inter. L3*; D *R*/*ante* punctus *add.* est *R*/punctus: punctum *R*/reflexionis: re *FP1*/*post* ita[2] *add.* patet *R* 218 amplius *om. R*/centrum: centro *O* 219 et[1] *om. O*/D *om. O* 220 oculi: circuli *L3*/*post* dico *scr. et del.* in h *F*; *add.* ergo *O* 221 punctus qui: punctum quod *R* 222 si enim: sm *FS*; *alter. in* sive *L3*; sin *R*/*ante* sumatur *add.* autem *P1C1ER*/punctus: punctum *R* 223 et *inter. a. m. C1*/*post* non *add.* iam *R*/perveniet: pervenient *S*/*post* perpendicualrem *add.* AH *R* 224 propter . . . perpendicularem (225) *om. O*/interpositionem: interpositione *P1*/non: si *FP1*/reflectitur *corr. ex* reflectatur *P1*; reflectetur *R* 226 alio: aliquo *FP1S*/potest: possit *R* 227 movebitur . . . B[2] *mg. a. m. E*/quidem: quod *P1* 228 reflectitur: reflectetur *R*/per[1]: quod *S*/*post* per[3] *scr. et del.* d *S* 229 igitur TB *om. L3*; *inter. a. m. C1*/TB *om. R* 230 TG: AG *S*/est[1] *om. FP1*/ab *inter. L3* 234 *post* nisi *scr. et del.* p *L3* 235 *post* perpendicularem *add.* DG *R*/autem: aliquod *O* 236 dicatur: ducatur *O*/formam: forma *OC1*/mittere *om. FP1*; *mg. a. m. C1* 237 *ante* improbatio *scr. et del.* tem *C1*/improbatio *corr. ex* probatio *L3*; improbo *R*/aut *corr. ex* autem *C1*/solidum . . . si (238) *mg. a. m. C1*/aut rarum *corr. ex* ? *O* 238 procedet: procedit *O*; *corr. ex* procedat *F*; *corr. ex* procedit *E* 239 regredietur: regradietur *FP1*; *corr. ex* regreditur *E*

propter soliditatem non poterit transire et ad visum pervenire.

[2.56] Si autem punctum illud fuerit rarum, forma eius regrediens a speculo super perpendicularem miscebitur ei, et adherebit, nec reflectetur ad visum.

[2.57] Quod autem forma cuiuscumque puncti in hac perpendiculari inter G et D sumpti non possit ab alio puncto speculi ad visum reflecti modo supradicto potest probari. Similiter, forma puncti inter A et D sumpti nec reflectitur ad visum per perpendicularem nec per aliam, quoniam puncta centrum visus et superficiem eius interposita sunt valde rara, unde nec mittitur eorum forma nec reflectitur ut sentiatur. Et quoniam quodlibet punctum preter D in superficie visus sumptum opponitur speculo non ad rectum angulum, videbitur quodlibet super perpendicularem ab eo ad speculum ductam, et eius ymago ultra speculum equidistans a superficie speculi, sicut ipsum punctum. Et quoniam D videtur continuum cum aliis superficiei visus punctis, et ymago eius continua cum aliis ymaginibus, videbitur ymago D tantum distans a superficie speculi quantum distat D ab eadem.

[2.58] Palam ergo quod cuiuscumque puncti in speculo visi ymago videbitur super perpendicularem, et elongatio ymaginis et visi corporis a superficie speculi eadem.

[2.59] **[PROPOSITIO 3]** Amplius, forma puncti visi in speculo plano non reflectitur ad eundem visum nisi ab uno puncto tantum. Sit enim A [FIGURE 5.2.3, p. 563] centrum visus, B punctum visum, ZH speculum. Si ergo dicatur quod a duobus punctis speculi reflectitur forma B ad visum, sit unum punctum D, aliud E. Et ducatur linea a puncto viso ad visum, scilicet AB, que quidem linea aut erit perpendicularis supra speculum, aut non.

[2.60] Si non fuerit perpendicularis, scimus quod illa linea est in superficie reflexionis ortogonali super superficiem speculi, et in una

240 soliditatem *corr. ex* liditatem *L3*/et *inter. a. m. E* 241 regrediens: regradiens *P1*; egrediens *O* 242 et adherebit *inter. L3*/adherebit: adhebit *F* 244 quod: quia *O*/*post* forma *scr. et del.* eius *O* 245 *post* sumpti *add.* si *C1*/non: si *FP1S* 247 nec[1]: non *R*/ad visum *om. O*/per[1] *om. FP1SL3*; *inter. OE* (*a. m. E*)/*ante* perpendicularem *add.* hanc *C1*/perpendicularem: perpendicularis *FP1SL3* 248 quoniam: quando *P1O*/*post* puncta *add.* inter *R* 249 mittitur: mittetur *O*; mutatur *E* 250 *post* quoniam *inter.* ad *L3*/quodlibet *alter. in* ad quodlibet *O* 251 speculo: speculationi *FP1S*; speculum *OL3*/quodlibet *inter. a. m. E* 252 eius ymago *transp. R* 253 speculi *om. R*/*post* speculi *scr. et del.* sint *C1*/*post* punctum *add.* E *F* 254 *post* aliis *add.* punctis *C1*/superficiei *inter. a. m. E*/punctis: punctus *FP1S*; *corr. ex* punctus *L3*; *om. C1* 258 super *inter. a. m. E* 259 *post* speculi *add.* est *OR* 260 forma *corr. ex* formarum *L3* 261 enim: est *S* 262 visus: visum *FP1*; *inter. E* 263 speculi reflectitur *corr. ex* reflectitur speculi *C1*/reflectitur: reflectatur *R* 264 *post* visum *add.* A *R*/*post* D *add.* et *FP1*/aliud: alium *FP1*/et *om. O*/ducatur: producatur *O*/*post* puncto *scr. et del.* ni *F* 265 AB: BA *R* 268 ortogonali: ortogonalis *O*/super *om. S*/et in *inter. O*/una: unaquaque *F*

sola tali. Quoniam si in duabus, erit communis duabus superficiebus ortogonalibus, et sumpto in ea puncto, et ducta ab illo linea in alteram superficierum super lineam communem huic superficiei et superficiei speculi, erit quidem hec linea ortogonalis supra speculum. Similiter, ab eo puncto ducatur linea in alia superficie super lineam communem ei et superficiei speculi, et erit hec linea ortogonalis supra speculum, quare ab eodem puncto erit ducere duas perpendiculares.

[2.61] Cum ergo BA sit in una sola superficie ortogonali, et tria puncta A, B, E sint in eadem superficie ortogonali, et erunt AE et EB in illa superficie ortogonali in qua est AB, similiter EB et DB. Quare EA, EB in eadem superficie cum DA, DB. Sed angulus AEH est equalis angulo DEB, et angulus HEA maior angulo ADE, quia extrinsecus, quare BED maior ADE. Sed BDZ equalis ADE, et BDZ maior BED, quare ADE maior BED, et dictum est quod minor. Restat ergo ut a solo puncto fiat reflexio.

[2.62] Si vero AB sit perpendicularis supra speculum, iam dictum est quod unicum est punctum in linea a centro visus ad speculum ortogonaliter ducta cuius forma reflectitur a speculo ad visum. Et iam probatum est quod ymago illius puncti ab uno solo reflectitur puncto, quare propositum.

[2.63] **[PROPOSITIO 4]** Amplius, inspecto aliquo puncto ab utroque visu, una tantum et eadem ymago apparet utrique, et in loco predicto. Verum planum est quod forma puncti non reflectitur ad utrumque visum ab eodem puncto speculi. Si enim linea reflexionis ad unum visum procedens angulum teneat cum perpendiculari erecta super superficiem speculi equalem angulo quem tenet linea accessus forme ad speculum cum eadem perpendiculari, non poterit in eadem superficie sumi linea alia que equalem angulum huic efficiat cum

269 erit . . . duabus *inter. L3* 270 sumpto: sumpta *O*/*post* ea *add.* eo *P1*/puncto: puncta *S*/ducta: ducto *FP1SO*/illo: illa *FP1*/alteram: altera *C1*; *alter. ex* alterum *in* altera *L3* 272 quidem *om. R*/hec *om. FP1*/linea . . . ducatur (273) *om. P1*/supra: super *L3C1R*/*post* speculum *add.* quare *O* 273 eo: eodem *OL3C1E*/alia: illa *FP1S*/*post* superficie *scr. et del.* superficie *F* 274 ei: huic superficiei *R*/et[1] *om. E*/et[2] *om. R*/hec *om. FP1S*/supra: super *R* 275 *post* perpendiculares *add.* ad superficiem speculi *R* 276 ortogonali *corr. ex* ortogonalis *L3* 277 ortogonali *corr. ex* ortogonalis *L3*/et[1,2] *om. R*/EB *corr. ex* EDB *F*; EDB *SL3C1E* 278 in . . . AB *om. R*/similiter . . . DB *om. O*/EB *alter. in* AD *a. m. F*; AD *P1*/*post* DB *add.* DA *R*/quare EA EB (279) *mg. a. m. C1* 279 *post* EB *add.* sunt *R*/est *om. L3* 280 *post* angulo[1] *scr. et del.* DH *S*/DEB: DEH *L3*; BED *R*/HEA: AEH *R*/extrinsecus: exterior *R* 281 *post* maior *inter.* angulo *E*/equalis . . . BDZ[2] *om. FP1S*/et . . . BED *om. R* 282 ADE: AED *FP1SL3E*/BED *corr. ex* BDE *F*; ADE *E*/*post* quod *scr. et del.* in *O*/restat: constat *O* 284 supra: super *R* 285 *post* est[1] *inter.* in secunda figura *a. m. L3*; *add. C1* 287 uno: illo *FP1* 288 *post* quare *add.* patet *R*/propositum: per ipsum *O* 289 inspecto: in speculo *SL3* 290 *post* utrique *add.* visui *R* 291 verum: unde *L3C1ER*/planum: palam *O* 292 speculi: speculo *S*/si: quia *R*/*post* enim *add.* una *C1* 293 teneat: tenet *R*/super superficiem (294) *corr. ex* superficiem *O* 294 quem: quam *FP1S* 296 linea alia *transp. R*/que: non *S*

perpendiculari. Unde ab hoc puncto non reflectetur linea aliqua ad alium visum. Oportet igitur ut a diversis speculi punctis fiat reflexio.

[2.64] Sint illa puncta T, Z [FIGURE 5.2.4, p. 564]. Et sit speculum planum QE; A punctus visus; B, G duo visus; AD perpendicularis. Palam ergo quod BT et AT et ET, AD sunt in eadem superficie ortogonali super superficiem speculi. Similiter, AD, AZ, GZ sunt in eadem superficie ortogonali, et DT linea communis superficiei ADTB, et DZ communis superficiei ADZG. Si BT, GZ fuerint in eadem superficie ortogonali, erit TDZ linea una, et perpendicularis AD aut erit inter duas perpendiculares predictas ad superficiem speculi a duobus visibus, aut extra [FIGURE 5.2.4a, p. 564].

[2.65] Utrumlibet sit, linea reflexionis BT secabit ex perpendiculari AD ultra speculum partem equalem parti que est AD. Similiter, GZ secabit ex eadem perpendiculari partem ultra speculum equalem illi parti. Igitur ille due linee reflexionis secabunt perpendicularem ultra speculum in eodem puncto. Ergo ymago puncti A in eodem perpendicularis puncto percipietur ab utroque visu, quare unica tantum erit ymago et eadem, et in eodem loco que esset uno tantum visu adhibito.

[2.66] Si vero puncta T, Z non fuerint in eadem superficie reflexionis ortogonali super speculum [FIGURE 5.2.4b, p. 564], eadem tantum erit probatio, cum utraque linea reflexionis secet ex perpendiculari partem equalem parti superiori, et erit sectio linearum reflexionis cum perpendiculari in eodem puncto, quare propositum.

[2.67] Si vero fuerit punctus A in perpendiculari ducta ab uno visu ad superficiem speculi tantum, secundum eundem visum comprehenditur ultra speculum in puncto perpendicularis tantum elongato a superficie speculi quantum distat A ab eadem, quia forma A videtur continua cum formis aliorum punctorum que quidem videntur in locis

297 unde: unum *S*/reflectetur: reflectitur *O* 298 alium: alterum *R*/ut *om. S*/speculi punctis *transp. ER* 300 A punctus visus: punctum visum A *R*/punctus: punctis *FP1S*; *corr. ex* puncta *O*/B . . . visus: duo visus B G *R*/AD perpendicularis *transp. R* 1 et[1]: ET *F*; *om. SR*/AT: AD *O*/et[2] *om. P1O*/et ET *om. R* 2 super *inter. O*/super . . . ortogonali (3) *rep. S*; *mg. a. m. E* (AD AZ *transp. E*)/speculi *om. FP1S*/AD AZ GZ: GZ AZ AD *R*/AZ: EZ *FP1C1* 3 DT linea *transp. R*/ADTB: ADDB *FP1S*/*post* ADTB *add.* et superficiei speculi *R*/*post* DZ *add.* est linea *C1*; *add.* linea *R* 4 *post* communis *inter.* linea *a. m. L3*/*post* ADZG *add.* et superficiei speculi *R*/*post* si *add.* iam *R*/eadem: eam *F* 5 *post* una *add.* recta *R* 6 perpendiculares: lineas *O*/predictas: ductas *C1*; productas *R* 8 utrumlibet: utrumque *S*; utramque *O*; utramlibet *E*/sit *om. O*/reflexionis *om. R* 9 speculum *corr. ex* speculi *C1* 11 igitur ille *transp. R*/igitur . . . due: ille due igitur *E*/ille *mg. a. m. C1* 12 *post* A *scr. et del.* pe *C1*/eodem: eadem *FP1S*/perpendicularis (13): particularis *S*; *corr. ex* particularis *L3* 13 percipietur: participetur *FP1S* 14 visu: viso *FP1SL3C1*; *corr. ex E* 16 tantum *om. P1*; tamen *R*/*post* erit *scr. et del.* puncta *C1* 17 cum: quod *R*/secet: secat *FP1S* 18 equalem parti *transp. O* 19 *post* quare *add.* patet *R* 20 punctus: punctum *R*/visu: viso *FP1SOL3E* 21 tantum: tamen *O*/comprehenditur (22): comprehendetur *R* 22 perpendicularis *corr. ex* perpendiculari *C1*; perpendiculari *E*

similibus. Et ab alio visu comprehenditur ymago A in eodem perpendicularis puncto, quare et sic utrique visui unica tantum apparet ymago puncti A, et in eodem eiusdem perpendicularis puncto, quod est propositum.

[2.68] **[PROPOSITIO 5]** In speculis spericis exterioribus patebit quod dicimus. Sit A [FIGURE 5.2.5, p. 565] punctus visus, B centrum visus, G punctum reflexionis. Palam quod BG, AG sunt in superficie ortogonali super superficiem contingentem speram in puncto G. Linea communis superficiei reflexionis et superficiei spere est circulus, et sit ZGQ. Linea contingens hunc circulum in puncto reflexionis sit PGE. Perpendicularis super hanc lineam sit HG. Planum quod HG perveniat ad centrum spere. Quod si non, cum linea a centro spere ducta ad punctum G sit etiam perpendicularis super lineam PGE, erit ab eodem puncto in eadem parte ducere duas lineas perpendiculares super unam lineam.

[2.69] Sit autem centrum spere N, et ducatur linea a puncto viso ad centrum spere, scilicet AN, que quidem erit perpendicularis super superficiem contingentem speram in puncto spere per quem transit. Et quoniam planum quod BG secat speram, cum sit inter HG, GP que continent rectum angulum, concurret cum linea AN. Et cum perpendicularis HG sit in superficie reflexionis, erit centrum spere in eadem, et ita AN in eadem superficie cum HG.

[2.70] Sit ergo concursus BG cum AN D. Planum quoniam D erit locus ymaginis, et hec quidem intelligenda sunt quando linea ducta a puncto viso ad centrum visus non fuerit perpendicularis super speculum.

[2.71] **[PROPOSITIO 6]** Amplius, linea PGE [FIGURE 5.2.6, p. 565] secat lineam AN. Sit punctus sectionis E, et dicitur punctus iste finis contingentie. Dico quod in hoc situ linea a centro spere ad locum ymagi-

25 visu: viso *FP1SE*/comprehenditur: comprehendetur *R*/in: ab *FP1SL3*; *corr. ex* ab *a. m. C1* 26 perpendicularis *corr. ex* perpendiculari *C1*/et *om. O*; *inter. E*/sic: sit *FP1SL3* 27 perpendicularis *om. FP1L3E* 29 spericis *mg. a. m. F*; *om. SL3C1E*/exterioribus: extra politis *R* 30 dicimus: diximus *R*/*post* sit *add.* autem *O*/punctus visus: punctum visum *R* 31 *post* in *add.* eadem *R* 32 ortogonali: ortogonalis *P1SL3*/contingentem speram *transp. R* 33 circulus: circumferentia *R* 35 HG[1]: HZ *P1*/planum quod HG *om. S*/perveniat: perveniet *OR*; *corr. ex* perveniant *E* 36 punctum (37): centrum *FP1*; *corr. ex* centrum *L3E* (*a. m. E*) 38 eadem parte: partem eadem *S*; eandem partem *L3C1ER*/ducere *inter. L3*; *mg. a. m. C1*/duas *inter. a. m. O* 39 sit *corr. ex* si *E* 40 su-per *om. F* 41 quem *corr. ex* quam *F*; quod *R* 42 *post* planum *add.* est *R*/HG: GH *FP1* 45 *post* eadem[1] *add.* superficie cum HG (*post* HG *scr. et del.* si ergo) *C1*/HG: BG *SO* 46 *post* AN *add.* in puncto *C1*; *inter.* punctus *E*; *add.* punctum *R*/D[1] . . . D[2] *om. O*/*post* D[1] *inter.* punctus *a. m. L3*/quoniam: quod *R* 47 et *inter. O*/hec: hoc *P1*/intelligenda: intelligendum *FP1S*/sunt: est *FP1*/quando: quoniam *FE*; *corr. ex* quoniam *L3*/linea ducta *transp. FP1* 48 non: si *S* 50 linea *mg. a. m. C1* 51 secat *corr. ex* fiet *O*/punctus[1,2]: punctum *R*/E *om. S*/iste: istud *R* 52 in hoc situ: nihil *P1*; nihil situ *scr. et del. F*

nis ducta maior est linea a loco ymaginis ducta ad locum reflexionis: id est, DN maior DG.

[2.72] Quoniam angulus BGH equalis angulo HGA, sed angulus BGH equalis angulo NGD. Ergo angulus HGA equalis eidem, et EG perpendicularis super HGN, quare angulus AGE equalis est angulo EGD. Igitur proportio AG ad DG sicut AE ad ED.

[2.73] Protrahatur a puncto A equidistans DG, et concurrat cum linea HN in puncto H. Erit igitur angulus NGD equalis angulo GHA. Sed angulus NGD equalis est angulo AGH. Ergo angulus GHA est equalis eidem, quare duo latera AG, HA sunt equalia. Igitur proportio AH ad DG sicut AG ad idem. Sed proportio AH ad DG sicut AN ad DN, quare proportio AN ad DN sicut AG ad DG. Igitur proportio AN ad AG sicut DN ad DG. Sed AN est maior AG, quia respicit angulum maiorem recto in triangulo ANG. Igitur DN maior DG, quod est propositum.

[2.74] **[PROPOSITIO 7]** Amplius, dico quod linea ducta a fine contingentie, qui est E, usque ad speram perpendiculariter, id est pars linee EN, minor est semidyametro.

[2.75] Sit F [FIGURE 5.2.7, p. 566] punctus in quo AN tangit superficiem spere. Dico ergo quod EF minor est NF.

[2.76] Quoniam, ut dictum est, proportio AG ad DG sicut AE ad ED, sed AN ad DN sicut AG ad GD. Igitur AN ad DN sicut AE ad ED. Igitur AN ad AE sicut DN ad DE. Sed AN maior AE, quare DN maior DE, quare DN maior EF, quare NF maior EF, quod est propositum.

[2.77] **[PROPOSITIO 8]** Amplius, sit G [FIGURE 5.2.8, p. 566] centrum visus, D centrum spere, DZG perpendicularis a centro visus ad speram. Dico quod nullius puncti forma reflectitur per hanc perpendicularem nisi puncti eius quod est in superficie visus.

54 *post* DN *add.* est *O* 55 *post* quoniam *add.* enim *R*/*post* BGH *add.* est *ER*/*post* equalis *add.* est *O*/HGA: NGD *E*/sed . . . NGD (56) *scr. et del. E* 56 *post* equalis[1,2] *add.* est *R*/NGD *corr. ex* NDG *FL3*; NDG *SO*/eidem *inter. O* 57 *post* angulus *scr. et del.* HGN *F*/equalis est *transp. L3C1* 58 EGD *corr. ex* EDG *FL3*; EDG *SE*/*post* EGD *add.* est enim equalis angulo BGP qui opponitur EGD *C1*/DG: GD *R* 59 *post* puncto *add.* A linea *mg. F*; *scr. et del.* que *F*/A *om. P1*/*post* equidistans *add.* ipsi *R* 60 NGD *corr. ex* NSDG *F*; NDG *P1*; *corr. ex* NDG *L3*/NGD . . . angulus[1] (61) *om. S* 61 NGD *corr. ex* NGDG *F*; NGDG *S*; *corr. ex* NDG *L3*; NDG *E*/est[1] *om. C1*/*post* AGH *scr. et del.* quare duo latera AG HA *P1*/ergo . . . eidem (62) *mg. a. m. F*; *om. SOL3C1ER* 62 AG HA: AGNHA *S*/*post* equalia *scr. et del.* igitur proportio AG ad DG *F*/igitur . . . DG (64) *om. FP1* 63 idem: eandem *R*/*post* idem *add.* GD *C1*/*post* sicut[2] *scr. et del.* AG ad DG sed proportio AN ad DG est *C1*/*post* AN *scr. et del.* ad *E* 64 proportio[1] *om. R* 66 in *inter. L3*/ANG: AGN *R*/igitur: G *S* 68 contingentie (69): contingente *FP1* 69 qui: que *O*/E *inter. a. m. F*/*post* est[2] *add.* EF *R* 71 punctus: punctum *R*/tangit: secat *R* 72 est *om. S* 74 ad[2] *inter. O*/GD: DG *E*/AN: NA *O* 76 EF[1]: DF *R*/quare . . . EF[2] *mg. a. m. C1*; *inter. a. m. E* (*post* maior *add.* est *E*) 78 DZG: GZD *SO* 79 nullius: nullus *FP1L3*/puncti *om. O* 80 puncti *scr. et del. F*; punctum *O*

[2.78] Punctorum enim forme post centrum visus sumptorum non reflectuntur per eam propter causam supradictam. Similiter nec puncta inter superficiem visus et speculum sumpta. Dico etiam quod nullum punctum huius perpendicularis reflectitur ab alio puncto speculi.

[2.79] Si enim dicatur quod ab alio puncto, sit illud punctum A. Erit linea GA linea reflexionis, et a puncto illo intelligemus lineam ad A, que est linea per quam movetur forma. Et includunt hee due linee angulum super A, quem quidem angulum necessario dividet dyameter DA, cum sit perpendicularis super punctum A, quia perpendicularis dividit angulum ex linea motus forme et linea reflexionis per equa. Et ita dyameter DA concurret cum perpendiculari GD inter punctum sumptum et G. Et ita due linee recte in duobus punctis concurrent et superficiem includent.

[2.80] Restat ergo ut solius puncti qui est in superficie visus forma reflectatur a speculo ad perpendicularem, et videatur in primo ymaginis loco propter eius cum aliis continuitatem.

[2.81] **[PROPOSITIO 9]** Amplius, GA, GB [FIGURE 5.2.9, p. 567] sint linee a centro visus ducte contingentes speram, et signetur circulus super quem superficies hiis lineis inclusa secat speram. Erit AB portio apparens ex hoc circulo. Dico ergo quod loca ymaginum que per reflexiones ab hac portione factas comprehenduntur quedam sunt intra speculum, quedam in superficie speculi, quedam extra speculum. Et singulum horum est determinandum.

[2.82] Ducatur a puncto G linea secans circulum, et pars eius que est corda arcus circuli sit equalis semidyametro circuli. Sit linea illa GHK, et corda equalis semidyametro sit HK. Et producatur a puncto H perpendicularis, que sit DHM. Dico quod forma reflexa a puncto H locus eius erit intra speram.

81 forme: forma *FP1L3E*/post centrum: posterius *O*/*ante* visus *add.* quoniam *O* 82 per eam: preter eum *O* 83 inter: intra *E*/etiam: ergo *L3*/nullum: nullius *O* 84 huius *rep. P1*; *inter. E*/*post* perpendicularis *add.* huius *E*/speculi *om. O* 85 si . . . puncto *mg.* (dicatur: reflectatur; quod *om.*) *O*/illud: illum *P1* 86 GA *corr. ex* GH *F*; GHA *S*; *corr. ex* GHA *L3*; *corr. ex* G *a. m. C1*/linea *om. P1*/intelligemus: intelligamus *R* 87 A *corr. ex* DA *L3*/includunt *corr. ex* includuntur *S*/due linee *transp. FL3E* 88 quem *corr. ex* quoniam *F*; que *P1*/necessario *om. O*/*post* dividet *add.* per equalia *R* 89 *post* perpendicularis[1] *scr. et del.* super perpendicularem *F*; *add. L3*; *scr. et del.* p *S*/A *inter. O* 90 et[2] *inter. a. m. C1* 91 GD: G *P1*; DG *O* 92 recte *om. P1* 93 includent: inducent *FP1SE*; *corr. ex* inducent *L3* 94 solius *corr. ex* solus *P1*/puncti *om. P1* 95 ad: et *S*; per *inter. O*; per *R*/primo: proprio *R* 96 *post* aliis *add.* punctis *R* 98 sint *corr. ex* sunt *C1*/*post* sint *add.* due *P1*/*post* centro *scr. et del.* a centro *F* 99 superficies: superficiem *S* 100 apparens: apparentque *FP* 101 reflexiones *corr. ex* reflexionis *F*; *corr. ex* flexiones *a. m. C1* 102 quedam[1] . . . speculum[2] *mg. a. m. F* 103 *ante* et *scr. et del.* intra *F*/singulum: unumquodque *R*/horum *inter. a. m. C1*/*post* horum *scr. et del.* h *O* 105 sit equalis *transp. S*/*post* semidyametro *add.* magni *C1*/circuli[2] . . . semidyametro (106) *inter. a. m. L3* 106 producatur: protrahatur *FP1* 107 quod *om. S*/forma reflexa: forme reflexe *R* 108 eius *inter. a. m. C1*; *om. R*/erit: est *R*/erit . . . linea (109) *mg. a. m. E*

[2.83] Ducatur a puncto H linea equalem tenens angulum cum MH angulo MHG, et sit OH. Reflectentur quidem puncta huius linee a puncto H ad visum, et non alterius. Sumatur ergo aliquod eius punctum, et sit O, et ducatur ab eo linea ad centrum spere, que sit OD. Erit quidem OD perpendicularis super superficiem contingentem speram super punctum eius per quod transit OD. Verum angulus OHM equalis est angulo, ex ypothesi, MHG, quare similiter equalis est angulo contraposito, scilicet KHD. Sed KHD est equalis KDH, quoniam respiciunt equalia latera.

[2.84] Igitur angulus OHM equalis est angulo KDM, quare linee KD, OH sunt equidistantes. Ergo in infinitum producte numquam concurrent. Et linea OD secabit lineam interiacentem KD, OH, et ita quodcumque punctum sumatur in linea OH, linea ducta ab illo puncto ad punctum D secabit lineam reflexionis intra speram, que quidem linea erit perpendicularis super speram, sicut est OD. Quare ymago cuiuscumque puncti linee OH apparebit intra speram.

[2.85] Amplius, arcus circuli interiacens punctum H et punctum per quem transit perpendicularis a centro visus ducta est HZ. Dico quod, a quocumque puncto huius arcus fiat reflexio, locus ymaginis erit intra speram.

[2.86] Probatio: sit I [FIGURE 5.2.9a, p. 567] punctus sumptus, et ducatur linea a centro visus secans circulum super punctum illum, que sit GIS, et ducatur perpendicularis a puncto hoc que sit DIT. Et fiat linea PI equalem tenens angulum cum IT angulo TIG. Palam quod sola puncta linee PI reflectuntur a puncto I ad visum. Palam etiam quod linea IS maior est linea KH, quare maior SD. Igitur angulus SDI maior est angulo SID, quare est maior angulo GIT, quare maior angulo TIP.

110 OH: PH *R* 111 *post* visum *add.* G *R* 112 O: P *R*/sit: si *FP1*/OD: PD *R*/erit . . . OD (113) *inter. L3* 113 quidem *om. R*/OD: PD *R*/perpendicularis: perpendiculariter *FP1*; *corr. ex* perpendiculariter *a. m. E* 114 OD: PD *R*/*ante* verum *add.* et coniungatur DK *R*/OHM: PHM *R*/ equalis est (115) *transp. R* 115 ex ypothesi *om. SOC1R*/contraposito (116): circa posito *S* 116 scilicet *om. R*/scilicet KHD *om. P1*/*post* KHD[1] *scr. et del.* est equalis HDB *F*/sed KHD *om. S*; *inter. L3E* (*a. m. E*)/sed . . . KDH (KDH: KDM) *mg. a. m. F*/KDH *corr. ex* DH *a. m. C1*/quoniam: quia *SOC1* 118 OHM: PHM *R*/equalis est *transp. L3*/est *om. E*/KDM: KDH *O* 119 OH *corr. ex* OG *FL3E*; HO *C1*; PH *R*/sunt . . . OH (120) *om. S*/ergo *inter. O*/in *om. FP1L3*; *inter. a. m. E*/*post* producte *scr. et del.* in *E*/numquam: numquid *FP1* 120 OD: PD *R*/*post* interiacentem *add.* inter *R*/*post* KD *add.* et *R*/OH: PH *R* 121 quodcumque *corr. ex* quodcum *a. m. C1*/OH: PH *R* 122 intra: inter *FP1* 123 erit perpendicularis *transp. ER*/OD: PD *R* 124 OH: PH *R* 125 interiacens: interiacentis *FP1L3*/*post* interiacens *add.* inter *R*/H et punctum *inter. L3* 126 quem: quod *R*/est: esto *R* 127 quocumque: quolibet *O*/huius: huiusmodi *C1*/*post* reflexio *scr. et del.* huius *O* 128 intra: inter *FP1OL3* 129 probatio *om. R*/punctus sumptus: punctum sumptum *R* 130 illum: illud *R* 131 GIS: GI *R*/perpendicularis: perpendiculariter *C1*/a puncto: per punctum *R* 132 tenens *om. P1*/*post* IT *add.* et *P1*/TIG *corr. ex* TG *E* 133 *post* I *scr. et del.* palam *E*/etiam *inter. O* 134 quare *mg. F*/quare . . . SID (135) *om. S*/*post* quare *add.* est *OC1*/*post* maior *add.* angulo (*post* angulo *scr. et del.* quare maior angulo) *L3*/SD: ID *P1E*; *corr. ex* GSD *L3* 135 SID: SIT *E*/*post* quare[2] *add.* est *R*/*post* maior[2] *inter.* est *a. m. E*

[2.87] Igitur linea PI et SD numquam concurrent, et linea ducta a puncto quocumque PI linee ad punctum D secabit lineam SI intra speram, que SI est linea reflexionis. Et omnis linea ducta a quocumque puncto PI linee erit perpendicularis super speram, sicut est PD. Et cum locus ymaginis sit in concursu perpendicularis a puncto viso et linee reflexionis, erit ymago cuiuslibet puncti linee PI intra speram. Palam ergo quod omnium ymaginum arcus HZ locus proprius erit intra speculum, quod est propositum.

[2.88] Amplius, sumpto quocumque puncto arcus HB, dico quod quedam eius ymago erit intra speculum, quedam in superficie speculi, quedam extra speculum.

[2.89] Sumatur aliquod eius punctum, et sit N [FIGURE 5.2.9b, p. 567], et ducatur linea a puncto G secans circulum, que sit GNQ. Et ducatur perpendicularis DNF, et protrahatur linea equalem angulum tenens cum perpendiculari angulo FNG, et sit EN. Quoniam linea NQ minor est linea KH, est etiam minor linea QD, et ita angulus QDN minor est angulo DNQ, quare minor angulo GNF, quare etiam minor angulo ENF. Igitur linea EN et DQ concurrent. Sit ergo concursus in puncto E. Palam quod linea EQD est perpendicularis super speram, et secat lineam GNQ, que est linea reflexionis, in puncto Q, qui est punctus spere. Quare ymago puncti E, cum fuerit reflexio super punctum N, apparebit in puncto Q, et est in superficie spere.

[2.90] Si vero in linea EN sumatur punctum ultra E, utpote R, perpendicularis ducta ab eo ad centrum spere, sicut RD, secabit lineam reflexionis, que est GNQ, ultra punctum Q. Et est extra speram, quare ymago cuiuslibet puncti linee EN ultra E sumpti erit extra superficiem speculi.

136 SD: DS *O*/concurrent: concurrunt *FP1L3E* 137 puncto *corr. ex* puncta *F*/puncto quocumque *transp. L3*/secabit: secat *ER*/lineam: linea *L3*/SI: sumptam *O*; *om. C1* 138 que SI *transp. O*/*post* linea[1] *scr. et del.* ducta *F*/a *corr. ex* ei *a. m. F* 139 *post* linee *add.* ad punctum D *R*/erit perpendicularis: erunt perpendiculares *FP1* 140 a: cum *E* 141 intra speram *mg. O* 142 ergo *om. O*/quod *om. FP1* 147 N: enim *S* 148 ducatur: deducatur *L3* 149 *post* linea *inter.* tenens *a. m. O*/*post* angulum *scr. et del.* G *L3* 150 tenens *om. O*/*post* perpendiculari *add.* FN *mg. a. m. F*; *add. P1*/*post* sit *inter.* linea illa *L3*/*post* quoniam *add.* ergo *O* 151 minor *corr. ex* maior *FL3* (*mg. a. m. F*)/linea[1] *om. R*/angulus QDN *transp. R* 152 DNQ: QND *FP1*/DNQ . . . angulo (153) *mg. a. m. F*/*post* minor[1] *add.* est *FP1*/*post* GNF *add.* contraposito *FP1*/quare[2]: ergo *FP1*/*ante* etiam *add.* erit *FP1* 153 ENF *corr. ex* GNF *S*/linea *om. R* 155 secat *alter. in* secabit *a. m. F*/linea *inter. L3*/*post* reflexionis *scr. et del.* que est linea *F*/qui est punctus: quod est punctum *R*/qui . . . Q (157) *om. S* 156 E: G *O*/N *corr. ex* enim *L3* 157 apparebit: apparet *FP1*/et est *om. C1* 158 EN: NE *R*/R: K *FP1*; *corr. ex* ? *O* 159 sicut: que sit *R*/RD: KD *FP1*/*post* lineam *add.* GNQ *R* 160 que est GNQ *om. R*/est[1] *inter. a. m. E* 161 puncti: punctum *FP1*

[2.91] Si vero in linea EN citra punctum E sumatur aliquod punctum, perpendicularis ab eo ducta ad speculum secabit lineam GNQ intra speram, quoniam in puncto quod sit inter N et Q. Quare ymago cuiuslibet puncti linee EN inter E et N sumpti apparebit intra speram.

[2.92] Eadem penitus erit probatio, sumpto quocumque alio arcus BH puncto. Et ita ymago cuiuslibet puncti arcus BH una sola est in superficie speculi; aliarum quedam in speculo, quedam extra. Et quod demonstratum est in arcu ZB eodem modo potest patere in arcu ZA, et eadem penitus erit demonstratio, cuiuscumque circuli spere sumatur portio visui opposita et perpendiculari GD equaliter divisa.

[2.93] Unde visu immoto et perpendiculari GD manente, si moveatur equidistans perpendiculari linea GHK, secabit ex spera motu suo portionem circularem, et cuiuslibet puncti huius portionis ymago apparebit intra speram.

[2.94] Si vero linea contingens GB moveatur equidistanter perpendiculari visus, secabit ex spera portionem predictam maiorem, et a quolibet puncto excrementi unius super aliam refertur ymago cuius locus erit in superficie spere; et aliarum quedam intra speram, quedam extra.

[2.95] Scimus ex hiis quod in hoc speculo quelibet ymago apparet in dyametro spere, aut intra, aut extra, aut in superficie. Et omnis dyameter in quo appareat ymago aliqua in superficie spere aut extra dimissior est puncto spere quem tangit linea contingens a centro visus ducta, id est ultimo puncto portionis apparentis. Scimus quod quelibet linea reflexionis secat speram in duobus punctis: in puncto reflexionis et in alio.

163 citra: intra *O*; circa *E* 164 *ante* perpendicularis *add.* utpote C *mg. a. m. F*; *add. P1*/perpendicularis: perpendiculariter *C1*/ducta *corr. ex* ducatur *E*/*post* secabit *add.* sicut GD *mg. a. m. F*; *inter. P1*/*post* lineam *scr. et del.* reflexionis *S* 165 intra *corr. ex* ultra *mg. a. m. F*/quoniam *om. FP1*/*post* puncto *inter.* V *a. m. F*; *add. P1* 166 EN *om. O*/inter: citra *P1*/inter E *mg. a. m.* (inter: citra) *F*/E: Q *O*/et N *om. FP1*/N *corr. ex* EN *L3* 167 *post* sumpto *add.* a *L3*/alio *om. P1* 168 puncto . . . BH *inter. a. m. E*/ita *om. O*/ymago: ymaginem *S*; ymaginum *O*; *om. C1*/*post* est *add.* imago *R* 170 ZB *corr. ex* EB *O*/potest patere *transp. C1*/ZA: CA *O* 171 *post* penitus *scr. et del.* ZA et eadem penitus *F* 172 et: a *P1*; *alter. in a. m. F*/perpendiculari: perpendicularem *S*; per axem *O*; perpendicularis *C1*/GD: G et D *S*; GZD *O*; *alter. in* GZD *C1*/equaliter: earum *O* 173 immoto: in moto *S*; in motu *L3*/et *om. O*/GD *corr. ex* GGD *F*; G et D *S*; GZD *OR*; *corr. ex* GSD *L3* 174 equidistans: equidistantem *F*; equidistanter *OR*/*post* perpendiculari *add.* visus *R*/GHK: GH *P1R*; *alter. in F* 175 huius *om. P1S*; huiusmodi *O*/ymago *inter. O* 177 contingens GB *transp. R*/*post* contingens *inter.* scilicet *FO* 178 predictam: predicta *R* 179 unius *corr. ex* minus *O*/*post* unius *add.* portionis *R*/refertur: reflectitur *R* 180 superficie *om. O*/spere: spera *O*/quedam: quodam *F* 182 speculo: spero *S*/*post* ymago *scr. et del.* cuius locuius erit in superficie *S* 183 in[1] *inter. O*/dyametro *corr. ex* dextro *FL3* (*mg. a. m. F*); dextro *SC1*/*post* intra *add.* speram *R* 184 quo appareat: qua apparet *R*/ymago aliqua *transp. C1*/in[2] *inter. O* 185 dimissior: demissior *SR*/quem: quod *R*/linea *corr. ex* lineam *O* 186 id . . . puncto: in ultimum punctum *R*/*post* est *inter.* dimissior *a. m. F*/*post* scimus *add.* etiam *OR* (*inter. O*)/quelibet *corr ex* quecumque *a. m. C1*

[2.96] **[PROPOSITIO 10]** Restat ut loca ymaginum certius determinemus. Dico quod, sumpto dyametro, si ad ipsum ducatur linea secans speram a centro visus cuius pars interiacens punctum sectionis spere et punctum dyametri quem attingit est equalis parti dyametri interiacenti punctum illud et centrum, punctus ille non est locus alicuius ymaginis.

[2.97] Verbi gratia, sit AG [FIGURE 5.2.10, p. 568] circulus spere, H visus, ED dyameter spere, sive perpendicularis. Et HZ sit linea secans speram super punctum F et concurrens cum ED in puncto Z, et sit ZF equalis ZD. Dico quod Z non est locus alicuius ymaginis.

[2.98] Palam enim quod non est locus ymaginis alterius quam alicuius puncti ED, quoniam ymago cuiuslibet puncti est super dyametrum ab eo ad centrum spere ductum. Et quod locus ymaginis alicuius puncti ED non sit in Z sic constabit.

[2.99] Ducatur perpendicularis a puncto D super punctum F, que sit DFN, et super punctum F fiat angulus equalis angulo NFH, et sit QFN. Palam igitur quod angulus QFN est equalis angulo ZFD. Sed ZFD est equalis angulo ZDF, quia respiciunt equalia latera. Igitur QFN est equalis angulo ZDN, quare linea FQ est equidistans linee ED.

[2.100] Igitur in infinitum producte numquam concurrent. Igitur nullius puncti ED forma movebitur ad punctum F per QF, et non potest esse locus ymaginis alicuius puncti in puncto Z nisi forma eius moveatur ad F per lineam QF. Eadem erit improbatio, sumpto quocumque dyametro, quare propositum.

[2.101] Amplius, dico quod nullus punctus linee ZD potest esse locus alicuius ymaginis.

189 *post* restat *add.* iam *R* 190 sumpto: sumpta *R*/dyametro *corr. ex* dextro *O*/ipsum: ipsam *R*/ducatur *corr. ex* dicatur *L3* 191 *post* interiacens *scr. et del.* inter *C1* 192 dyametri[1]: dextri *FP1S*; *corr. ex* dextri *OL3C1* (*a. m. C1*)/quem: quam *R*/attingit *corr. ex* contingit *S* 193 *post* interiacenti *add.* inter *R*/punctus ille: punctum illud *R*/locus *corr. ex* locuius *mg. a. m. F* 196 dyameter: semidyameter *R*/et *om. FP1* 198 equalis: vel *FL3*/*post* equalis *add.* vel *P1*/ZD: EZD *FP1L3* 199 est *inter. a. m. E*/*post* est *scr. et del.* e *S*/alterius quam *om. S*; *mg. a. m. C1* 200 *post* alicuius *scr. et del.* quoniam alterius *C1*/puncti ED *om. S*/*post* puncti *add.* linee *R*/*post* ED *scr. et del.* quoniam ymago cuiuslibet puncti ED *L3*/quoniam . . . ED (202) *om. FP1*/cuiuslibet: cuiuscumque *S*; cuiusque *OC1* 201 dyametrum: dextrum *SL3*; *corr. ex* dextrum *O*/ductum: ductam *R* 203 a puncto D *scr. et del.* (D: B) *F*; *om. P1*/que *inter. a. m. F*; *om. L3*; et *R*/que . . . F (204) *om. SC1* 204 DFN: DFNA *P1L3*; *corr. ex F*/angulo *corr. ex* alio *a. m. C1*/sit[2] *inter. a. m. E* 205 quod *inter. O*/angulus: angelus *FE*/*post* QFN[2] *add.* non *S*/ est equalis *transp. L3R*/est . . . angulo: equalis angulo est *FP1E*/angulo: A *S*/*post* angulo *add.* nec *S*/sed ZFD (206) *om. L3* 206 ZDF: ZDFQ *F*/quia . . . latera *om. R*/*post* equalia *scr. et del.* linea *O* 208 in *om. FSC1*/producte *corr. ex* producere *L3* 209 nullius *corr. ex* nullus *a. m. C1*/et *inter. FO* (*a. m. F*); *om. SL3C1E*/*ante* non *inter.* autem *L3E* (*a. m. E*); *add. C1*/*post* potest *add.* autem *R* 210 ymaginis alicuius *corr. ex* alicuius ymaginis *C1*/alicuius: alius *FP1*/eius *inter. a. m. E* 211 F *inter. C1*/erit *om. C1*/*post* erit *inter.* etiam *a. m. E*/improbatio . . . quocumque: probatio sumpta quacumque *R*/dyametro (212) *corr. ex* dextro *O* 212 *post* quare *add.* est *OR* (*inter. O*) 213 nullus punctus: nullum punctum *R*/locus . . . ymaginis (214): alicuius ymaginis locus *C1*

[2.102] Sumatur enim punctus P, et ducatur linea HP secans speram in puncto B. Et ducatur perpendicularis DBM, et angulo MBH fiat angulus equalis, qui sit TBM. Palam quod TBM est equalis PBD, et palam quod angulus DPH est maior angulo PZF, quia extrinsecus. Igitur duo alii anguli trianguli DPB sunt minores duobus aliis angulis trianguli DZF. Sed angulus PDB maior angulo ZDF. Restat ergo ut angulus DBP sit minor angulo DFZ. Sed angulus DFZ est equalis angulo ZDF, quare angulus DPB minor est angulo ZDF. Igitur multo minor angulo PDB, quare angulus TBM minor est angulo PDB. Igitur linee TB, ED numquam concurrent, et ita nulla ymago puncti B refertur ad punctum P. Similiter, nec ymago alterius puncti, et similiter de quolibet puncto linee ZD. Restat ergo ut tota ZD sit vacua a locis ymaginum.

[2.103] **[PROPOSITIO 11]** Amplius, sumpto quocumque dyametro inter lineas contingentie a visu ad speram ductas, preter dyametrum a centro visus ad centrum spere intellectum, et determinato in eo puncto quem diximus, qui est meta locorum ymaginum, dico quod in punctis tantum illius dyametri qui sunt inter superficiem spere et metam predictam sunt loca ymaginum punctorum illius dyametri.

[2.104] Verbi gratia, sint BZ, BE [FIGURE 5.2.11, p. 568] linee contingentie, B centrum visus, A centrum spere, BHA dyameter visualis, DA dyameter sumptus cum meta G punctus spere. Dico quod in sola puncta G et T interiacentia cadunt ymagines punctorum DA.

215 punctus: punctum *R* 216 *post* angulo *scr. et del.* B *C1* 217 TBM[1]: LH *FP1*; TBH *S*; *corr. ex* TB *L3*/TBM[2] *alter. ex* TBH *in* LBM *a. m. F*; BM *P1*; BH *S*; *corr. ex* TBH *L3*; MBH *C1E*/*post* palam *add.* quoniam *P1* 215 punctus: punctum *R* 216 *post* angulo *scr. et del.* B *C1* 217 TBM[1]: LH *FP1*; TBH *S*; *corr. ex* TB *L3*/TBM[2] *alter. ex* TBH *in* LBM *a. m. F*; BM *P1*; BH *S*; *corr. ex* TBH *L3*; MBH *C1E*/*post* palam *add.* quoniam *P1* 218 DPH: DPB *SC1*/PZF *alter. ex* ZF *in* DZF *O*/PZF . . . angulo (220) *om. FP1*/extrinsecus: exterior *R* 219 alii anguli *transp. E*/trianguli[1,2] *om. S* 220 *ante* DZF *add.* NBI *S*/DZF: ZDF *R*/angulus[1] *om. R*/PDB: DPB *C1*/*post* PDB *add.* est *ER*/angulo *corr. ex* angulos *C1* 221 DBP: DPB *FP1C1*; *corr. ex* DPB *S*/sed . . . ZDF *mg. a. m. E*/DFZ *corr. ex* DEF *S* 222 DPB: DBP *P1SOR*; *alter. in* DBP *L3* 223 PDB[1] . . . angulo *scr. et del. E*/quare: ergo *R*/angulus *om. R*/angulo *om. R*/*post* PDB[2] *scr. et del.* quare angulus *S*/linee *om. R*/*post* TB *inter.* et *O*/ED *inter. O* 224 concurrent: concurrunt *FP1L3E*/ymago . . . refertur: forma a puncto B reflectetur *R* 225 P *corr. ex* B *F*; B *P1*; H *R*/*ante* similiter[1] *add.* ut P sit locus imaginis *R*/*post* similiter[2] *scr. et del.* et *F* 226 *post* tota *add.* linea *ER* (*inter. E*) 227 sumpto quocumque: sumpta quacumque *R*/*post* quocumque *add.* primo *FP1*; *add.* puncto *L3*; *scr. et del.* puncto *E*/dyametro: dextro *FP1S*; *corr. ex* dextro *OL3* 228 preter *corr. ex* propter *P1* 229 intellectum *corr. ex* intelli *F*; intellectam *R*/eo: ea *R* 230 quem: quod *R*/qui: quod *R* 231 tantum: centrum *O*/qui: que *R*/metam *corr. ex* metram *F*; metram *P1* 232 *post* sunt *scr. et del.* sicut *P1* 233 sint: sunt *FP1L3*/contingentie (234): contingentes *R* 234 centrum[2] *corr. ex* centro *FL3* (*mg. a. m. F*); centro *P1O* 235 dyameter sumptus: deinde sumetur ymaginis *O*/*post* dyameter *scr. et del.* vis *P1*/sumptus cum: sumpta cuius *R*/cum meta *transp. O*/*post* meta *add.* sit *R*/G *om. O*; T *R*/*post* G *scr. et del.* dico *L3*; *scr.et del.* dico quod *E*/punctus: punctum *R*/punctus . . . dico: dico punctus spere *FS*/punctus . . . quod: dico quod punctus spere *P1*/*post* spere *add.* in quo diameter secat speram *R*/*post* puncta *add.* inter *R* 236 et *corr. ex* Z *S*; *om. R*/cadunt: sunt *inter. O*/*post* punctorum *add.* recte *R*

[2.105] Quod enim non cadent in puncto G vel extra superficiem spere palam per hoc quod supradictum est dyametrum in quo locus ymaginis erit in superficie spere aut extra demissiorem esse puncto contingentie; et cum dyameter DA sit inter lineas contingentie, non erit in eo locus ymaginis, aut in superficie spere aut extra. Quod autem in quodlibet punctum inter G et T sumptum cadat ymago constabit.

[2.106] Sumatur punctum, et sit Q, et ducatur linea BQ secans speram in puncto C. Et ducatur perpendicularis ACL, et angulo LCB fiat angulus equalis DCL, et educatur linea BT secans speram in puncto F, et ducatur perpendicularis AF. Igitur triangulus ACB continet triangulum AFB, quare angulus AFB maior angulo ACB. Restat ergo ut angulus AFT sit minor ACQ. Sed angulus AFT est equalis angulo FAT, quia equalia latera respiciunt. Igitur ACQ maior erit angulo CAQ, quare LCB maior CAQ, unde DCL maior CAQ. Igitur CD, AQ concurrent. Sit D concursus. Forma igitur puncti D reflectatur in puncto C per lineam CB, et locus ymaginis eius est Q. Et eadem est probatio, sumpto quocumque puncto inter G et T.

[2.107] **[PROPOSITIO 12]** Restat ut assignemus loca ymaginum in sectione spere occulta visui.

[2.108] Sint ergo AC, AG [FIGURE 5.2.12, p. 568] linee contingentes portionem apparentem, A centrum visus, B centrum spere, ADBZ dyameter visualis, et ZCG circulus spere in superficie linearum contingentie. Et protrahatur a centro ad punctum contingentie dyameter BG. Palam quod angulus ZBG est maior recto, cum enim in triangulo BAG angulus BGA sit rectus, erit angulus GBA minor recto, quare ZBG maior.

237 *post* quod *scr. et del.* n *C1*/cadent: caden *S*; cadit *E*; cadant *R*/puncto: punctum *R*/*post* superficiem *scr. et del.* terre *P1* 238 quod *rep. C1*/dyametrum: scilicet locus *O*/quo: qua *R*/ *post* quo *add.* est *ER* 239 erit *om. R*/demissiorem *corr. ex* demissionem *F* 240 contingentie^2 *corr. ex* contingentes *a. m. E*; contingentes *R*/non: nunc *E* 241 eo: ea *R* 242 quodlibet *corr. ex* quolibet *O*/punctum *inter. E*/cadat *corr. ex* cadit *E*/*post* ymago *add.* sic *R* 244 C: P *R*/et *om. O*/ducatur: educatur *O*/ACL: APL *R*/angulo: angulus *F*; *corr. ex* angulus *L3*/LCB: LPB *R* 245 angulus equalis *transp. R*/DCL: DCB *FP1S*; DEB *O*; DPL *R* 246 et ducatur *om. O*/ triangulus: triangulum *R*/ACB: APB *R*/continet *corr. ex* contineat *E* 247 AFB^1 *corr. ex* AB *L3*/ *post* AFB^1 *scr. et del.* quare angulus *E*/*post* AFB^2 *add.* est *P1*/*post* maior *add.* est *R*/ACB: APB *R* 248 *post* minor *add.* angulo *SO*/ACQ: APQ *R*/angulus^2: angelus *F*/est *om. FSL3*/equalis *corr. ex* inequalis *FL3*; inequalis *S*/*post* equalis *add.* est *mg. a. m. F* 249 FAT: fiat *F*/ACQ: APQ *R*/maior erit *transp. ER*/*ante* angulo *add.* FAT ergo et *R*/angulo CAQ *om. FP1SO*; *inter. L3*; *mg. a. m. C1E*/ CAQ: PAQ *R* 250 quare LCB: FAQ quare *O*/LCB: LPB *R*/*ante* maior^1 *add.* erit *O*/*post* maior *add.* est *R*/CAQ^{1,2}: PAQ *R*/unde . . . CAQ^2 *mg. a. m. L3*/DCL: DPL *R*/CD: PD *R* 251 D^2 *om. S*/ reflectatur *alter. in* reflectetur *O*; reflectetur *R*/in: a *R*/C: P *R* 252 CB: CD *FP1*; PB *R*/eius *inter. a. m. E* 253 T: G *FP1*; D *SO* 254 ymaginum *corr. ex* ymaginis *P1* 256 sint: sunt *FP1L3*/ *post* AC *add.* et *SO* 257 *post* centrum^2 *scr. et del.* de *C1*/ADBZ: ABZ *O*; DZ *E*/dyameter (258): dyametrum *FP1* 258 et *om. SC1ER*; *inter. O*/ZCG: ZOG *FP1* 259 et *om. O* 260 ZBG: et BG *P1*/enim *om. S* 261 angulus BGA *om. P1*/BGA: BAG *FSL3*; *corr. ex* BAG *a. m. E*/GBA *corr. ex* GA *O*; BGA *L3*/quare *corr. ex* quando *F*/*post* quare *add.* erit *O*/maior sit (262): minor fit *FP1*

Sit ergo HBG rectus. Erit ergo HB equidistans linee contingentie AG. Igitur producte numquam concurrent, et quilibet dyameter inter H et G concurret cum linea AG.

[2.109] Ducatur a puncto A linea secans speram, que sit AMO, ita quod corda, que est MO, sit equalis semidyametro OB, et concurrat dyametro BO cum linea AG in puncto T. Dico quod in quolibet puncto TO est locus ymaginis, et in nullo alio puncto dyametri TB est locus ymaginis, et sunt O, T termini locorum ymaginis.

[2.110] Sumatur enim punctum, et sit K, et ducatur ANK secans speram in puncto N. Et ducatur perpendicularis BNX, et angulo XNA fiat angulus equalis per lineam FN. Palam quod FN non cadet inter B et T, quoniam aut secans speram aut secat contingentem AP in duobus punctis. Igitur forma puncti F movebitur per FN ad punctum N, et reflectetur ad A per lineam AN, et apparebit ymago eius in puncto K. Et eadem probatio, sumpto quocumque alio puncto.

[2.111] **[PROPOSITIO 13]** Amplius, dico quod in arcu OG, quicumque sumatur dyameter, continebit loca ymaginum et intra speculum, et unam in superficie speculi, et alias extra speculum.

[2.112] Sumatur ergo punctus L [FIGURE 5.2.13, p. 569], et protrahatur dyameter BL usquoque secet AP in puncto E. Et producatur linea AL secans speram in puncto R. Palam quod RL minor est LB, quia est minor MO, que est equalis semidyametro. Si ergo ab A ducatur linea ad dyametrum LB cuius pars interiacens circulum et dyameter sit equalis parti dyametri a puncto in quod cadit ad centrum, cadet quidem

262 ergo[1] *inter. a. m. E*/*post* erit *scr. et del.* perpendicularis *P1*/HB *om. P1*/*post* AG *inter.* HB *L3* 263 concurrent *corr. ex* concurrunt *F*/et[1] *inter. E*/quilibet: quelibet *R* 264 *post* G *add.* tunc *O*/concurret: concurrent *S*; continet *O*; *corr. ex* concurrent *C1* 265 *post* ducatur *add.* ante *S*; *add.* autem *OC1* 266 semidyametro: semicirculo *FP1*/OB: OH *FP1*; *om. O* 267 dyametro *om. O*; *corr. ex* linea *E*; semidiameter *R*/dyametro BO: DIS ab O *FP1* 268 TO: DO *P1*/et *om. FP1SL3*; *inter. OC1* (*a. m. C1*)/et . . . ymaginis[1] (269) *inter. a. m. E*/dyametri: dicit *FP1L3*; diametrum *S*; *alter. ex* dicit *in* diametrum *O*/TB: EH *P1*; OB *SC1E* 269 ymaginis[2]: ymaginum *OC1R* 270 ducatur ANK *transp. ER* (ducatur *inter. a. m. E*)/*post* ducatur *add.* linea *O*/ANK: AK *O* 271 ducatur *corr. ex* dicatur *C1*/angulo *corr. ex* alio *mg. a. m. F*/*post* angulo *add.* alio *P1*/XNA: YNA *FSOL3*; NA *P1* 272 FN[2]: NF *R*/cadet: cadit *FP1* 273 et *om. R*/T *corr. ex* D *a. m. F*; TC *P1*; G *R*/*post* quoniam *add.* sic *R*/*post* aut[1] *inter.* esset *L3*/secans: secaret *OR*; secat *C1*; *alter. ex* secat *in* secaret *E*/secat: secaret *OR*; *alter. in* secaret *a. m. E*/AP: AG *R* 274 per: super *FP1* 275 AN *corr. ex* AM *F*/eius *om. FP1* 276 *post* probatio *add.* est *R* 277 quicumque (278): quecumque *R* 278 *post* continebit *inter.* quedam *O*/et *om. O*/*post* speculum *inter.* quasdam *L3E* (*a. m. E*); *add. R* 279 *post* unam *inter.* tantum *O*/speculi *corr. ex* puncti *FOC1* (*mg. a. m. C1*); puncti *P1S*; circuli *E*; *om. R* 280 ergo *inter. a. m. E*/punctus: punctum *FP1R* 281 BL *inter. O*; RBL *R*/usquoque: usquequo *FP1SL3*; quousque *R*/AP: GP *O*; AT *R*/in: a *FP1*; etiam *S* 282 R: L *O*; *om. S*/RL: YL *inter. O*/LB: TB *R*/quia *corr. ex* quod *a. m. E* 283 MO: LM *O*/que *corr. ex* qui *O*/est *om. SO*/semidyametro: semicirculo *FP1*/*post* ducatur *add.* semidyameter *mg. a. m. F*; *add. P1* 284 ad *inter. O*/LB: BL *R*/*post* interiacens *add.* inter *R*/dyameter: dyametrum *ER* 285 quod: quo *O*/*post* cadit *add.* usque *R*/cadet: cadit *C1*/quidem *om. R*

inter L et B. Si enim inter L et E ceciderit, erit RL maior LB, et omnis linea interiacens centrum et illam equalem erit maior parte dyametri que terminatur, secundum probationem assignatam in explanatione mete ymaginum.

[2.113] Sit ergo punctus in quem linea equalis cadet I. Dico quod in quolibet puncto in EI sumpto est locus ymaginis. Et erit eadem demonstratio que fuit in TO.

[2.114] Igitur quedam ymagines in dyametro EB sortiuntur loca intra speculum, quedam extra speculum, una sola in superficie, scilicet puncto L. Et ita poteris demonstrare in quolibet dyametro partium OG transeunte.

[2.115] **[PROPOSITIO 14]** Amplius, sumpto quocumque dyametro in arcu OH, locus ymaginis in eo erit extra speculum.

[2.116] Sumatur dyameter BQ [FIGURE 5.2.14, p. 570], et concurrat cum contingente in puncto P. Et ducatur linea ANQ secans speram in puncto N. Iam dictum est quod MO est equalis OB, sed NQ est maior MO, quare NQ est maior QB. Et linea ducta ad circumferentiam ad dyametrum PB equalis parti BP ipsam et centrum interiacenti non cadet inter Q et B. Si enim ceciderit, secundum supradictam probationem erit NQ minor QB.

[2.117] Restat ergo ut linea equalis cadat inter P et Q. Et quod non cadat in punctum P palam per hoc quod angulus PGB rectus. Igitur PB maius PG. Cadet ergo citra P.

[2.118] Sit punctus in quem cadit S. Erit igitur S meta locorum ymaginum, et quilibet punctus inter P et S erit locus ymaginum, et est eadem probatio quam supra.

286 inter[1]: intra *FP1SOC1E*/L et B: B et L *L3*/inter[2] *om. C1*/RL: L *O*; *corr. ex* LR *E*/*ante* maior *inter.* illa *O*/et[3] *om. R*/*post* omnis *add.* enim *R* 287 *post* interiacens *add.* inter *R*/*post* illam *add.* partem linee reflexionis illi parti diametri *R* 288 que: qua *R*/terminatur: terminantur *FP1*; terminat *SO*/explanatione: expletione *FP1*; explatitione *S*; exploratione *O*; *corr. ex* expletione *a. m. C1* 290 sit *corr. ex* si *F*/ergo punctus *transp. FP1*/punctus: punctum *R*/quem: quod *R*/cadet: cadit *R*/I:L *FP1L3* 291 quolibet: quocumque *C1*/in *om. FP1SL3E*; linee *R*/EI *om. S*/sumpto *om. FP1L3ER*/est locus *transp. E* 292 TO: DO *FP1*; SD *O* 293 intra (294): inter *FP1O* 294 *post* scilicet *add.* in *SOC1R* 295 quolibet: qualibet *R*/partium: per puncta arcus *R* 297 sumpto quocumque: sumpta quacumque *R* 298 in arcu OH *corr. ex* m ar cuoh *mg. a. m. F*/*post* OH *add.* m ar cuoh *P1*/locus . . . P (300) *mg. O* 299 dyameter: dyametrum *SC1* 300 in[1] *inter. E*/ANQ: AUQ *OL3C1ER* 1 N: U *R*/NQ *alter. in* UQ *a. m. E*; UQ *R*/est *inter. a. m. E* 2 MO . . . maior *mg. a. m. L3*/NQ: UQ *L3R*; *alter. in* UQ *a. m. E*/*post* maior *add.* OB id est *R*/QB: BQ *R*/ad[1] *alter. in* a *O*; a *R*/circumferentiam: circulo *O*; circumferentia *R* 3 BP: PB *SOL3C1ER*/ipsam . . . interiacenti: interiacenti inter ipsam et centrum *R*/interiacenti: interiacentem *E* 5 NQ: ut *O*; *alter. in* UQ *a. m. E*; UQ *R*/minor *alter. in* maior *O* 6 cadat: cadit *L3* 7 quod *inter. O*; quia *R*/*post* PGB *add.* est *R*/PB *corr. ex* PG *L3* 8 *post* maius *add.* est *R*/cadet *corr. ex* cadat *F*/citra *alter. ex* circa *in* intra *O*/*post* citra *add.* punctum *L3C1ER*/P: PAP *FP1* 9 punctus in quem: punctum in quod *R*/erit igitur S *om. FP1*/S *inter. O* 10 quilibet punctus: quodlibet punctum *R*/ymaginum *corr. ex* ymaginis *P1*/est eadem (11) *transp. R* 11 quam *alter. in* que *a. m. E*; que *R*

[2.119] Palam ex hiis quod ymagines dyametrorum arcus HO omnes sunt extra; ymagines dyametri FB una in superficie, que est in O, alie omnes extra, scilicet in TO; omnes autem ymagines dyametri arcus OG, quedam intra, quedam extra, quedam in superficie.

[2.120] **[PROPOSITIO 15]** Amplius, in arcu HZ [FIGURE 5.2.15, p. 570] non potest sumi dyameter in quo est locus ymaginis, quoniam nullus dyameter ibi sumptus concurrit cum contingente AP.

[2.121] Et a quocumque puncto illius talis dyametri ducatur linea ad speram. Cadet quidem in portione GZC et nulla in portione GDC, nisi secando speram. Ergo nulla forma puncti alicuius talis dyametri veniet ad portionem visui apparentem.

[2.122] Quod autem dictum in arcu GH potest eodem modo demonstrari in parte arcus CZ eam respiciente. Et sumpto arcu citra Z equali HZ, in nullo dyametro illius arcus erit ymaginis locus.

[2.123] Idem est demonstrandi modus in quocumque circulo, quare, si linea HB moveatur, eodem manente angulo HBZ, signabit motu suo portionem spere in dyametris cuius nullus sit ymaginis locus. Si vero HB immota, moveatur OB, describetur portio cuius omnes ymagines extra; verum dyametrum TB una in superficie, alie extra. Moto autem arcu OG, fiet portio cuius quedam ymagines in superficie, quedam extra speculum, quedam intra.

[2.124] Verum visus non comprehendit que ymagines in superficie spere aut que extra, nec certificatur in comprehensione earum nisi quod

12 ymagines *om. O*/dyametrorum: dyametrum *O*/omnes: OS *FP1SO*; *corr. ex* OG *a. m. E* 13 *post* extra *add.* superficiem speculi *R*/ymagines: imaginum *R*/dyametri: dicit *FP1*; dyametrum *S*/FB: FY *R*/superficie *corr. ex* se *O*/*post* superficie *add.* speculi *R*/in O *corr. ex* LO *a. m. E*/O: L *R*/*post* O *inter.* nec *O*/*post* alie *add.* intra scilicet in IL alie *R* 14 TO: DO *FP1*; LE *R*/omnes . . . ymagines: omnium autem imaginum *R*/*post* ymagines *scr. et del.* extra *P1*/*post* dyametri *add.* et *FP1SO*; *scr. et del.* et *L3C1* 15 *post* intra *add.* speculum *R* 17 quo: qua *R* 18 nullus: nulla *R*/sumptus: sumpta *R*/sumptus concurrit *corr. ex* concurrit sumptus *C1*/concurrit: concurrunt *S* 19 quocumque puncto *transp. C1*/dyametri ducatur *transp. FP1* (dyametri: dyameter *F*) 20 portione[1,2]: portionem *R*/in[2] *inter. E* 21 ergo: quare *R*/puncti alicuius *transp. R* 23 dictum in arcu: in arcu dictum *L3*/*post* dictum *add.* est *C1ER*/GH: GHZ *R* 24 demonstrari *corr. ex* demonstra *F*/respiciente: respicientie *FP1*; respicientem *O*/et *inter. a. m. E*/arcu: arca *S*; *om. O*/citra: cui *FL3*; in *P1*; circa *SO*; circu *C1*; *corr. ex* circa *E*/Z *alter. ex* AZ *in* CZ *mg. F*; *alter. ex* CD *in* C *P1*; et *S*; AZ *L3* 25 nullo: nulla *R* 26 idem est *transp.* (idem *inter.*) *O*/est *rep. S* 27 si linea: similia *FP1S*; *corr. ex* similia *L3*/HB moveatur *corr. ex* moveatur HB *C1*/*post* HB *scr. et del.* immota *S*/signabit: significabit *E*/*post* signabit *scr. et del.* n *P1* 28 dyametris cuius *transp. R* 29 immota *alter. in* mota *O*; *alter. in* in meta *a. m. E*/*ante* moveatur *scr. et del.* mota *F*; *inter.* non *O*/OB: OH *SER*/portio: proportio *FP1* 30 *post* extra[1] *add.* speculum sunt *R*/verum . . . extra[2] *om. R*/TB: TH *FP1*/alie: alia *O* 31 fiet: fiat *FP1L3E*/*post* ymagines *add.* sunt *R*/in superficie: et sic *FP1*/extra (32): intra *SO* 32 intra: extra *SO*/*post* intra *add.* quedam extra speculum *R* 33 *post* ymagines *add.* sint *OR* (*inter.O*) 34 que *inter. O*

sunt ultra portionem apparentem. Iam ergo determinata sunt in hiis speculis ymaginum loca.

[2.125] **[PROPOSITIO 16]** Amplius, puncti visi forma non potest in hoc speculo ad visum reflecti nisi a solo puncto speculi.

[2.126] Sit enim punctus B [FIGURE 5.2.16, p. 571], A centrum visus, et non sit A in perpendiculari ducta ad centrum. Dico quod B refertur ad A ab uno solo puncto speculi, et unam solam pretendit visui ymaginem in hoc speculo.

[2.127] Palam quod ab aliquo puncto potest reflecti forma eius. Sit illud G, et ducantur BG, AG. Et sit N centrum spere, et ducatur dyameter BN secans superficiem spere in puncto L, et termini portionis opposite visui sint D, E. Et secet linea AG perpendicularem in puncto Q, qui est locus ymaginis.

[2.128] Palam quod A, N, B sunt in eadem superficie ortogonali super speram. Et cum omnes superficies ortogonales super speram in quibus fuerit BN secent se super BN, et non possit superficies in qua linea BN extendi ad punctum A nisi una tantum, palam quod A et B sunt in una superficie ortogonali tantum super speram, non in pluribus. Et cum necesse sit quod punctus visus et A sint in eadem superficie ortogonali super punctum reflexionis, palam quod non fiet reflexio puncti B ad visum nisi in circulo spere qui est in superficie ANB. Sit ergo circulus DGE. Dico iterum quod a nullo puncto huius circuli, preter quam a G, fiet reflexio.

[2.129] Si dicatur quod a puncto L, cum BN sit perpendicularis, et AL non sit perpendicularis, et forma per perpendicularem veniens nec-

35 sunt[1]: sint *FP1*/apparentem *corr. ex* apparens *O* 36 *post* loca *add.* A *C1* 37 *post* visi *add.* in *S*/forma *inter. a. m. E* 38 *post* ad *add.* unum *R*/a: ab *R*/*ante* solo *add.* uno *R*/*post* solo *add.* uno *C1* 39 enim *om. O*/punctus: punctum *R*/*post* punctus *inter.* visus *a. m. O*; *add.* qui erit visus *C1*; *add.* visum *R* 40 et *inter. S*/non: enim *P1*/sit *om. FP1*/A *inter. L3*/perpendiculari: perpendiculariter *FP1*/*post* centrum *add.* spere *ER* (*inter. a. m. E*)/refertur: reflectitur *R* 41 puncto speculi *transp. ER* (puncto *inter. a. m. E*)/speculi *om. O*/pretendit: ostendit *R*/visui *alter. ex* ? *in* ipsa *O* 43 ab *om. S* 44 N: enim *S*; *corr. ex* enim *L3* 45 termini *corr. ex* TL *mg. F*; *om. P1*/opposite visui (46) *transp. R* 46 linea: lineam *FP1SOR*; *corr. ex* lineam *L3*/qui: quoniam *S*; quod *R* 47 ymaginis: ymaginum *FP1L3ER* 48 sunt: sint *R* 49 *post* speram[2] *scr. et del.* est *O* 50 fuerit: fuit *O*; fuerint *R*/se *om. O*; *inter. a. m. E*/super BN: superficiem BNA *O*/qua linea (51): quo lineam *SO* 51 linea *om. FP1*/linea BN *transp. R*/BN: EBN *P1*/ad punctum *rep. P1*/et *inter. O*/B: BN *OC1*; *alter. in* N *E*/*post* B *add.* et N *L3R* (*inter. L3*) 52 ortogonali *om. FP1*/ortogonali tantum *transp. R*/ortogonali . . . superficie (53) *mg. S*/*post* speram *inter.* et *O*/non *alter. ex* NN *in* N *F*; N *S* 53 quod . . . visus: ut omne punctum visum *R*/*post* quod *inter.* omnis *E* 54 reflexio *corr. ex* refer *mg. F* 55 circulo: circulum *FP1*/qui: quoniam *F*; EM *P1*/ANB: AMB *P1* 56 ergo *om. FP1*/iterum: esse *scr. et del. O*; igitur *R*/a nullo: autemlo *S* 57 a G *om. FP1*/reflexio *corr. ex* reflexionibus *F*; *corr. ex* inflexio *O*; inflexio *L3*/*post* reflexio *add.* a puncto G *P1* 58 *post* si *add.* enim *R*/L: R *FP1*/*post* sit *add.* super superficiem speculi *R*/perpendicularis: particularis *S* 59 perpendicularis: particularis *S*/per *om. FP1*; *inter. L3E*/perpendicularem: perpendicularis *FP1*; partem *S*

essario per perpendicularem reflectatur, planum quod non refertur B ad A a puncto L. Iterum nec ab alio puncto arcus LE. Quia ad quodcumque punctum illius arcus ducatur linea a puncto B, tenebit cum contingente illius puncti angulum obtusum ex parte E, et linea ducta a puncto A ad illud punctum tenebit cum contingente illa angulum acutum ex parte L. Quare si ab illo puncto fieret reflexio, esset angulus acutus equalis obtuso.

[2.130] Iterum a nullo puncto GL potest fieri reflexio. Sumatur enim punctum quodcumque, et sit Z, et ducatur linea AZO secans perpendicularem in puncto O. Et ducatur linea contingens circulum in puncto Z, que cadit necessario inter BG et BL, et sit MZ. Et sit FG linea contingens circulum in puncto G. Palam ex superioribus quod proportio BN ad NQ sicut BF ad FQ. Eodem modo erit proportio BN ad NO sicut proportio BM ad MO. Sed maior est proportio BN ad NQ quam BN ad NO. Igitur maior est proportio BF ad FQ quam BM ad MO, quod plane impossibile, cum BF sit minus BM, et FQ maius MO. Restat ergo ut a puncto Z non fiat reflexio.

[2.131] Verum quod ab aliquo puncto arcus GD non fiat reflexio sic constabit. Sumatur quodcumque punctum, et sit T. Educatur linea BT, et linea ATH secans BN in puncto H. Et ducatur contingens circulum in puncto T, que sit CT. Erit ergo proportio BN ad NH sicut BC ad CH, et BN ad NQ sicut BF ad FQ. Sed maior BN ad NH quam BN ad NQ. Ergo maior BC ad CH quam BF ad FQ, quod plane falsum, cum BF maior BC, et CH maior FQ. Restat ergo ut a nullo puncto arcus GD fiat reflexio puncti B, quare quodlibet punctum ab uno solo puncto spere refertur ad visum. Ergo una sola erit linea reflexionis cuiuslibet puncti visi, quare unica unius puncti ymago.

60 per: propter *F*; *om. O*; *corr. ex* propter *a. m. E*/perpendicularem: partem *S*; perpendiculariter *O*/planum: palam *R*; palam *mg. a. m. E*/refertur: reflectetur *R*/B ad A (61): BA DA *FS* 61 iterum nec *om. R*/*ante* ab *add.* planum etiam est quod non reflectetur *R*/alio: aliquo *FP1* 62 *post* puncto *scr. et del.* A ad illud punctum tenebit cum contingente *S* 63 contingente *corr. ex* contingentie *S*/puncti . . . illa (64) *mg. a. m. E* 64 illud: illum *FP1* 67 iterum: item *SC1*/*post* a *scr. et del.* viso *P1*/*post* puncto *add.* arcus *C1R*/potest: post *L3*/enim *corr. ex* EN *F* 68 AZO: AZD *FP1S* 69 perpendicularem: partem *S* 70 Z *corr. ex* C *a. m. F*; C *P1*; *om. S*; *corr. ex* E *L3*/cadit: cadet *SOC1*/sit[2] *om. R*/linea contingens (71) *om. R* 71 circulum *corr. ex* circulo *F*/*post* circulum *add.* contingat *R*/proportio *corr. ex* portio *E* 72 BN[1]: DN *E*/erit *om. ER*/BN[2] . . . proportio[2] (73) *om. S* 74 maior est *om. P1*/est *alter. in* erit *a. m. E* 75 BF: LF *F*/minus: minor *R*/maius: maior *R* 77 verum . . . reflexio *om. P1*/ab *inter. O* 78 educatur: et ducatur *R*/BT: DT *C1* 79 ATH: ATB *C1*/BN . . . CH (80) *mg. a. m. C1*/et[2] *om. R*/et ducatur: educatur *O* 80 CT: PT *R*/*post* ergo *add.* ex superioribus *R*/BC: HC *S*; BP *R*/CH *corr. ex* BH *P1*; PH *R* 81 NQ *corr. ex* QN *L3*/maior . . . NH: BN ad NH maior est *R*/ad[3]: A *S*/NH *corr. ex* NF *O*; NK *C1*/BN[3] *om. FP1SO*; *inter. L3C1* 82 *post* maior *add.* est proportio *R*/BC: BP *R*/CH *corr. ex* BH *P1*; PH *R*/ad[2] *om. F*/cum: quoniam *S*/*post* BF[2] *add.* sit *R* 83 BC: BP *R*/CH: PH *R*/*post* CH *scr. et del.* et *L3* 84 spere refertur (85): speculi reflectitur *R* 85 puncti *corr. ex* puncto *O* 86 unica *corr. ex* mututa *mg. a. m. F*; mutata *P1*; linea *L3*

[2.132] Si autem punctus B fuerit in perpendiculari visuali, palam quod reflectitur ab uno solo puncto, quoniam per perpendicularem tantum, et unica erit eius ymago et propter continuitatem aliorum punctorum in loco ymaginis proprio.

[2.133] **[PROPOSITIO 17]** Amplius, si in aliquo dyametro sumantur duo puncta ex eadem parte centri, locus ymaginis puncti centro propinquioris erit remotior a centro spere loco ymaginis puncti remotioris a centro spere. Et locus reflexionis puncti propinquioris centro erit remotior a centro visus quam puncti remotioris a centro spere.

[2.134] Verbi gratia, dico quod locus ymaginis puncti C [FIGURE 5.2.17, p. 572] remotior est a centro loco ymaginis puncti B, et punctus reflexionis puncti C remotior ab A puncto puncto reflexionis puncti B, qui est punctus G. Dico quoniam punctus C non reflectitur nisi ab aliquo puncto arcus GL.

[2.135] Palam enim quod non reflectetur ab aliquo puncto arcus LE, nec a puncto L, nec a puncto G, cum B reflectitur ab eo. Et si dicatur quod ab aliquo puncto arcus GD, sit ille punctus T, et sit CT linea per quam forma movetur ad speculum. Et ducatur perpendicularis NT per que necessario dividet angulum CTA per equalia, et ducatur perpendicularis NGK. Erit angulus NTA maior NGA. Restat ergo angulus PTA minor angulo KGA, quare angulus CTP minor angulo BGK. Sed angulus CTP valet angulum TNC et angulum TCN, quia extrinsecus. Et angulus BGK valet angulum GNB et angulum GBN. Erunt ergo duo anguli TNC, TCN minores duobus angulis GBN, GNB, quod est impossibile, cum angulus TNC contineat GNB tanquam partem et angulus TCN sit maior GBN.

87 si . . . ymago (89) *om. S* / autem punctus *corr. ex* a puncto *O* / punctus: punctum *R* / fuerit *om. FP1* 88 reflectitur: reflectetur *R* / puncto *inter. a. m. E* / quoniam *om. R* / per *om. FP1O*; *inter. L3E* (*a. m. E*) / *post* per *add.* quod *R* / perpendicularem: perpendicularis *FP1R*; *corr. ex* perpendicularis *a. m. E* 89 eius ymago *transp. FP1* 91 alio: aliquo *L3*; aliqua *R* / sumantur duo puncta (92): duo puncta sumantur *FP1* 92 eadem . . . centri: parte centri eadem *R* / puncti *om. R* / centro . . . puncti (93) *mg. a. m.* (centro propinquioris (93) *transp.*) *C1* 94 et . . . spere (95) *om. R* 95 centro[1] . . . remotior *inter. L3* / visus . . . centro[3] *rep. P1* / *post* remotioris *scr. et del.* verbi gratia *O* 96 C *inter. O*; P *R* 97 et *inter. O* / punctus: punctum *R* 98 C: P *R* / remotior: remotius *R* / ab: a *L3E* / A *om. E* / A puncto *transp.* (A *inter.*) *L3* / *post* A *add.* visus *C1* / *post* puncto[1] *add.* visus *R* / puncto[2] *om. P1OE* / reflexionis *corr. ex* remotionis *O* 99 qui: quod *R* / punctus[1,2]: punctum *R* / quoniam: quod *ER* / punctus[2]: punctum *FP1* / C *om. FP1*; P *R* / reflectitur: reflectetur *OC1* 100 aliquo: alio *SO* / puncto *inter. L3* / *post* arcus *add.* puncto *L3* 101 reflectetur: reflectitur *R* 102 LE . . . arcus (103) *mg. a. m. E* / nec[1]: nisi *ER* / reflectitur: reflectetur *R* 103 ille *alter. in* illud *O* / ille punctus: illud punctum *R* / CT: TP *R* 104 et ducatur: educatur *O* / NT: PTU *R* 105 per[1] *om. R* / CTA: PTA *R* / et ducatur *corr. ex* educatur *O* 106 NGK: NGH *S* / NGA: NQA *F* 107 PTA: UTA *R* / KGA *corr. ex* GA *P1* / CTP: PTU *R* / BGK *corr. ex* GP *O* 108 CTP: OTP *S*; PTU *R* / valet: valeat *FP1* / TNC: DNC *F*; TCN *O*; TNP *R* / angulum[2]: angulus *FP1*; *corr. ex* angulus *L3*; *om. R* / TCN: TNC *OE*; TPN *R* 109 extrinsecus: exterior *R* 110 TNC: tunc *S*; TNP *R* / TCN *inter. L3*; TPN *R* / *post* GBN *add.* et *C1* 111 TNC: PNT *R* / *post* contineat *add.* angulum *R* 112 TCN *corr. ex* CTN *a. m. F*; CTN *P1*; TPN *R*

[2.136] Restat ergo ut punctus C non reflectatur nisi a punctis G et L intermediis. Et omnes linee a puncto A per hec puncta ducte ad dyametrum BN cadunt in puncta a centro spere remotiora puncto Q, et cadunt in puncta spere a centro visus magis elongata puncto G, et ita propositum.

[2.137] **[PROPOSITIO 18]** Amplius, dato speculo et dato puncto viso, est invenire punctum reflexionis.

[2.138] Sit enim B [FIGURE 5.2.18, p. 572] punctus visus, A centrum visus, et ducantur ab eis due linee ad centrum speculi. Si fuerint ille linee equales, erit facile invenire. Quoniam sumetur circulus spere in superficie illarum linearum, et scimus quod ab uno solo puncto illius circuli fit reflexio. Dividetur ergo angulus quem continent in centro ille due linee per equalia.

[2.139] Et ducatur linea dividens angulum extra speram. Erit quidem perpendicularis super lineam contingentem hunc circulum in puncto per quem transit. Et si ducantur ad illud punctum due linee, una a centro visus alia a puncto viso, efficient cum perpendiculari illa et duabus primis lineis duos triangulos, quorum duo latera duobus lateribus equalia, et angulus angulo. Et ita punctus circuli per quem transit illa perpendicularis est punctus reflexionis, quod est propositum.

[2.140] Si vero linea a puncto viso ad centrum spere ducta fuerit inequalis linee a centro visus ad idem centrum ducte, oportet nos quedam antecedentia proponere, quorum unum est:

[2.141] **[PROPOSITIO 19]** Sumpto circuli dyametro et sumpto in circumferentia puncto, est ducere ab eo ad dyametrum extra productum

113 ut: quod *E*/punctus C: punctum P *R*/*post* punctis *add.* inter *R* 114 ducte: ducta *S* 115 *ante* BN *add.* super *O*/cadunt . . . et *om. R*/*post* remotiora *add.* a *O*; *scr. et del.* a *E*/puncto *corr. ex* puncta *F*/*post* puncto *add.* L *FP1* 116 elongata *corr. ex* elevata *P1*/*post* elongata *add.* a *FP1L3*; *scr. et del.* a *E*/G: P *P1*/*post* G *add.* amplius dato PN speculo et dato puncto viso est invenire punctum reflexionis *FP1* (PN *om. F*)/*post* ita *add.* patet *R* 117 *post* propositum *add.* est *P1* 118 amplius . . . reflexionis (119) *om. FP1* 120 enim: N *P1S*/B *inter. a. m. C1*/punctus visus: punctum visum *R*/ 121 et ducantur: educantur *O*/*post* fuerint *add.* due *R*/ille linee (122) *transp. P1* 122 quoniam *corr. ex* quando *L3*; *corr. ex* qui *a. m. E*/*post* in *scr. et del.* spere *E* 123 *post* superficie *add.* duarum *R*/uno: alio *S*; unico *C1ER*; angulo *L3*; *alter. ex* angulo *in* aliquo *O*/solo puncto *transp. C1* 124 circuli: cum *scr. et del. O*/*post* fit *add.* unius puncti *R*/dividetur *alter. in* dividatur *a. m. C1*; dividatur *R* 125 ille due *transp. R* 128 quem: quod *R*/illud: illum *FP1* 129 visus *om. S*; *inter. O*/alia *om. R* 130 duos triangulos: duo triangula *R*/*post* latera *add.* a *P1* 131 punctus: punctum *R*/quem: quam *S*; quod *R* 132 transit *inter. O*/transit . . . perpendicularis: perpendicularis illa transit *R*/punctus: punctum *R*/quod est propositum *om. FP1R* 133 linea *om. O*/*post* linea *scr. et del.* perpendicularis *F*/*post* puncto *scr. et del.* linea *O*/*post* viso *inter.* linea *O* 134 *post* inequalis *add.* R *S*/*post* a *add.* quod est propositum *FP1* 135 antecedentia: accidentia *FP1S*; *corr. ex* accidentia *OL3*/praponere: preponere *SR* 136 sumpto[1]: sumpta *R* 137 productum: productam *R*

lineam que a puncto in quo secat circulum usque ad concursum cum dyametro sit equalis linee date.

[2.142] Verbi gratia, sit QE [FIGURE 5.2.19a, p. 573] data linea, GB dyameter circuli ABG, A punctus datus. Dico quod a puncto A ducam lineam que a puncto in quo secaverit circulum usque ad dyametrum GB sit equalis linee QE, quod sic constabit. Ducantur due linee AB, AG que aut erunt equales aut inequales.

[2.143] Sint equales, et adiungatur linee QE linea talis ut illud quod fiet ex ductu totius cum adiuncta in adiunctam sit equale quadrato AG, et sit linea adiuncta EZ. Cum igitur illud quod fiet ex ductu QZ in EZ sit equale ei quod fit ex ductu AG in se, erit QZ maior AG. Si enim EZ fuerit equalis aut maior AG, est impossibile ut ductum QZ in EZ sit equale quadrato AG. Si autem minor, palam quod QZ maior AG.

[2.144] Producatur ergo AG ad equalitatem, et sit AGT. Et posite pede circini super A, fiat circulus secundum quantitatem AGT, qui quidem circulus secabit dyametrum BG, et secet in puncto D. Et ducatur linea AD, que secabit necessario circulum, si enim esset contingens in puncto A, esset equidistans BG, et numquam concurreret cum ea. Secet igitur in puncto H, et ducatur linea GH.

[2.145] Palam, cum ABGH sit quadrangulum intra circulum, duo anguli oppositi, scilicet ABG, AHG, valent duos rectos. Sed AGB est equalis ABG, cum respiciant equalia latera, ex ypothesi. Erit igitur angulus AHG equalis angulo DGA, et angulus HAG communis triangulo totali ADG et partiali AHG. Restat ergo ut angulus HDG sit equalis angulo HGA, et triangulus sit similis triangulo, quare proportio DA ad

138 *post* puncto *scr. et del.* secat *F* / concursum *corr. ex* centrum *P1*; *corr. ex* circulum *C1* 140 sit *om. S* 141 dyameter: diametrus *C1* / punctus *corr. ex* punctis *P1* / punctus datus: punctum datum *R* / ducam: ductam *F*; *corr. ex* ductam *P1L3* 142 secaverit: secavit *O*; secuerit *R* 143 ducantur *corr. ex* dicantur *F* 145 sint: sicut *S* / adiungatur *corr. ex* adiungantur *OC1* / illud: illum *FP1*; istud *C1* 146 fiet: fit *R* / totius: eius *inter. O* 147 illud: illum *F* / fiet: fit *R* / ex . . . fit (148) *om. FP1* 148 QZ: quia *S* / *post* AG^{2} *add.* et EZ minor eadem *R* 149 equalis *corr. ex* equales *O* / impossibile: impossibilis *FP1SL3*; *corr. ex* impossibilis *C1* / ductum: ductus *R* 150 equale: equalis *R* / quod *inter. a. m. E* / QZ *corr. ex* QD *P1* / *post* QZ *add.* est *R* 151 ergo AG *mg. a. m. F*; *transp.* (AG *mg. a. m.*) *L3* / AG *om. SOC1E* / *post* AGT *scr. et del.* et posito pede super A fiat circulus secundum quantitatem *F* / et . . . AGT (152) *mg. a. m. F* / posite: posito *R* 52 circini: circuli *P1* / secundum: super *C1* 154 AD: ad A *O* / que *om. O* / secabit necessario *transp. SOC1* / esset contingens: contingeret *R* 155 esset: esse *FP1S* / concurreret: concurrent *S*; *corr. ex* concurrent *P1*; *alter. ex* concurrent *in* concurret *OC1*; concurret *E* 157 palam: planum *C1* / cum *corr. ex* autem *O* / ABGH: ABH *P1* / *post* circulum *add.* ABG AHG *R* / duo *om. R* 158 anguli . . . valent: angulos oppositos valere *R* / est equalis (159) *om. P1* 159 *post* equalis *add.* angulo *R* / ABG *inter. O* / equalia latera *corr. ex* equalater *P1* / *post* ypothesi *scr. et del.* erit igitur angulus ABG equalis angulo DGA et duos rectos seu ABG est equalis ABG cum respiciant equalia latera ex ipothesi *O* 160 AHG: AGH *FP1* / HAG *om. P1*; AHG *L3* / *post* communis *scr. et del.* commune *O* 161 *post* partiali *scr. et del.* H *F* / angulus *corr. ex* triangulus *C1* 162 triangulus sit similis: triangulum simile *R*

AG sicut AG ad AH. Igitur quod fit ex ductu DA in HA est equale quadrato AG. Sed DA equalis TA; igitur est equalis QZ. Et erit AH equalis EZ et DH equalis QE, que est data linea, et ita propositum.

[2.146] Si vero AB et AG non sint equales [FIGURE 5.2.19c, p. 573], protrahatur a puncto G linea equidistans AB que sit GN, et sumatur linea quecumque, que sit ZT, et fiat super punctum Z angulus equalis angulo AGD per lineam ZF. Et ducatur a puncto T linea equidistans ZF, et sit TM, et ex angulo TZF secetur angulus equalis angulo NGD per lineam ZM, quoniam hec linea necessario concurret cum TM, cum sit inter equidistantes. Et sit punctus concursus M. Restat igitur angulus MZF equalis angulo AGN.

[2.147] Et a puncto T ducatur linea equidistans linee ZM, que sit TO, que quidem necessario concurret cum FZ. Et sit concursus in puncto K. Et sumatur linea cuius proportio ad lineam ZT sicut BG ad EQ, lineam datam, et sit I. Deinde fiat super punctum M sectio piramidalis quemadmodum docet Ablonius in libro secundo de piramidalibus, propositione quarta, et sit UCM, que quidem sectio non secet lineas KO, KF. Et in hac sectione ducatur linea equalis linee I, scilicet MC, et producatur usque ad lineas KT, KF, et sint puncta sectionum O, L. Igitur, sicut ibidem probabitur, erit OM equalis CL .

[2.148] Et a puncto T ducatur linea equidistans CM, que sit TF, et super punctum A fiat angulus equalis angulo ZFT per lineam AND. Palam quod hec linea concurret cum GD, cum angulus AGN sit equalis FZM angulo, et angulus GAN angulo ZFT. Igitur AD linea aut erit contingens circulo aut secabit ipsum, quoniam, si non fuerit contingens, et arcus AB fuerit maior arcu AG, secabit arcum AB, et si AB fuerit minor, secabit arcum AG.

163 *post* AG[1] *add.* est *mg. a. m. F*; *add.* sit *C1*/sicut AG *om. P1*; *mg. a. m.* (*post* sicut *add.* proportio) *F*/sicut . . . AH *inter. a. m. E*/quod: cum *L3*; *corr. ex* cum *C1*/HA: AH *FP1ER*; *corr. ex* AG *L3* 164 *post* DA *add.* est *R* 165 DH: TH *F*; *alter. in* TH *P1*/*post* ita *add.* quod *C1*; *add.* est *R* 166 et . . . AB (167): fuerit maior AG et linea ducta ab A fuerit contingens ducatur AG equidistans AB *mg. O* 167 *post* linea *scr. et del.* reflexionis *P1*/GN *corr. ex* NG *a. m. E* 168 que *om. C1R*/fiat . . . Z: super punctum Z fiat *R* 170 ZF: et F *S*/et[1] *inter. L3E*/et[1] . . . TZF *om. P1*/TZF: ZF *S*/angulus *corr. ex* angulos *F*; angulis *C1*/NGD: NDG *S*; *corr. ex* NDG *L3*; DGN *C1ER* 171 quoniam: que *O*; *om. R*/hec *om. P1*/*post* hec *add.* igitur *R*/concurret: concurrit *R* 172 et *om. R*/punctus: punctum *R* 174 et *inter. a. m. C1* 175 que *om. F* 176 *post* sicut *scr. et del.* et *C1*/EQ: QE *R* 177 I: L *FP1*; Z *S*/piramidalis: pyramidis *L3* 178 Ablonius: Apollonius *R*/piramidalibus: piramidibus *O* 179 UCM: UC *O*/secet: secat *R* 180 I:L *C1E*/I scilicet *om. FP1L3*/scilicet *inter. OC1*/MC: LMC *FP1L3*; IMO *S* 181 ad *om. FP1SE*; *inter. O*/*post* et *add.* si *F*; *add.* sicut *S*/sectionum: sectionis *O*/O: D *C1* 182 sicut ibidem: sunt idem *P1*/probabitur *corr. ex* probatur *E*; probatur *R* 184 ZFT *corr. ex* FT *a. m. C1*/AND *corr. ex* AFT *P1* 186 angulo[1] *om. O*/ZFT: ZFZ *F*/erit contingens (187): tanget *R* 187 circulo: circulum *OR*/ipsum: circulum *C1*/fuerit contingens: tetigerit *R* 188 arcu AG *inter. a. m. E*/arcum . . . arcum (189) *om. P1*/et[2] . . . AG (189) *rep.* (AG[1]: AB) *S*; *mg. a. m. E*

[2.149] Sit igitur contingens in puncto A. Cum angulus GAN sit equalis angulo ZFT, et angulus AGN angulo FZY, erit tertius tertio equalis, et erit triangulus AGN similis triangulo ZFY. Similiter, cum AGD sit equalis angulo FZT, erit triangulus AGD similis triangulo FZT. Igitur que est proportio AN ad AG ea est proportio FY ad FZ, et que est proportio AG ad GD ea FZ ad ZT, quare que est proportio AN ad GD ea est FY ad ZT.

[2.150] Verum, cum TM sit equidistans FL, et FT equidistans ML, erit FT equalis ML, quare erit equalis CO, cum sit MO equalis LC. Sed MO est equalis YT, cum sit ei equidistans, et YM equidistans TO. Restat ergo FY equalis CM. Sed CM equalis I. Erit igitur FY equalis I. Sed proportio I ad ZT sicut BG ad EQ. Igitur proportio AN ad GD sicut BG ad EQ.

[2.151] Verum angulus GAN est equalis angulo GBA, sicut probat Euclides in tertio. Sed angulus NGD est equalis angulo ABG, cum NG sit equidistans AB. Igitur angulus NGD equalis est angulo NAG, et angulus NDG communis, quare tertius tertio equalis, quare triangulus NDG similis triangulo ADG. Igitur proportio AD ad GD sicut GD ad ND, quare quod fit ex ductu AD in DN est equale quadrangulo DG.

[2.152] Verum quadratum AD est equale ei quod fit ex ductu BD in DG, sicut probat Euclides, et quadratum AD est equale ei quod fit ex ductu AD in DN et ei quod fit ex ductu AD in NA. Et illud quod fit ex ductu BD in DG est equale quadrato DG et ei quod fit ex ductu BG in GD, sicut probat Euclides. Ablatis ergo equalibus, restat ut quod fit ex ductu AD in AN sit equale ei quod fit ex ductu BG in DG. Igitur proportio secundi ad quartum sicut tertii ad primum, quare proportio

190 sit[1] *om. R*/igitur *om. FP1*/igitur contingens: tangat igitur *R*/contingens: congingens *F*/in *om. FP1S*; *inter. O*/*post* cum *add.* igitur *R* 191 angulo[2] *om. FP1* 192 triangulus: triangulo *O*; triangulum *R*/similis: simul *S*; simile *R*/triangulo *om. O*/*post* cum *add.* angulus *C1* 193 FZT[1] *corr. ex* ZFT *L3*/*post* FZT[1] *scr. et del.* erit tertius tertio *S*; *add.* et *S*/erit . . . FZT[2] *om. FP1*/triangulus: triangulum *R*/similis: simile *R* 194 AN *mg. C1*/est[2] *om. E*/FY *om. P1*; *corr. ex* AN *L3*/et *inter. P1* 195 *post* ea *add.* est *R*/que *om. FP1*/AN *corr. ex* NA *E*/ad[3] *inter. O* 196 ea: eadem *P1*/*post* est *add.* proportio *FP1*/FY: SY *F* 197 FL *corr. ex* A *L3*/*post* FT *add.* sit *ER*/ML: LM *R* 198 erit[1]: est *R*/erit[1] . . . ML *om. S*/FT: ET *P1*/ML: LM *R*/*post* ML *scr. et del.* et FT equalis LM ergo et *C1*; *scr. et del.* equalis ML *E*/sit MO *transp. R* 199 ei *om. FP1*; *inter. L3*; ipsi *R*/*post* equidistans[1] *scr. et del.* totum sit MO equalis LC sed MO est equalis *S* 200 *post* CM[2] *add.* est *R*/erit *om. R*/igitur: quare *R*/*post* FY[2] *add.* est *R* 201 ZT: TZ *L3*/*post* sicut *scr. et del.* sicut *F* 204 in tertio *om. O* 205 angulus *corr. ex* anguli *F* 206 *post* tertio *add.* est *R*/triangulus: triangulum *R* 207 similis: simile *R*/ADG: AGD *C1*/ad[1] *om. P1S*/GD[1]: DG *R* 208 ND: DN *R*/*post* AD *scr. et del.* ad ZB *O*/DN *alter. ex* CND *in* ND *O*/quadrangulo: quadrato *ER* 209 fit *mg. F*/BD: IBD *E* 210 *post* Euclides *add.* 35a propositione *E*; *add.* 36 propositione *R*/equale: equalis *S*/ex . . . fit[1] (211) *om. P1* 211 *ante* AD[1] *scr. et del.* AD *O*/*post* ductu[2] *add.* BG in GD sicut probat Euclides et quadratum AD est equalis ei quod fit ex ductu AD in DN *S*/AD[2] . . . DG[2] (212) *om. S* 212 *post* equale *scr. et del.* quadrangulo *C1* 213 GD: DG *FP1*/sicut . . . DG (214) *mg. E*/Euclides *corr. ex* eudocles *C1*/ut *om. L3* 214 in[1]: I *P1*/AN *corr. ex* AM *F*/ductu[2] *om. R*/DG: GD *R*/igitur *alter. ex* L *in* ergo *L3* 215 proportio secundi: prime linee ad secundam *R*/ad quartum *om. R*/quartum: quartam *FP1*/*ante* sicut *add.* est *R*/tertii: tertiae *R*/primum: quartam *R*

AN ad DG sicut BG ad AD. Sed iam dictum est quod proportio AN ad GD sicut BG ad EQ. Igitur EQ equalis AD, quod est propositum.

[2.153] Quod si AD non fuerit contingens circulum sed secans, et fuerit AG maior AB [FIGURE 5.2.19d, p. 574], secabit quidem AG. Secet ergo in puncto H, et ducatur linea AG.

[2.154] Palam quod duo anguli AHG, ABG valent duos rectos. Sed angulus NGD est equalis angulo ABG. Igitur angulus AHG et angulus NGD sunt equales duobus rectis. Quare angulus NGD est equalis angulo NHG, et angulus NDG communis, quare tertius angulus tertio angulo equalis est, et triangulus HGD similis triangulo NDG. Igitur proportio HD ad DG sicut proportio DG ad DN, quare illud quod fit ex ductu HD in DN est equale quadrato GD.

[2.155] Sed quod fit ex ductu AD in DH est equale ei quod fit ex ductu BD in DG, sicut probat Euclides, et illud quod fit ex ductu AD in DH est equale ei quod fit ex ductu DH in DN et DH in AN. Et quod fit ex ductu BD in DG est equale ei quod fit ex ductu BG in GD et quadrato GD. Ablatis igitur equalibus (scilicet quadrato GD et eo quod fit ex ductu DH in DN) restat ut illud quod fit ex ductu DH in AN est equale ei quod fit ex ductu BG in DG, quare proportio secundi ad quartum (id est AN ad GD) sicut tertii ad primum (id est BG ad DH). Sed iam probatum est quod proportio AN ad DG sicut BG ad EQ. Igitur EQ est equalis DH, et ita propositum.

[2.156] Si vero AG erit minus AB (et secet ad arcum AB), sit sectio punctus H [FIGURE 5.2.19e, p. 574], et ducatur linea HG. Palam quod angulus NGD est equalis angulo ABG. Sed anguli ABG, AHG sunt equales, quia cadunt in eundem arcum. Igitur angulus NGD equalis est angulo AHG, et angulus NDG communis, quare tertius tertio equalis, et

216 DG: GD *SOC1R* 217 igitur EQ *om. S*; *inter. L3*/*post* EQ^2 *add.* est *ER* 218 fuerit contingens: tetigerit *R*/secans: secuerit *R* 219 *post* quidem *add.* ad arcum *P1*; *add.* arcum *R* 220 ergo in puncto: in puncto ergo *C1*/H *corr. ex* AG *O*/et ducatur: educatur *O*/AG: HG *OR* 221 AHG: AH *S*/ABG *inter. O* 222 NGD: NDG *L3*/est equalis *transp. R*/*post* est *scr. et del.* q *P1*/angulo *om. ER*/*post* angulus *add.* NGD /AHG *inter.* (*ante* AHG *inter.* et) *O*/et angulus NGD (223) *om. O* 223 NGD^2: NDG *L3* 225 equalis est *transp. OER* (est *inter. O*)/est *om. SL3C1*/triangulus: triangulum *R*/HGD: HDG *SO*/similis: simile *R* 226 ad DG *om. S*/*post* DG^1 *add.* est *R*/*post* quare *add.* et *L3* 227 est . . . GD: valet quadratum DG *E*/GD: DG *R* 228 ductu: conductu *FP1S*; *corr. ex* conductu *C1*/AD *inter. a. m. E*/DH: HD *R*/*post* equale *scr. et del.* quadrato *S* 229 ductu^1: conductu *L3*/fit *inter. a. m. E* 230 equale: equalis *O*/DH^2 . . . ductu^2 (231) *rep.* (ei^2: est) *S*/DH^3 *corr. ex* H *C1* 232 GD^{1,2}: DG *R*/igitur equalibus *corr. ex* equalibus igitur *E*/equalibus *corr. ex* qualibet *O*/GD^2: DG *C1R* 233 in^1 . . . BG (234) *om. P1*/ut *inter. a. m. E*/DH^2 *corr. ex* DG *L3*/est: sit *R* 234 secundi *om. P1*; secunde linee *R* 235 tertii ad primum: tertie ad primam *R* 236 igitur EQ *inter.* (igitur: quare) *O* 237 DH: BH *L3*; *corr. ex* HD *C1*/*post* ita *add.* est *R* 238 erit minus: sit minor *R*/et . . . AB^2 *inter. L3*/AB sit *rep. O*/sectio punctus (239): sectionis punctum *R* 239 punctus: puncti *O* 240 sunt: fuit *E* 241 arcum *corr. ex* arcus *O*/NGD: NDG *S*; *corr. ex* NDG *L3*/equalis est *transp. R* 242 AHG: DHG *O*/communis *om. S*

trianguli similes. Igitur proportio HD ad GD sicut GD ad DN, quare quod fit ex ductu HD in DN est equale quadrato GD.

[2.157] Sed quod fit ex ductu HD in DA est equale ei quod fit ex ductu BD in DG, et quod fit ex ductu HD in DA est equale ei quod fit ex DN in HD et AN in HD. Et ductus BD in DG valet quadratum DG et ductum BG in DG. Igitur, remotis equalibus, erit ductus HD in NA sicut BG in DG. Igitur proportio AN ad DG sicut BG ad HD. Sed iam dictum est quod proportio AN ad DG est sicut BG ad EQ. Igitur EQ est equalis HD, quod est propositum, quia a puncto A dato duximus lineam secantem circulum, et a puncto sectionis ad dyametrum est equalis linee date.

[2.158] **[PROPOSITIO 20]** Amplius, a puncto dato in circulo extra dyametrum eius est ducere lineam per dyametrum ad circulum, ut pars eius a dyametro ad circulum sit equalis linee date.

[2.159] Verbi gratia ABG [FIGURE 5.2.20, p. 575] sit circulus datus, BG dyameter, A punctus datus, HZ linea data. Dico quod a puncto A est ducere lineam transeuntem per dyametrum BG cuius pars a dyametro ad circulum sit equalis linee HZ.

[2.160] Probatio: ducantur linee AB, AG, et super punctum H fiat angulus equalis angulo AGB per lineam MH, et super idem punctum fiat angulus equalis angulo ABG per lineam HL. Et a puncto Z ducatur equidistans linee HM, que sit ZN, que quidem secabit HL, et a puncto Z ducatur linea equidistans HL, que sit ZT, et secet HM in puncto T. Et a puncto T ducatur sectio piramidalis TP, quam assignabit Ablonius in libro piramidis, que quidem sectio non continget aliquam linearum ZN, HL inter quas iacet. Similiter fiat sectio piramidalis ei opposita inter easdem lineas, que sit CU.

[2.161] Cum igitur linea minima ex lineis a puncto T ad sectionem CU ductis fuerit equalis dyametro BG, circulus factus secundum hanc minimam lineam, posito pede circini super punctum T, continget sec-

243 trianguli similes: triangula similia *R* / GD[1,2]: DG *R* 244 GD: DG *R* 245 ex *inter. O* / ex ductu *om. FP1S; inter. L3; mg. a. m. C1* / ductu *om. O* / fit[2] *om. E* 246 BD: DH *C1* / quod[1] *om. S* / ductu[2] *om. SOC1* / ex[2] *mg. a. m. C1* / *post* ex[2] *add.* ductu *R* 247 quadratum: quantum *FP1S; corr. ex* quantum *OL3* 248 ductus: ductio *P1; alter. ex* reductio *in* ductio *F* / NA: HNA *S; corr. ex* HNA *L3E*; AN *R* / *post* NA *add.* equa et ductus *mg. O* 249 sicut *om. O* / *post* sicut *scr. et del.* ad EQ igitur EQ est equalis HD quod est propositum *S* / BG[1] . . . DG[2] *mg. O* / *post* BG[2] *scr. et del.* in DG *S* / *post* ad[2] *scr. et del.* HG *P1* 250 est[2] *om. SOC1* 251 quia: quare *L3ER* / duximus *corr. ex* diximus *L3* 252 equalis: equale *FP1L3* 256 ad: a *O* 257 circulus datus *transp. R* 258 punctus datus: punctum datum *R* / *post* HZ *scr. et del.* et *C1* 261 probatio: preter ea *O; om. R* 262 AGB *corr. ex* ABG *L3* / idem punctum *transp. FP1* 263 AGB *corr. ex* ABG *L3* / HL *corr. ex* HHL *F* / Z: et *S* 265 linea *om. FP1L3E* / ZT: GT *E* / et[2] . . . T (266) *om. P1* 266 piramidalis: piramidis *FP1L3ER* / quam *om. S* / assignabit *alter. in* assignat *OC1; alter. in* assignavit *E*; assignavit *R* / Ablonius: Apollonius *R* / *post* Ablonius *scr. et del.* et *O; scr. et del.* UZ *C1* 267 *post* libro *add.* secundo *O* / piramidis: piramidum *SOR* / linearum: lineam *FP1L3E* 268 piramidalis: piramidis *FP1L3ER* 270 cum *scr. et del. L3* / *post* cum *scr. et del.* sit *C1* / *post* igitur *add.* si *FL3; add.* similia *S* 271 ductis: ductus *FP1* / ductis . . . CU[1] (273) *om. S*

tionem CU. Si vero minima ex lineis a puncto T ad sectionem CU ductis fuerit minor dyametro BG, circulus factus modo predicto secundum quantitatem BG secabit sectionem CU in duobus punctis.

[2.162] Sit ergo CT minima et equalis dyametro BG, que quidem secabit ZQ et HF, cum ducatur ad sectionem que eas interiacet. Et ducatur a puncto Z equidistans huic que quidem secabit HM et HL sicut sua equidistans. Secet ergo in punctis M, L, et sit MZL, et punctus sectionis in quo CT secat ZN sit Q, et super dyametrum GB fiat angulus equalis angulo HLZ, qui sit DGB. Et ducantur due linee AD, BD.

[2.163] Palam cum angulus GAB sit rectus, alii duo anguli trianguli AGB valent rectum, quare angulus LHM est rectus, et est equalis angulo GDB. Et angulus HLM est equalis angulo DGB. Igitur tertius tertio, et triangulus similis triangulo, quare proportio GB ad BD sicut LM ad MH.

[2.164] Sed quoniam angulus ADB equalis est angulo BGA, quia cadunt in eundem arcum, et angulus BGA equalis angulo MHZ, ex ypothesi, erit angulus ADB equalis angulo MHZ. Et iam habemus quod angulus GBD est equalis angulo HMZ. Ergo tertius tertio, et triangulus DEB similis triangulo MHZ. Sit E punctus in quo linea AD secat dyametrum BG. Igitur proportio BD ad DE sicut MH ad HZ. Igitur proportio BG ad DE sicut LM ad HZ.

[2.165] Verum Ablonius probat quod, cum fuerint due sectiones opposite piramidales inter duas lineas, et producetur linea ab una sectione ad aliam, pars eius que interiacet unam sectionem et unam ex lineis est equalis alii parti que interiacet aliam sectionem et aliam lineam, quare QC equalis TF. Sed TQ est equalis MZ, cum sit ei equidistans et inter

273 CU^{2}: CP *S*/ductis *corr. ex* ductus *F*; ductus *P1* 274 maior: minor *R* 275 CU *om. R* 276 CT: TC *OL3C1ER* 277 secabit *om. O*/ZQ: ZN *R*/HF: HL *R*/*post* que *scr. et del.* iacet *L3*; *add.* inter *R*/eas *inter. L3* 278 que *inter. L3*/et *om. R* 279 *post* equidistans *scr. et del.* huic que quidem *S*; *add.* TC *R*/*post* et^{2} *add.* in *O*/punctus *corr. ex* punctis *P1*; punctis *O*; punctum *R* 280 CT: TC *SL3C1ER*/ZN *corr. ex* ZM *L3* 281 HLZ *alter. in* HLM *F*; HLM *P1R*/DGB *corr. ex* DBG *L3*/due linee *R*/BD: DB *C1* 282 *post* palam *add.* quod *E*; *add.* ergo *R*/trianguli: scilicet *P1* 283 LHM *corr. ex* HHM *O* 284 DGB *corr. ex* GDGB *L3* 285 triangulus similis: triangulum simile *R*/*post* triangulo *scr. et del.* simul *E*/GB: BG *R*/*post* BD *add.* est *R* 287 equalis est *transp. SOC1*/BGA *corr. ex* BGB *S*; *corr. ex* GBA *L3* 288 ex ypothesi (289) *om. R*/ex . . . MHZ (289) *inter. a. m. E* 289 ypothesi: pothesi *S*/erit: est *R*/*ante* angulus *add.* ergo *R*/ADB *om. P1* 290 GBD: DAB *S*; *corr. ex* DAB *L3*; DBG *C1R*; *alter. ex* DAG *in* DAB *E*/tertius tertio *transp. C1*/triangulus: triangulum *R* 291 similis: simile *R*/*post* similis *scr. et del.* tib *S*/*post* sit *add.* autem *R*/punctus: punctum *R* 292 igitur^{2} . . . HZ (293) *om. OR*; *mg. a. m.* (proportio BG *transp.*) *L3* 294 Ablonius: optimum *FP1*(vel Ablonius *inter. a. m. F*); *corr. ex* oblonus *O*; OB cum *L3*; obtum *alter. in* obtusum *alter. in* Ablonius *inter. a. m. E*/*ante* probat *add.* 12 *FP1*; *add.* NZ *L3E*/probat *inter. a. m. E*/fuerint: fiunt *L3*/sectiones *corr. ex* sectionis *O* 295 piramidales: piramidum (vel piramidales *mg. a. m.*) *F*; piramidis *P1*; piramides *L3*; piramidi *E*/piramidales . . . lineas *om. R*/inter duas: interpositas *C1*/producetur: producitur *FP1*; producatur *R*/ab *inter. L3*; a *ER*/una *om. L3ER* 296 *post* interiacet *add.* inter *R* 297 alii: alteri *C1*/*post* interiacet *add.* inter *R*/*post* aliam^{1} *add.* in *FP1*/*post* lineam *scr. et del.* in sectionem *P1* 298 *post* equalis^{1} *add.* et *R*/TF: DF *E*/sed TQ est *inter. a. m. O*/sed . . . TF (299) *om. S*/TQ: TF *P1*/ei: enim *E*; illi *R*/et *inter. a. m. E*; *om. R*

duas equidistantes. Igitur MZ equalis FC, et ZL equalis TF. Igitur ML equalis TC, quare proportio GB ad ED sicut TC ad HZ, et cum TC sit equale BG, erit ED equalis HZ, quod est propositum.

[2.166] Si autem linea a T ad sectionem CU ducta et minima fuerit minor dyametro BG, producatur ultra sectionem donec sit equalis. Et secundum quantitatem eius fiat circulus qui quidem secabit sectionem in duobus punctis a quibus linee ducte ad T erunt equales BG. Et a puncto Z ducatur equidistans utrique, et tunc erit ducere a puncto A, modo predicto, duas lineas equales linee date. Erit idem penitus probandi modus.

[2.167] **[PROPOSITIO 21]** Amplius, dato triangulo ortogonio, et dato puncto in uno laterum angulum rectum continentium, est ducere a puncto illo lineam ad aliud latus continentium rectum lineam secantem tertium oppositum recto, ita quod pars huius linee interiacens punctum sectionis et latus in quo non est punctus datus se habeat ad partem lateris oppositi recto que est a sectione ad latus in quo est punctus datus sicut data linea ad datam lineam.

[2.168] Verbi gratia ABG [FIGURE 5.2.21, p. 576] est triangulus datus, cuius angulus ABG rectus, et in latere GB est punctus datus D extra triangulum aut intra. Dico quod a puncto D est ducere lineam secantem latus AG et concurrentem cum latere AB ita quod pars eius interiacens latera AB AG sit eiusdem proportionis ad partem lateris AG que est ab illa linea usque ad punctum G sicut se habet E ad Z, que sunt date linee.

[2.169] Probatio: sit punctus D in ipso triangulo ABG, et ducatur ab eo linea equidistans AB, que sit DM. Et fiat circulus super tria puncta G, M, D, et protrahatur linea AD. Et quoniam planum quod angulus GMD est equalis angulo GAB, erit maior angulo GAD. Secetur ex eo

299 FC: FZ *P1*/et . . . TF *om. O* 300 *post* TC[1] *add.* quare proportio TC ad HZ sicut ML ad HZ *R*/ ED *corr. ex* OED *F*/HZ *corr. ex* HE *a. m. F*; *corr. ex* HTL *P1* 1 equale: equalis *R*/ED *om. S*; *inter. L3E* (*a. m. E*)/est *mg. E* 2 linea *corr. ex* aliam *S*/a *om. R*/T: TC *R* 4 *post* quidem *add.* circulus *R* 5 duobus punctis *transp. C1* 6 Z: T *S*; *corr. ex* ZT *C1*/*post* utrique *scr. et del.* et *P1*/A . . . predicto (7) *om. O* 7 *ante* duas *inter.* dato *O*/lineas *om. SOC1*/*post* date *inter.* et *O*/erit: eritque *inter. a. m. C1*; eritque *mg. a. m. E*; eritque *R* 8 modus: motus *P1* 11 ad *om. E*/latus: laterum *R*/ continentium *corr. ex* continendum *O*/lineam[2] *om. R* 12 oppositum *alter. in* rectum *a. m. E*/quod: ut *R*/interiacens: interiacentis *FP1L3* 13 punctus datus: punctum datum *R*/habeat *corr. ex* habet *a. m. E* 14 oppositi *corr. ex* oppositum *C1*/a *corr. ex* de *a. m. E*; de *R* 15 punctus datus: punctum datum *R* 16 ABG . . . datus: est triangulum datum ABG *R* 17 *post* angulus *rep. et del.* in (13) . . . lineam (15) *O*/ABG *corr. ex* ATG *O*/est: cum *O*/*ante* punctus *inter.* fuerit *a. m. O*/ punctus datus: punctum datum *R*/D *om. O* 19 et: ad *S*/quod: ut *R*/*post* interiacens *add.* inter *R* 20 eiusdem: eius *FP1L3ER* 21 E ad Z *corr. ex* EA DZ *FL3* 23 probatio *om. R*/punctus: punctum *R* 24 sit: sunt *S*/fiat . . . D (25): super tria puncta G M D fiat circulus *R*/super *corr. ex* circa *a. m. C1* 25 M D *corr. ex* D M *C1*/et[2] *scr. et del. E*; *om. R*/*post* planum *add.* est *R* 26 GAB: GNB *E*/angulo *om. R*/*post* eo *scr. et del.* a *C1*

equalis per lineam MN, et sit DMN, et fit H linea ad quam se habeat AD sicut se habet E ad Z. Et a puncto N, qui est punctus circuli, ducatur linea ad dyametrum GM equalis linee H secundum supradicta, et sit NL, et punctus in quo secat circulum sit C. Et ducatur linea GC, et a puncto D ducatur linea ad punctum C, que, cum cadat inter duas equidistantes, tenens angulum acutum cum altera, si producatur, necessario concurret cum alia. Concurrat igitur, et sit punctus concursus Q.

[2.170] Palam quod angulus GMD est equalis angulo GCD, quia cadunt in eundem arcum, et angulus GMD est equalis angulo GAB. Restat igitur ut angulus GCQ sit equalis angulo GAQ. Sit T punctus in quo DQ secat AG, et angulus GTC est equalis angulo ATQ. Igitur tertius tertio, quare triangulus ATQ similis est triangulo TCG. Igitur proportio QT ad TG sicut AT ad TC.

[2.171] Verum angulus NMD est equalis angulo TAD et angulo NCD, quare NCD equalis TAD. Et angulus CTL communis duobus triangulis, quare tertius tertio, et triangulus similis triangulo, scilicet TLC triangulo TAD. Igitur proportio TA ad CT sicut proportio AD ad LC, quare erit proportio AD ad LC sicut QT ad TG. Sed LC est equalis linee H, et proportio AD ad H sicut E ad Z. Igitur proportio QT ad TG sicut E ad Z, quod est propositum.

[2.172] Si vero D sumatur in illo latere extra triangulum [FIGURES 5.2.21a, 5.2.21b, pp. 576 and 577], producatur a puncto D equidistans AB, et sit DM, et ducatur AG donec concurrat cum DM in puncto M. Et fiat circulus transiens per tria puncta G, D, M, et ducatur linea AD. Erit quidem angulus GAD maior angulo GMD. Fiat ei equalis, et sit NMD, et a puncto N, qui sit punctus circuli, ducatur linea equalis H linee ad quam H se habeat AD sicut E ad Z, et sit NCL, et hoc super dyametrum MG. Et concursus sit L.

27 MN *alter. ex* ANU *in* KM *S*; *corr. ex* NM *a. m. E* / *post* linea *scr. et del.* equidistans *P1* 28 E ad Z: EA DZ *scr. et del. F*; *corr. ex* EA DZ *L3*; E ad DZ *P1* / qui est punctus: quod est punctum *R* 29 linee: linea *FP1* / supradicta: supradictam *SOL3* 30 punctus: punctum *R* / C *inter. a. m. E* 31 cadat: cadit *C1* 32 equidistantes: equidem *S* / tenens: tenet *C1* 33 punctus: punctum *R* 34 Q *corr. ex* QI *F* 35 palam: planum *R* 36 cadunt: cadent *L3* / GAB *corr. ex* GABB *C1*; GMD *E* 37 ut *corr. ex* un *P1* / punctus: punctum *R* 38 *post* secat *scr. et del.* se *C1* 39 triangulus: triangulum *R* / ATQ: AQT *S* / similis: simile *R* / est *om. R* 40 QT *corr. ex* QTA *F* / TC *corr. ex* DC *F* 41 NMD: HMD *SL3* / et . . . TAD (42) *om. P1* 42 NCD^1: NOD *FSL3* / NCD^2: LCT *R* / NCD^2 . . . CTL *inter. a. m. E* / *post* NCD^2 *add.* est *C1* / *post* equalis *add.* angulo *O* / CTL: CTA *FP1* / *post* communis *scr. et del.* duo *S* 43 triangulus similis: triangulum simile *R* / scilicet: S *S* 44 TLC *om. FP1* / triangulo *corr. ex* triangulus *F* / TAD *corr. ex* DAD *mg. a. m. F* / *ante* igitur *add.* similis triangulo LCT *mg. a. m. F*; *add. P1* / TA: AT *C1* 45 quare . . . LC^1 *mg. a. m. L3* / erit *om. R* 46 AD *inter. L3* 48 D: DQ *S* / sumatur: sumuatur *F* / *post* sumatur *add.* quidem *O* / in illo *corr. ex* nullo *C1* / triangulum *corr. ex* angulum *E* 49 producatur: producto *SOC1* / *ante* a *add.* ducatur *OC1* / a puncto *om. S* / equidistans: equidistanter *FP1*; equidem *SOL3* 51 AD: et *S* 53 N: K *S* / qui sit punctus: quod est punctum *R* 54 *post* quam *add.* lineam *R* / et hoc *om. R*

[2.173] Cum igitur angulus NMD et angulus NCD valeant duos rectos, et angulus NMD sit equalis angulo TAD, erunt duo trianguli TCL, TAD similes. Et cum duo anguli GCD, GMD sint equales, erunt duo trianguli GCT, TAQ similes, et erit proportio AD ad CL, que est equalis H, sicut QT ad TG, et ita E ad Z sicut QT ad TG, quod est propositum.

[2.174] **[PROPOSITIO 22]** Amplius, duobus punctis datis, scilicet E, D, et dato circulo, est invenire punctum in eo ut angulum contentum a lineis a punctis predictis ad illud punctum ductis dividat per equalia linea circulum contingens in puncto illo.

[2.175] Verbi gratia ducatur a puncto E [FIGURE 5.2.22, p. 578] ad centrum circuli dati linea EG, et producatur usque ad circumferentiam, et sit ES. Deinde ducatur linea GD et sit MI linea in puncto C divisa ut proportio IC ad CM sicut EG ad GD. Et dividatur MI per equalia in puncto N, et ducatur perpendicularis NO. Et super punctum M fiat angulus equalis medietati anguli DGS per lineam MO. Palam quod erit minor recto, et angulus ONM rectus. Igitur MO concurret cum NO. Concurrat autem in O puncto, et a puncto C ducatur linea ad triangulum que sit CKF ita ut proportio KF ad FM sit sicut proportio EG ad GS. Et super punctum G fiat angulus equalis angulo MFK per lineam usque ad circulum ductam, que sit AG, et sit angulus AGE. Et ducantur due linee AG, DG. Dico quod A est punctus quem querimus.

[2.176] Ducatur linea EA. Cum ergo MFK sit equalis angulo AGE, et proportio KF ad FM sicut proportio EG ad GA, cum GA sit equalis GS, erit triangulus AGE similis triangulo MFK. Igitur, angulus FMK est equalis angulo EAG, et angulus AEG equalis angulo MKF.

56 et[1] . . . NMD (57) *om. FP1*; *mg. a. m. L3*/*post* duos *scr. et del.* DM in puncto *O* 57 duo *inter. a. m. E*/trianguli: triangula *R* 58 similes: similia *R*/sint: sunt *L3* 59 trianguli: triangula *R*/similes: similia *R*/CL *corr. ex* AD *C1* 60 ad[1] *om. O*/et . . . TG[2] *om. S*/*post* ita *add.* est *ER*/E: est *FP1*; *inter. a. m. C1*/quod: que *F* 63 est . . . punctum: invenire punctum est *C1*/ut angulum *corr. ex* ut A *a. m. O*/*post* contentum *scr. et del.* contingens (65) . . . dati (67) (circuli dati: circumdat) *O* 64 a lineis: alienis *S*/lineis: leneis *F*/illud: illum *FP1*; aliud *O*/*post* ductis *scr. et del.* lineis *P1* 65 puncto illo *transp. R* 68 *post* ducatur *rep. et del.* a (66) . . . EG (67) (circuli dati: circumdati) *S*/linea[1]: line *S*/GD: DG *C1*/MI: IM *C1*/in . . . divisa: divisa in puncto C *R*/*post* divisa *inter.* ita *a. m. F*/*post* ut *add.* sit *R* 69 IC: LC *FP1*; AT *S*/*post* CM *inter.* sit *a. m. F*; *add. P1*/et *om. R*/MI: IM *C1*/per equalia *om. P1* 71 DGS *corr. ex* DG *S* 72 angulus *om. R*/MO *om. R* 73 O *om. S*/O puncto *transp. OL3C1ER*/a . . . ducatur: ducatur a puncto C *R* 74 CKF: CHF *P1*; *corr. ex* CKFF *C1*/FM: *corr. ex* MF *E*; MF *R* 75 *post* equalis *scr. et del.* e *C1* 76 ductam: ductum *O*/sit[2] *om. O*/*post* angulus *scr. et del.* ductus *C1*/AGE *corr. ex* AEG *F*/et[2] . . . DG (77) *scr. et del.* (AG: AGA) *F*; *om. P1* 77 due *om. R*/DG: ADG *S*; AD *C1R*/*ante* dico *add.* igitur *C1*/A *inter. S*/*post* est *scr. et del.* est *S*/punctus quem: punctum quod *R* 78 *ante* MFK *add.* angulus *R*/MFK *corr. ex* FMK *F*; FMK *P1*; MFG *S* 79 FM: MF *R*/proportio[2] *om. R*/EG: GE *R*/GA[1] *corr. ex* HG *L3*/cum GA sit *om. P1*/GA sit *transp. FL3E* 80 GS *corr. ex* GP *F*/triangulus: triangulum *R*/similis: simile *R*/igitur . . . FMK *mg. a. m. F*/FMK *om. P1* 81 MKF *alter. in* MFK *S*

[2.177] Igitur a puncto A ducatur linea tenens cum linea AE angulum equalem angulo NMK, et sit linea AZ, que necessario concurret cum linea GE, quoniam que est proportio KF ad FM ea est EG ad GA, et angulus GAZ equalis angulo FMC. Igitur, sicut linea MO concurret cum FK in puncto F, concurret AZ cum GE. Sit concursus in puncto Z, et producatur AZ usque ad punctum Q ita ut linea AZ se habeat ad ZQ sicut MC ad CI, et ducatur linea EQ.

[2.178] Deinde a puncto A ducatur equidistans EQ, que sit AT. Erit quidem angulus AQE equalis angulo QAT, et quoniam duo anguli ZEA, EAT sunt minores duobus rectis, concurret AT necessario cum EZ. Sit concursus punctus T. Palam quod angulus AEG est equalis angulo MKF. Ducta a puncto E linea perpendiculari super AZ, que sit EL, erit angulus AEL equalis angulo MKN, cum angulus EAL sit equalis angulo KMN, et angulus ALE equalis angulo MNK, quia uterque rectus. Restat ergo ut angulus LEZ sit equalis angulo NKC, et angulus ELZ rectus equalis angulo KNC. Restat ut angulus EZL sit equalis angulo KCN. Igitur angulus EZQ equalis angulo KCI.

[2.179] Palam ergo quod triangulus EAG similis triangulo FMK, et triangulus EAL similis triangulo KMN, et triangulus ELZ similis triangulo KNC, et triangulus EAZ triangulo KMC. Ergo proportio AZ ad ZE sicut MC ad CK, et proportio QZ ad ZA sicut IC ad CM, et proportio QZ ad ZE sicut IC ad CK, quare triangulus QZE similis triangulo ICK, et triangulus QLE similis triangulo IKN. Erit proportio NM ad NI sicut AL ad LQ, et ita AL equalis LQ, et EQ erit equalis EA,

82 igitur: iam *R* 83 sit: fiat *O* 84 GE: EG *R*/que *inter. O*/FM: MF *ER*/ea: eadem *C1*/*post* EG *add.* que *FP1*; *scr. et del. L3* 85 *post* equalis *add.* est *ER*/MO: MC *C1*/concurret: concurrit *R* 86 F: C *OC1*/*post* F *add.* sic *R*/GE: G *O*; EG *R* 87 *ante* ZQ *scr. et del.* punctum *P1*/ZQ: QZ *C1* 88 CI *alter. ex* CY *in* CU *F*; CU *P1*; Q *S* 89 deinde: item *FP1*/*post* ducatur *scr. et del.* EQ *F*/EQ: EI *SO*/AT: GAT *S*; *corr. ex* GAT *L3*/*ante* erit *add.* et *S* 91 *post* EAT *scr. et del.* sicut *L3* 92 punctus: punctum *R*/T *inter. O*/quod *alter. in* quoniam *a. m. F*; quoniam *P1* 93 *post* ducta *add.* autem *R*/perpendiculari *om. S* 94 *post* angulus[1] *scr. et del.* equalis angulo *S*/AEL *corr. ex* AL *a. m. F* 95 KMN *corr. ex* MKN *L3*/KMN . . . angulo (96) *om. S*/et . . . MNK: et anguli ADN GZL equales MNK *inter. O*/angulo *om. R* 96 ut angulus *om. R*/LEZ *corr. ex* LEF *a. m. F*; *corr. ex* LZ *E*/sit *om. R*/angulo *om. P1L3E*/NKC *corr. ex* NKCE *L3*/ELZ *corr. ex* ELS *a. m. F* 97 KNC *alter. ex* MNKC *in* NKC *S*/EZL: ELZ *E*/angulo[2] *om. R*/KCN *corr. ex* KCM *F*; KCM *P1* 98 angulus *om. R*/*post* EZQ *add.* est *L3*/*post* equalis *add.* est *FP1*/KCI: KCU *L3* 99 triangulus: triangulum *R*/EAG: EAT *O*/similis: simile *R*/*post* similis *inter.* est *a. m. E*; *add. R*/FMK . . . triangulo (100) *inter. a. m. L3* 100 triangulus[1,2]: triangulum *R*/similis[1,2]: simile *R*/*post* triangulo *scr. et del.* KAN ei *F*/KMN *inter. a. m F*/ELZ *corr. ex* EHZ *F* 101 triangulo[1] *om. R*/triangulus: triangulum *R*/triangulus . . . quare (103) *rep.*; et[2] (102): igitur *S*/EAZ *corr. ex* KNC *F*; *corr. ex* EZ *E*/*post* EAZ *add.* similis *mg. L3*/triangulo[2] *corr. ex* triangulus *F*/AZ *corr. ex* AZT *C1* 102 QZ: AZ *R*/ad ZA *mg. a. m. F*/ZA: ZE *FP1L3*; *corr. ex* ZE *E*; ZQ *R*/IC: MC *P1R*/CM: CK *FP1L3*; CI *R*/et[2]: igitur *R*/et[2] . . . CK (103) *om. FP1L3*; *mg. a. m.* (IC *alter. in* AC) *E* 103 ZE: EZ *R*/triangulus: triangulum *R*/QZE: EQZ *S*/similis: simile *R* 104 ICK *corr. ex* ACK *O*/triangulus: triangulum *R*/similis: simile *R*/IKN *corr. ex* IKM *L3*; INK *R*/*post* IKN *inter.* ergo *a. m. F*; *add. P1*/*post* erit *add.* ergo *R* 105 *post* EQ *inter.* ergo *a. m. F*

et angulus EQZ equalis angulo LAT, et angulus EZQ equalis angulo AZT. Igitur tertius tertio equalis, et triangulus EZQ similis triangulo ZAT, quare proportio QZ ad ZA sicut EZ ad ZT, et sicut EQ ad AT, et sicut AE ad AT. Sed QZ ad ZA sicut EG ad GD. Igitur AE ad AT sicut EG ad GD.

[2.180] Fiat autem supra punctum A angulus equalis angulo GAE, qui sit UAG. Palam quod angulus GAL est medietas anguli UAT, sed est medietas anguli DGU, quare angulus UAT est equalis angulo DGU. Sed anguli TAU, TUA sunt minores duobus rectis, cum AT et UT concurrant, quare duo anguli TUA, DGU sunt minores duobus rectis. Igitur AU concurret cum DG.

[2.181] Dico quod concurret in puncto D, quoniam efficiet cum lineis UG, GD triangulum similem triangulo AUT, habebunt enim angulum AUG communem, et angulus TAU equalis angulo UGD. Igitur proportio AU ad AT sicut UG ad lineam quam secat AU ex GD, et proportio EA ad AU sicut EG ad GU, cum sit angulus UAG equalis angulo GAE.

[2.182] Cum ergo eadem sit proportio EA ad AT sicut EG ad GD, proportio EA ad AT sit compacta ex proportione EA ad AU et AU ad AT. Erit proportio EG ad GD compacta ex eisdem, quare erit compacta ex proportione EG ad GU et GU ad lineam quam secat AU ex GD. Sed est compacta ex proportionibus EG ad GU et GU ad GD. Igitur linea quam secat AU ex GD est linea GD. Igitur AU secat GD in puncto D.

[2.183] Producatur ergo a puncto A contingens que sit AH. Erit ergo GAH rectus. Sed GAL medietas anguli DGU. Igitur angulus LAH est medietas anguli DGE, cum illi duo valeant duos rectos. Sed cum angulus TAU sit equalis angulo DGU, erit angulus TAD equalis DGE.

106 *post* EQZ *add.* est *SOC1*/EZQ: EQZ *ER* 107 AZT: ZAT *R*/et . . . ZAT (108) *om. R* 108 sicut[1] . . . ZA (109) *rep. S*/*post* EZ *rep. et del.* ad . . . EZ *L3* 109 GD: DG *R*/igitur . . . GD (110) *om. L3* 111 autem *inter. F*/supra: super *R*/angulo: angulus *S* 112 UAT . . . anguli (113) *inter. a. m. L3*/sed . . . DGU (113) *inter. a. m. E* 113 anguli *om. ER*/DGU *corr. ex* DGA *O* 114 TAU: UAT *R*/*ante* TUA *add.* et *R*/minores *corr. ex* maiores *FL3*; maiores *P1*/cum . . . rectis (115) *mg. a. m.*; concurrant (115) *inter.*; TUA (115): TAU; *ante* DGU (115) *add.* et *E*/UT: TU *R* 115 TUA *corr. ex* TAU *L3C1*; DUG *O*/*ante* DGU *add.* et *P1L3R*/*post* DGU *scr. et del.* sed anguli TAU *C1* 116 AU *inter. O* 117 *post* cum *scr. et del.* E *F* 118 similem: simile *R*/AUT: aut *P1*; AUD *S*/angulum *corr. ex* angulus *E* 119 *post* TAU *add.* est *R*/*post* equalis *inter.* est *O*/UGD *corr. ex* UDG *L3*; DGU *R* 120 AU[1]: AB *S*/GD: DG *R*/et *inter. a. m. C1* 121 GU *corr. ex* AD *L3* 123 ad[1] *rep. L3*/sicut *om. FP1*; *inter. a. m. L3C1*; et *inter. O*/sicut . . . AT (124) *om. S*/*post* GD *add.* et *OR*; *inter. a. m. C1E* 124 *post* proportio *add.* autem *FP1* (*mg. a. m. F*)/compacta: composita *R*/proportione *corr. ex* proportio *F*/*post* EA[2] *scr. et del.* AU *F*/et AU *om. S* 125 compacta: composita *R*/ex *corr. ex* cum *O*/eisdem: ijsdem *R*/quare . . . ex[1] (126) *om. P1* 126 GD: DG *R*/*post* GD *scr. et del.* e *F* 127 est: EA *S*/et GU *mg. F* 128 ex *inter. F*/GD: DG *R* 129 AH *corr. ex* HA *E*; HA *R* 130 GAH: GHA *E*/*post* GAH *scr. et del.* respectu *P1*/*post* GAL *add.* est *FP1R* (*inter. F*) 131 est *inter. FOE* (*a. m. E*); *om. SL3C1*/medietas *corr. ex* medietatas *L3*/*post* DGE *add.* et *S* 132 TAD *corr. ex* TAG *a. m. F*/TAD . . . anguli (133) *om. S*/DGE *alter. ex* DGDGC *in* DGC *P1*

Igitur angulus LAH est medietas anguli TAD, et angulus EAL medietas anguli EAT. Igitur angulus EAH medietas anguli EAD, quare AH dividit angulum EAD per equalia, quod est propositum.

[2.184] Si vero AU, cum sit angulus super punctum A equalis angulo GAE, non cadat super lineam ES extra circulum vel intra, sit ergo equidistans [FIGURE 5.2.22a, p. 578]. Igitur angulus UAG equalis est angulo AGE. Sed idem est equalis angulo GAE, quare angulus GAE equalis est angulo AGE. Igitur EG equalis AE. Similiter, angulus TAD erit equalis angulo ATG, quia coalternus. Sed iam dictum est quod angulus TAD est equalis angulo DGT. Igitur angulus ATG est equalis angulo DGT, et similiter duo anguli ADG, DGT sunt equales. Igitur duo anguli ADG, TAD sunt equales.

[2.185] Sequetur ergo ex hiis quod linea quam secat AU ex DG sit equalis linee AT. Et iam dictum est quod EG equalis AE. Igitur proportio EG ad lineam quam secat AU ex DG sicut AE ad AT. Sed iam dictum est quod AE ad AT sicut EG ad GD. Igitur linea quam secat AU ex DG est GD, et cum TAD sit equalis angulo DGT, erit LAH medietas anguli TAD, sicut dictum est supra, et EAL medietas EAT. Erit ergo EAH medietas anguli EAD, quod est propositum.

[2.186] **[PROPOSITIO 23]** Amplius, dato circulo cuius G [FIGURE 5.2.23, p. 579] centrum, et dato in eo dyametro GB, et dato E puncto extra circulum, est ducere a puncto E ad dyametrum GB lineam secantem circulum ita quod pars eius a circulo usque ad dyametrum sit equalis parti dyametri interiacentis ipsam et centrum.

[2.187] Verbi gratia, ducatur a puncto E perpendicularis super dyametrum, et sit EC, et ducatur linea EG. Et sumatur linea QT equalis linee EC, et fiat super QT portio circuli ut quilibet angulus cadens in hanc portionem sit equalis angulo EGB, et compleatur circulus. Et a medio puncto QT ducatur perpendicularis ex utraque parte usque ad circulum. Erit quidem dyameter huius circuli. Et a puncto Q ducatur

133 anguli *om. FP1* 134 EAT *mg. a. m. F*/*ante* igitur *add.* erit *FP1*; *scr. et del.* DGU *S*/EAH: EAG *P1*/*post* EAH *inter.* est *O*/quare . . . EAD (135) *mg. a. m. C1* 135 per *corr. ex* pars *F* 136 super: supra *C1* 137 cadat: cadit *FP1C1*; *corr. ex* cadit *a. m. E* 138 equalis . . . AGE (139) *rep. P1* 139 AGE: DGE *SO*; *corr. ex* GE *a. m. E*/sed . . . AGE (140) *rep.* (angulo^{1}: angulus; AGE^{1}: DGE) *S* 140 equalis est *transp. R* 141 quia coalternus *om. R* 142 DGT *corr. ex* DGN *O*; *alter. in* DGE *L3*/igitur . . . DGT^{1}(143) *mg. a. m. E* 143 DGT^{2} *om. S* 144 *post* anguli *scr. et del.* in puncto *P1*/TAD: ATG *R* 145 sequetur: sequitur *C1* 146 iam *corr. ex* am *S*/*post* equalis *inter.* est *a. m. E*; *add.* sit *R* 147 ad^{1} *corr. ex* a *O*/*post* DG *add.* est *R*/sicut: sit *O*/sed . . . AT (148) *mg. a. m. C1* 148 quod: ut *R*/sicut: sic *R*/igitur . . . GD (149) *om. P1* 149 *post* est *add.* DG et *S*/GD: DG *C1R*/angulo *om. R* 150 *post* sicut *add.* iam supradictum *O*/dictum *inter. a. m. E*/supra *om. O* 151 *ante* EAD *scr. et del.* EAD *P1* 152 G centrum (153) *transp. R* 153 dato: data *R*/GB: BG *R* 154 GB: BG *R* 155 quod: ut *ER* 156 parti *corr. ex* parte *F*/interiacentis: interiacenti inter *R* 158 EG *corr. ex* AEG *P1* 159 linee *om. ER* 161 perpendicularis . . . parte: ex utraque parte perpendicularis *R*

linea ad hunc dyametrum secans eum in puncto F, et producatur usque ad P punctum circuli ita ut FP sit equalis medietati GB, et ducatur linea PT et linea TF. Et ducatur a puncto P linea equidistans dyametro, que sit PU. Concurrat cum TF in puncto U, et a puncto U ducatur equidistans TQ, que sit UO. Et a puncto T ducatur perpendicularis super PQ, que sit TN, et a puncto T ducatur equidistans PQ, que sit TS, et a puncto U perpendicularis super PQ, que sit UH. Deinde ex angulo BGE secetur angulus equalis angulo QPU, qui sit BGD, et ducatur linea EDZ. Dico quod DZ est equalis ZG.

[2.188] Et ducatur a puncto D perpendicularis super BG, que sit DI, et ducatur a puncto D contingens, que sit DK. Palam, cum dyameter FL sit perpendicularis super QT et super OU, et PU sit equidistans ei, erit angulus OUP rectus. Et cum OU dividatur a dyametro per equalia et ortogonaliter, erit FO equalis FU, quare angulus FOU equalis angulo FUO. Sed, cum duo anguli POU, OPU valeant rectum, erit angulus FUP equalis angulo FPU, quare FP equalis FU, et ita equalis FO. Et ita PO equalis BG, et equalis GD, et ita proportio EC ad GD sicut TQ ad PO.

[2.189] Sed cum angulus KDG sit rectus equalis angulo GID, et angulus IGD communis, erit triangulus IGD similis triangulo KGD, et erit proportio GD ad DI sicut GK ad KD. Sed angulus KGD equalis angulo OPU, et KDG rectus equalis OUP, et ita triangulus KDG similis triangulo OUP, et proportio KG ad KD sicut OP ad OU. Igitur DG ad DI sicut OP ad OU. Ergo proportio EC ad DI sicut QT ad OU.

[2.190] Sed proportio QT ad OU sicut TF ad FU, cum triangulus TFQ sit similis triangulo OFU. Verum angulus UTS equalis angulo HFU, quia coalternus ei, et angulus UST rectus equalis angulo FHU.

163 hunc *om. FP1*; hanc *R*/eum: eam *R* 164 ad *om. FP1*/P *om. O*/P punctum *transp. ER*/*ante* ita *scr. et del.* ita *O*/FP *corr. ex* P *a. m. F*/medietati: mediedietati *F* 166 *post* PU *inter.* que *a. m. F*; *add.* que *P1*; *add.* et *L3C1* (*inter. L3*)/concurrat: concurrent *O*; *alter. in* concurrens *E*/*ante* cum *add.* que *ER*/TF *corr. ex* T *mg. a. m. F*/a *mg. F* 167 T *inter. SO* 168 PQ: PG *P1* 169 U: A *P1*/perpendicularis: perpendiculariter *FP1*/BGE *corr. ex* BG *C1* 170 qui: que *R*/dico *om. S* 172 perpendicularis . . . D (173) *mg. a. m. E* 173 *post* palam *add.* quod *FP1*/cum *corr. ex* quod *L3* 174 sit perpendicularis *transp. C1*/et[1] *inter. F*/sit[2]: sint *FP1*/sit equidistans *transp. C1*/equidistans: equidistantes *FP1*/ei *corr. ex* eis *F*; eis *P1* 175 angulus OUP *transp. L3*/dyametro *inter. a. m. E* 176 quare: et *R*/equalis[2] *corr. ex* sequalis *F* 177 OPU *corr. ex* OPOU *S*/valeant *corr. ex* valent *E* 178 quare . . . FU *om. P1*/*ante* FU *add.* est *R* 179 proportio *om. R* 181 KDG: KDH *P1*/*post* rectus *add.* est *FP1* (*mg. a. m. F*); *inter.* ei *L3* 182 IGD[1,2]: LGD *L3*/triangulus: triangulum *R*/IGD[2] *inter. O*/similis: simile *R*/KGD: KDG *OR*; *corr. ex* KDG *E* 183 erit *om. R*/*post* KD *scr. et del.* sicut OB ad NO *C1*/*post* KGD *add.* est *R* 184 *post* et[1] *add.* angulus *R*/KDG[1] *corr. ex* KGD *L3*; KGD *E*/triangulus: triangulum *R*/KDG[2]: KGD *FP1SL3C1ER*/similis: simile *R* 185 KG: GK *R*/OU: UO *C1*/DG . . . ergo (186) *om. S* 186 DI[1]: D *O*; GL *L3*; GI *E*/ergo . . . FU (187) *mg. a. m. E*/*post* proportio *scr. et del.* que est equalis QT *C1*/EC: hec *S*/QT: TQ *R* 187 sed . . . OU *om. S*; *rep. L3*/*post* FU *scr. et del.* ergo proportio que est equalis QT ad DI sicut QT ad OU sed proportio QE ad OI sicut TF ad UT *E*/triangulus: triangulum *R* 188 similis: simile *R* 189 HFU *corr. ex* FH *C1*/UST: UFT *E*/*post* UST *add.* est *P1*

Erit triangulus UST similis triangulo HUF, et ita proportio TU ad UF sicut SU ad UH, quare proportio TF ad UF sicut SH ad UH. Sed TN equalis SH, cum sit equidistans ei, et sint inter duas equidistantes. Igitur proportio TF ad UF sicut TN ad UH, quare proportio QT ad OU sicut TN ad UH, et EC ad DI sicut TN ad UH.

[2.191] Sed cum angulus GID sit rectus equalis angulo PHU, et angulus IGD equalis angulo HPU, erit triangulus IGD similis triangulo HPU, et proportio ID ad GD sicut HU ad UP, quare proportio EC ad GD sicut TN ad UP. Sed cum angulus CGE sit equalis angulo NPT, et angulus GCE rectus equalis PNT, erit proportio GE ad EC sicut PT ad NT. Igitur proportio GE ad GD sicut PT ad UP.

[2.192] Et angulus DGE equalis angulo UPT. Igitur triangulus DGE similis triangulo UPT. Igitur angulus GDE equalis angulo PUT. Restat ergo angulus GDZ equalis angulo PUF, et angulus DGZ equalis angulo UPF, quare tertius tertio, et proportio DZ ad ZG sicut UF ad FP. Sed UF equalis FP. Ergo DZ equalis ZG, quod est propositum.

[2.193] **[PROPOSITIO 24]** Amplius, dato triangulo ortogonio ABG [FIGURE 5.2.24, p. 580] cuius angulus ABG rectus, et dato in BG vel AB puncto D, est ducere lineam a puncto D ad latus AG concurrentem in puncto qui sit Q et ex alia parte concurrentem cum alio latere ut ipsa totalis se habeat ad GQ sicut E est ad Z.

[2.194] Verbi gratia, ducatur a puncto D equidistans AB, que sit DM, et fiat circulus transiens per tria puncta D, M, G. Erit MG dyameter. Et ducatur linea AD, et sit H linea ad quam se habeat AD sicut E ad Z. Et cum angulus DMG sit equalis angulo BAG, secetur ex eo equalis angulo DAG, et sit CMD. Et ducatur MC usque contingat circulum in puncto

190 triangulus: triangulum *R*/UST: UFT *O*/similis: simile *R*/UF: US *FP1C1* 191 SU: FU *O*/quare . . . UH^{2} *om. R*/TN: TQ *FP1* 192 *post* equalis *scr. et del.* sed H *F*/SH: FH *O*/equidistans *corr. ex* equalis *mg. a. m. F*/equidistans ei *transp. R*/sint *corr. ex* sicut *L3* 193 TN: TQ *FP1* 194 et . . . UH^{2} *om. S*/UH^{2}: HU *E* 195 angulus *corr. ex* proportio *L3*/GID: GLD *L3* 196 HPU *corr. ex* HPK *a. m. F*; HUP *S*/erit . . . HPU (197) *mg. a. m. E*/triangulus: triangulum *R*/similis: simile *R*/triangulo HPU (197) *transp. R* 197 ID: IG *L3*/GD *alter. in* DG *a. m. E* 198 TN *corr. ex* TQ *a. m. F*/angulus *om. R*/*post* angulus *scr. et del.* ultis *E*/CGE: DGE *O*/NPT *alter. ex* ATP *in* APT *P1*; HPT *S* 199 proportio *om. R*/*post* ad^{1} *add.* GE *alter. in* GD *P1*/EC *alter. ex* GD *in* ET *mg. F* 200 NT: UP *F*; *corr. ex* PN *P1*; *alter. ex* UP *in* UT *S*; TN *R*/igitur . . . UP *mg. a. m. FE*/proportio *om. FP1R*/GE: EG *FP1*/GD: DG *FP1*/PT: TP *FP1* 201 et: sed *R*/*post* DGE^{1} *add.* est *R*/igitur . . . UPT (202) *om. S*/triangulus: triangulum *R* 202 similis: simile *R*/angulus: triangulus *FP1SOL3*; *corr. ex* triangulus *C1*/GDE *corr. ex* DGE *E* 203 PUF *alter. ex* PF *in* UP *F*; FUP *R*/et . . . UPF (204) *scr. et del. E*; *om. R* 204 UPF *corr. ex* PUF *S*/et *inter. a. m. C1*/*post* UF^{1} *add.* et *E*/sed . . . FP (205) *om. FP1*; *inter. a. m. L3* 205 *post* equalis^{1} *add.* est *ER* 207 AB *alter. ex* AG *in* ABG *F*; ABC *P1*; AG *SO* 208 lineam: lineas *FP1L3E*; *corr. ex* lineas *O*/*post* puncto^{2} *scr. et del.* A A *F*/AG *corr. ex* G *P1*/*post* puncto^{3} *add.* et *OL3*; *add.* Z *E* 209 qui: quod *R*/et *om. S* 210 sicut: sunt *S*/E *om. P1*/E est *transp. R*/est *om. S* 213 habeat: habet *ER* 214 DMG . . . angulo^{1} *mg. F*/angulo^{1} *corr. ex* angulus *L3*; *om. R*/BAG . . . angulo^{2} *om. S* 215 *post* DAG *rep. et del.* secetur (214) . . . DAG *C1*/usque: quousque *R*

C, a quo ducatur linea ad dyametrum MG et usque ad circulum ita quod LN sit equalis linee H. Et ducatur linea NG, et linea DN concurrens cum AG in puncto Q.

[2.195] Cum igitur angulus DMC sit equalis angulo DNC, quia super eundem arcum, erit angulus QNL equalis angulo DAQ, et angulus NQL equalis angulo DQA, quare triangulus NQL similis triangulo DQA. Ergo proportio AQ ad QN sicut AD ad NL.

[2.196] Verum, cum angulus DMG sit equalis angulo DNG, erit QNG equalis angulo TAQ. Sit T punctus in quo DN concurrit cum AB, et angulus TQA equalis angulo NQG. Erit triangulus TQA similis triangulo NQG, et erit proportio AQ ad QN sicut TQ ad QG. Ergo proportio TQ ad QG sicut AD ad LN. Sed NL equalis H, et AD ad H sicut E ad Z. Igitur TQ ad QG sicut E ad Z, quod est propositum.

[2.197] Potest autem contingere quod a puncto C erit ducere duas lineas similes CN, et tunc erit ducere duas lineas a puncto D similes TQ ut utriusque ad partem quam secat ex AG sit proportio sicut E ad Z, et erit eadem probatio.

[2.198] **[PROPOSITIO 25]** Predictis habitis, dato speculo sperico, erit invenire punctum reflexionis in eo.

[2.199] Verbi gratia, sit A [FIGURE 5.2.25, p. 581] centrum visus, B punctus visus, G centrum spere, et ducantur linee AG, BG. Et sumatur superficies in qua sunt hee due linee, et sumatur circulus communis huic superficiei et speculo. Invenietur ergo punctus reflexionis in hoc circulo.

[2.200] Et sumatur linea alia MK, et dividatur in puncto F ut FM se habeat ad FK sicut BG ad GA. Et dividatur MK per equalia in puncto O, et ducatur a puncto O perpendicularis, que sit CO, et ducatur a puncto K linea ad CO tenens cum CO angulum equalem medietati anguli BGA, que sit KC. Et a puncto F ducatur linea ad CK, que sit FP,

216 ducatur: ducat *L3*/et *om. R* 218 *post* Q *add.* et cum AB in puncto C *L3ER* (*inter. a. m. L3*) 220 angulo *om. FP1*/DAQ *corr. ex* DQA *S*/angulus^2 *om. R* 221 NQL *corr. ex* NQ *a. m. F*; NQ *P1*/triangulus: triangulum *R*/NQL *alter. ex* ACB *in* QNL *a. m. F*; ACB *P1*; AQB *S*/similis: simile *R* 222 proportio *om. R* 223 *post* verum *add.* est *SL3*; *scr. et del. OC1E*/sit . . . DNG *inter. a. m. E* 224 angulo *om. R*/TAQ: TAG *C1*/T: D *SE*; *inter. O*/punctus: punctum *R*/*post* in *scr. et del.* sicut *O*/concurrit: concurret *C1* 225 equalis *corr. ex* similis *a. m. E*/erit . . . NQG (226) *inter. a. m.* (TQA: TQD) *L3*/triangulus: triangulum *R*/similis: simile *R* 226 proportio *om. S*/QG *alter. ex* G *in* GQ *a. m. F*; GQ *P1*; QS *E*/ergo . . . QG (227) *om. S*; *mg. O* 227 LN: NL *R*/NL: LN *E*/H^1 *corr. ex* HI *L3*/et . . . H^2 *mg.* (AD *corr. ex* AT) *F*/AD: A *inter. L3* 228 igitur: G *S*/QG: GQ *C1*/est *om. S* 229 duas lineas (230) *transp. ER* 230 similes^1 . . . lineas^2 *om. S*/*post* similes^1 *scr. et del.* t *F*/CN: CLN *OR*; ELN *L3C1*; ELU *inter. a. m. E* 232 probatio: proportio *FP1* 234 in eo *om. FP1L3ER* 236 punctus visus: punctum visum *R* 237 superficies *corr. ex* res *O*; *mg. L3*/hee *om. FP1* 238 speculo: spere *O*/punctus: punctum *R* 239 linea alia *transp. FP1*/FM: MF *R* 240 dividatur: dividitur *O* 241 O^2 *om. P1*/O^2 . . . puncto (242) *mg. a. m. E* 242 tenens cum CO: contingens *inter. O*/CO: ea *R* 243 KC: HC *SE*

et concurrat cum CO in puncto S ita ut proportio SP ad PK sicut BG ad semidyametrum GD. Et ex angulo BGA secetur angulus equalis angulo SPK, scilicet DGB, et ducantur linee SK, BD.

[2.201] Erit igitur proportio BG ad GD sicut SP ad PK, et ita triangulus SPK similis triangulo BGD, et erit angulus SKP equalis angulo BDG. Sed forsan, secundum predictam, poterimus a puncto F ducere aliam lineam ad CK similem SP ut sit proportio eius ad partem quam secabit ex CK sicut SP ad PK, et tunc a puncto K ad OS ducetur alia linea quam SK alium cum CK angulum tenens maiorem vel minorem angulo CKS. Si maior ex hiis angulis non fuerit maior recto, non erit invenire punctum reflexionis. Sit igitur angulus CKS maior recto, et invenitur punctum sic.

[2.202] Erit angulus BDG maior recto. Ducatur contingens NDY, et cum angulus PKO sit minor recto, secetur ex angulo BDG equalis ei, qui sit QDG. Cum igitur angulus SPK sit equalis angulo QGD, erit triangulus FPK similis triangulo QGD, et erit angulus DQB equalis angulo KFS, et triangulus DQB similis triangulo KFS.

[2.203] Producatur autem DQ, et a puncto B ducatur perpendicularis super ipsam, que sit BZ. Erit igitur angulus BQZ equalis angulo SFO. Et angulus BZQ rectus equalis angulo SOF, et ita triangulus BQZ similis triangulo SFO.

[2.204] Ducatur DZ usque ad punctum I, et sit ZI equalis ZD. Palam ergo quod ZQ ad QB et QB ad QD sicut OF ad FS et FS ad FK, et ex hoc erit ZD ad QD sicut OK ad FK, et ita ID ad QD sicut MK ad FK, et ita IQ ad QD sicut MF ad FK, et IQ ad QD sicut BG ad GA.

244 concurrat *corr. ex* concurrant *C1*/S: C *P1*/proportio: portio *S*/*post* PK *add.* sit *P1R*; *add.* sit *mg. a. m. E*/*post* sicut *scr. et del.* per premissam *C1* 245 GD: GN *S*; G? *O*/ex *rep. S*/angulus *om. R* 246 SPK: FPK *R*/SK *corr. ex* SF *O* 247 *post* ad[1] *scr. et del.* semidyametrum GN BG *S*/PK: PR *E*/*post* PK *add.* et angulus BGD equalis angulo SPK *FP1* (et angulus *inter. F*) 248 triangulus: triangulum *R*/similis: simile *R*/BGD: BDG *E*/SKP: SPK *O* 249 BDG: BGD *O*/predictam: predicta *P1SC1E* 250 similem: simile *P1*/proportio eius *transp. SOC1*/*post* partem *add.* ad *F* 251 ducetur: ducere *SO* 252 alia linea: aliam lineam *O*/*post* linea *inter.* ut EK *L3*/SK: LK *S*; *corr. ex* S *C1*/*ante* alium *add.* ut EK *C1*/alium: aliquantum *O*/vel minorem (253) *inter. L3E* (*a. m. E*) 253 angulo: an *P1*/si *inter. O*/ex . . . maior (254) *om. O* 254 erit: licebit *R*/invenire *om. P1* 255 et *om. O*/et . . . sic *om. R*/invenitur: invenietur *SOC1*/*post* sic *add.* si *O* 256 BDG: BGD *L3*/*post* recto *add.* et invenitur punctum sic *R* 257 cum: quia *R*/sit: est *R*/BDG *corr. ex* HLG *a. m. E*/equalis ei *transp. SOC1* 258 cum . . . QGD *om. ER*/erit: est *R*/*post* erit *add.* igitur *ER* 259 triangulus: triangulum *R*/FPK: SPK *FE*/similis *mg. a. m. C1*; simile *R*/triangulo . . . equalis *rep.* (triangulo[2]: angulo) *S* 260 KFS[1] . . . DQB *mg. a. m. E*/triangulus: triangulum *R*/DQB: DQF *S*/similis: simile *R*/KFS[2]: FKS *FP1*; KSF *O*/*post* KFS *scr. et del.* erit angulus DQB similis triangulo KFL *E* 261 autem: a *scr. et del. L3*; *om. ER*/DQ: QD *C1*/a . . . ducatur: ducatur a puncto B *R*/perpendicularis: particularis *S* 262 super: supra *E*/igitur *om. FP1R*/angulus *corr. ex* triangulus *F* 263 et angulus BZQ *mg. a. m. E*/*post* ita *scr. et del.* circulus *E*/triangulus: triangulum *R*/similis: simile *R* 266 ergo *om. R*/OF: FO *C1*/FS[1,2]: SF *FP1*/ex hoc *mg. a. m. E* 267 erit *om. ER*/*post* erit *add.* etiam *C1*/*post* sicut[1] *scr. et del.* FO ad F *C1*/OK: CK *S*/*post* FK[2] *add.* et MF ad FK sicut ex ypothesi BG ad GA *mg. a. m. F*/et[2] . . . FK (268) *om. FP1*/ita[2] *inter. a. m. E* 268 sicut . . . QD[2] *mg. a. m. E*

[2.205] Ducatur autem linea BI et ei equidistans DL. Erit triangulus LDQ similis triangulo BQI, et proportio IQ ad QD sicut IB ad DL. Et cum IZ sit equalis ZD, et BZ perpendicularis, erit BD equalis BI, quare erit BD ad DL sicut BG ad GA.

[2.206] Ducatur autem a puncto D linea, que sit DH, equalem tenens angulum cum linea LD angulo BGA. Et cum HL et DL concurrant, erunt duo anguli LHD, LDH minores duobus rectis, et ita duo anguli AGH, DHG eis equales sunt minores duobus rectis, quare HD concurret cum GA. Dico quod concurret in puncto A.

[2.207] Palam quod GDN rectus equalis duobus angulis OCK, OKC, et angulus OKC equalis angulo GDQ. Restat angulus QDN equalis angulo OCK, et ita angulus QDN medietas anguli BGA, et ita medietas anguli HDL. Sed angulus QDB est medietas anguli BDL, quoniam proportio BQ ad QL sicut BD ad DL, cum triangulus DLQ sit similis triangulo BQI, et BD equalis BI. Restat igitur ut angulus NDB sit medietas anguli HDB, et ita BDN equalis NDH. Restat autem BDE equalis angulo HDG. Sed angulus HDG equalis angulo EDA contraposito, quare BDE equalis EDA, et ita D est punctus reflexionis. Ita dico si HD concurrat cum AG in puncto A, quod quidem sic patebit.

[2.208] Ducatur linea HT equidistans BD. Palam quod angulus BDE equalis est angulo HDG. Sed BDE est equalis angulo HTD, quare HT erit equalis HD. Sed proportio BD ad HT sicut BG ad GH, sicut probat Euclides. Igitur proportio BD ad DH sicut BG ad GH. Sed HD producta concurret cum GA, et fiet triangulus similis triangulo HDL, cum habeat angulum LHD communem, et angulus HDL sit equalis angulo HGA. Igitur proportio HD ad DL sicut HG ad lineam quam secat HD ex GA. Et proportio BD ad DL constat ex BD ad DH et DH ad DL. Igitur con-

269 DL: DI *S*/triangulus: triangulum *R* 270 LDQ: BOQ *L3*/similis: simile *R*/et[1,2] *inter. a. m. C1* 271 *post* IZ *scr. et del.* ad *L3*/erit *om. S* 273 autem *om. ER*/ 274 cum[1] *mg. a. m. F*/ cum linea *inter. L3*/linea *om. FP1* 275 duo anguli[1] *om. R*/et . . . rectis (276) *mg. a. m. L3* 276 AGH: AGD *P1*/DHG: ABG *S*; AHG *OC1*/sunt *corr. ex* sint *E* 277 concurret *corr. ex* concurrent *F*; concurrent *P1* 278 *post* quod *add.* angulus *SOL3C1ER*/*post* rectus *add.* est *R*/*post* OCK *rep. et del.* OCK *S*; *inter.* et *L3*; *add.* et *ER* /OKC *inter. L3* 279 restat *corr. ex* resta *F*/*post* restat *add.* ergo *S* 280 angulus *inter. a. m. E*; *om. R*/BGA *corr. ex* GBA *C1* 282 triangulus: triangulum *R*/similis: simile *R* 283 BI: BC *S*/igitur *inter. O*/NDB *alter. ex* NT *in* NDH *O* 284 *post* NDH *add.* producatur GD ultra D ad punctum F quia igitur anguli FDN GDN sunt recti ergo *R*/autem *om. FP1R*/BDE: BDF *R* 285 HDG[1] . . . equalis (286) *om. S*/HDG[2] *corr. ex* HD *O*/ EDA: FDA *R* 286 BDE: BDF *F*/EDA: FDA *R*/D *corr. ex* DA *L3*; illud *C1*/punctus: punctum *R* 287 HD: AD *R*/AG: HAG *S*; *corr. ex* HAG *O* 288 BDE *corr. ex* BDEE *F*; BDF *R* 289 equalis est *transp. C1*/sed . . . HTD *mg. a. m. E*/BDE: DBE *S*; BDF *R*/est[2] *om. SOL3C1* 290 BD . . . proportio (291) *om. S*/sicut[2] . . . Euclides (291) *om. R* 291 Euclides *om. FP1*; *corr. ex* equalis *L3*/ igitur: et *C1*/*post* DH *scr. et del.* sicut BG ad DH *O*/GH *corr. ex* DB *O*/HD: HG *O* 292 triangulus similis: triangulum similem *R*/habeat: habeant *R* 293 LHD *alter. ex* HD *in* GHD *F*; GHD *P1*; SHD *S*/HGA *corr. ex* HG *O*; *corr. ex* GA *a. m. C1* 295 DL[1]: LD *C1*/*post* ex *add.* proportione *R*/DH[1]: BH *P1*/et DH ad DL *mg. a. m. F*/ad DL[2] *rep. F*/igitur *inter. E*

stat ex BG ad GH et GH ad lineam quam secat HD ex GA. Sed BD ad DL sicut BG ad GA. Igitur proportio BG ad GA constat ex proportionibus BG ad GH et GH ad lineam quam secat HD ex GA. Sed constat ex proportionibus BG ad GH et GH ad GA. Igitur, GA est linea quam secat HD ex GA, et ita concurret cum ea in puncto A, quod est propositum.

[2.209] Si vero angulus CKS non fuerit maior recto, dico quod non fiet reflexio ab aliquo puncto speculi ad visum.

[2.210] Si enim dicatur quod potest, sit D punctus reflexionis, et producatur linea AD usque ad H punctum in dyametro BG. Et fiat angulus LDH equalis angulo AGB, et producatur contingens NDY, et fiat angulus QDN equalis medietati anguli AGB.

[2.211] Palam quod triangulus HDL similis est triangulo HGA, quare proportio DH ad DL sicut HG ad GA. Sed BD ad DH sicut BG ad GH, quod patebit per HT equidistans BD. Igitur BD ad DL sicut BG ad GA. Sed cum angulus BDE sit equalis angulo HDG, erit angulus BDN medietas anguli BDH. Sed NDQ medietas anguli HDL. Igitur BDQ medietas anguli BDL, quare proportio BQ ad QL sicut BD ad DL.

[2.212] Ducatur a puncto B equidistans DL, et sit BI, et concurrat DQ cum ea in puncto I. Et dividatur DI per equalia in puncto Z, et ducatur BZ. Erit triangulus BQI similis triangulo QDL. Igitur BQ ad QL sicut BI ad DL, et ita BI equalis BD. Et IQ ad QD sicut MF ad FK, et ita ID ad QD sicut MK ad FK, et ita DZ ad QD sicut OK ad FK, et ita ZQ ad QD sicut OF ad FK.

[2.213] Palam quod BZ est perpendicularis. Ducatur donec concurrat cum DG in puncto X, quod quidem possibile est, cum angulus DZX rectus, ZDX minor recto. Et palam quod proportio BG ad GD sicut SP ad PK. Cum ergo angulus CKS dicatur non esse maior recto, dico quod super punctum K fiet maior recto per lineam concurrentem cum CO in puncto a quo ducetur linea ad CK transiens per punctum F retinens proportionem ad partem CK sicut BG ad GD.

296 et GH *inter. a. m. E*/GH^2 *inter. a. m. C1*/*post* lineam *add.* HD *S*/secat HD *transp. C1*/*post* GA *add.* et proportio *O*/sed *scr. et del. O* 297 igitur . . . GA^2 *mg. a. m. C1*/ex *inter. E* 298 GH^1 *inter. a. m. E*/et GH *inter. L3*/*post* et *scr. et del.* F *O*/ad^2 . . . secat: quam secat ad lineam *L3*/quam . . . linea (299) *om. FP1*/HD: BD *L3*/ex . . . GA (300) *inter. a. m.* (sed: sicut; GH^1: BH) *L3*/sed . . . GA (300) *inter. a. m. E* 300 concurret: concurrit *SOC1* 2 quod: quoniam *SOC1* 3 aliquo: alio *OL3*; *corr. ex* alio *E*/speculi: speciei *L3* 4 *post* enim *scr. et del.* d *C1*/punctus: punctum *R* 5 ad *inter. a. m. E* 6 angulo *rep. S* 7 QDN *corr. ex* qui DN *E*/equalis *inter. O*/medietati anguli *corr. ex* anguli medietati *C1* 8 triangulus: triangulum *R*/HDL: DHL *C1*/similis: simile *R* 9 proportio *om. L3*/DH^1: DG *P1*/*post* BG *scr. et del.* ad *O* 10 HT equidistans: equidistantem HT *R*/*post* HT *scr. et del.* ad *E*/equidistans: equidistantem *E*/*ante* BD *add.* ipsi *R* 12 NDQ *corr. ex* DNDQ *F*/*post* NDQ *add.* est *R* 14 *post* ducatur *scr. et del.* a *F*/equidistans: equidem *S* 15 ea: E A *R*/per: in *R* 16 triangulus: triangulum *R*/similis: simile *R*/*post* igitur *add.* ut *R*/BQ ad (17) *rep. P1* 17 sicut: sic *R*/IQ *alter. ex* QI *in* LQ *F*; Q *S*; *corr. ex* Q *L3*/MF: MI *R*/FK *corr. ex* FE *a. m. E* 18 ID *corr. ex* AD *F*/MK *corr. ex* ME *E* 20 est: cum *FP1*/ducatur: producatur *R* 22 *post* DZX *add.* sit *R* 23 CKS *corr. ex* CES *E*/dicatur *corr. ex* ducatur *L3*/non *corr. ex* M *O* 24 dico . . . recto *om. S*/quod: quoniam *OC1* 25 ducetur: ducatur *C1* 26 CK: PK *R*

[2.214] Verbi gratia, planum, cum angulus QDN sit equalis angulo KCO, erit angulus QDG equalis angulo CKO. Fiat ergo super punctum K angulus equalis BDQ, et ponatur quod linea hunc angulum tenens concurrat cum CO in puncto S, et ducatur SFP. Planum est, cum angulus BZD rectus equalis angulo SOK, erit triangulus BZD similis SOK, et proportio BZ ad BD sicut OS ad SK. Sed QZ ad QD sicut OF ad FK. Erit ergo angulus ZBQ equalis angulo OSF, et angulus QBD equalis angulo FSK, quare triangulus BGD similis triangulo SPK. Igitur proportio SP ad PK sicut BG ad GD, quod est propositum.

[2.215] Amplius, impossibile est quod duo anguli supra MO constitituti sit uterque maior recto. Si enim uterque talium maior fuerit recto, cum supra idem centrum fiat angulus equalis angulo SKM, fiet supra idem centrum alius angulus diversus ab isto quem efficiet supra KM alia linea similis SK. Et ita a puncto D et ab alio puncto illius circuli fiet reflexio, quod est impossibile, cum iam probatum sit quod unus uni visui sit reflexionis punctus, et iam ostensum est quomodo inveniri possit.

[2.216] Duobus autem visibus, licet duo sint puncta reflexionis, tamen unica erit ymago sensuali sillogismo, et unicus ymaginis locus. Et hoc probabimus, quoniam due linee a centris oculorum ad centrum circuli ducte sunt equales.

[2.217] Si autem situs puncti visi respectu utriusque visus sit idem ut linee a puncto viso ad centra oculorum sint equales, facilis erit probatio, quoniam dyametri visuales secant ex circulo arcus reflexionis, et tenent angulos equales cum linea a puncto viso ad centrum spere ducta, et arcus hanc lineam et dyametros visuales interiacentes sunt

27 verbi gratia: quod autem hoc possibile *R*/*post* planum *add.* est *R*/angulo . . . equalis (28) *om. L3* 29 BDQ: BDG *P1*/hunc . . . tenens (30): tenens hunc angulum *SOC1* 30 CO: eo *SOC1*/SFP *corr. ex* FP *a. m. C1*/cum[2] *corr. ex* quod *inter. O* 31 *post* SOK *add.* quod *R*/triangulus: triangulum *R*/similis: simile *R* 32 SOK: SOE *E*/proportio *om. R*/*post* SK *add.* et BZ ad ZD sicut SO ad OK *R*/sed . . . FK (33) *inter. a. m. L3* 33 QBD: QBO *S* 34 FSK: FLK *F*; FOK *E*/triangulus: triangulum *R*/similis: simile *R*/SPK: SPQ *FP1* 35 PK: NK *L3* 36 quod . . . constituti (37): ut duorum angulorum super MO constitutorum *R*/MO *corr. ex* NO *O* 37 sit: sicut *SOL3*; sint *C1*/uterque[1]: utrique *S*/maior[1]: ma *O*/maior fuerit *transp. R*/fuerit recto (38) *transp. P1* 38 cum *om. S*/supra: super *R*/idem *om. OC1*/SKM *alter. ex* SKN *in* SFK *O* 39 supra[1]: super *FP1*/*post* isto *add.* angulo *L3*/efficiet: efficit *SL3R*/supra[2]: super *R* 40 KM: KN *S*; *corr. ex* K? *O*/SK *alter. in* SF *O*/ita: in *S*; *corr. ex* in *L3*/et *om. S*; *inter. L3*/alio: illo *S*/circuli (41) *corr. ex* culi *O* 41 quod[1]: que *S*; *corr. ex* que *L3*/sit: est *FP1*/unus: unum *R* 42 uni visui *transp. FP1*/*post* visui *scr. et del.* c *O*/punctus: punctum *R*/est: sit *R*/inveniri *corr. ex* invenire *F*; invenire *P1* 43 possit: possunt *O* 44 sint: sunt *FP1*/puncta reflexionis *transp. R* 45 tamen: inde *S*/unicus: unus *ER* 46 quoniam: quando *FP1R* 47 ducte: ducere *S* 48 autem: ergo *R*/*post* situs *add.* oculorum *E*/visi: visui *S*/respectu *om. S* 49 centra: centrum *SOC1E*/sint: sunt *L3*/facilis: facit *FP1*/facilis erit *transp.* (erit *inter.*) *O* 50 probatio: proportio *FP1S*; *corr. ex* proportio *L3*; ? *O* 51 a *inter. a. m. E* 52 *post* arcus *add.* inter *R*

equales. Et si sumantur puncta reflexionis, secundum supradictam probationem, arcus circuli interiacentes hec duo puncta et punctum circuli quod est in perpendiculari a puncto viso ducta erunt equales, quod facile patebit iterata superiori probatione.

[2.218] Et hoc sive puncta reflexionis sunt in eadem superficie reflexionis, sive in diversis; erunt tamen arcus illi equales, et linee ducte a centris oculorum ad puncta reflexionum equales, et linee a puncto viso ad eadem puncta equales. Et linee a centris oculorum ad puncta reflexionum procedentes necessario se secabunt, et evidens est probatio quod super idem punctum perpendicularis a puncto viso ducte erit sectio, et in hoc puncto utrique visui apparebit ymago et una sola, quod est propositum.

[2.219] Est autem ordinatio ymaginum sicut ordinatio punctorum visorum. Si enim in re visa sumatur linea a capitibus cuius ducantur due linee ad centrum spere, fiet triangulus in quo continebuntur ymagines omnium punctorum illius linee. Et si sit in linea illa punctus eiusdem situs, ymago puncti remotioris ab eo erit in dyametro remotiori ab eius dyametro, et propinquioris in propinquiori. Et ita observatur pars in ymaginibus sicut fuerit in punctis visis.

[2.220] Sumpta autem linea in qua est punctus eiusdem situs, quodlibet punctum illius linee eiusdem situs erit respectu duorum oculorum, secundum modum predictum, et unicam habebit ymaginem propter equalitatem angulorum illius linee cum lineis visualibus. Si autem sumatur linea que angulum quem continent due linee a centris oculorum ad punctum visum dividat per equalia, situs cuiuslibet puncti illius linee, quantumlibet producte, erit idem utrique visui sicut fuit alterius, et idem probationis modus.

[2.221] Preter has duas lineas non est sumere eundem observantem situm, unde cum punctum visum comprehendatur in perpendiculari, cadet ymago eius in diversis punctis illius perpendicularis, sed

54 probationem: proportionem *FP1SOL3E*/*post* interiacentes *add.* inter *R*/duo *om. R* 55 circuli: eius *FP1*; C *E*/quod *om. FP1*/viso: suo *S*/erunt: erit *S*; *corr. ex* erit *L3*; est *O* 56 superiori: superiore *R*/probatione: proportione *FP1SOL3C1* 57 et hoc sive *inter. a. m. O*/puncta: puncti *O*/sunt: sint *R*/*post* eadem *scr. et del.* linea *P1* 58 erunt: erant *S* 59 *post* a[1] *scr. et del.* punctis *P1*/reflexionum *corr. ex* reflexionis *S*; reflexionis *L3C1*/equales . . . reflexionum (61) *om. S*; *mg. a. m. L3* 60 puncta reflexionum (61) *transp. R* 61 reflexionum: reflexionis *C1*/procedentes: precedentes *SC1*/se *om. SE*; *inter. L3*/probatio: proportio *SL3* 62 quod: non *FP1*; *om. S*; *inter. a. m. L3*/super: supra *E*/ducte: duce *F* 63 *post* sectio *add.* ambarum linearum reflexionis *R* 66 capitibus cuius *transp. R*/*post* ducantur *rep. et del.* ducantur *F* 67 due *om. FP1*/triangulus: triangulum *R* 68 si *om. FP1*/linea illa *transp. C1R*/punctus: punctum *R* 69 *ante* eiusdem *add.* non *R*/*post* situs *add.* respectu ambarum visuum *R*/remotiori: remotiore *R* 70 propinquioris in *om. R*/propinquiori: propinquiore *R* 71 pars: situs partium *R*/fuerit: fuit *R* 72 punctus: punctum *OL3C1ER* 73 situs erit *transp. OC1* 76 *post* autem *scr. et del.* fuerit *F* 77 puncti illius (78) *transp. C1*/puncti . . . linee (78) *corr. ex* linee illius puncti *E* 78 illius: alius *S*; *om. R*/quantumlibet: quantum que *FP1*/erit: erunt *P1* 79 alterius: uni *R*/*post* idem *add.* est *R* 80 *post* duas *add.* erunt *P1*/*post* sumere *add.* aliam *R*

imperceptibiliter a se remotis. Et ymago cuiuslibet puncti a quotcumque videatur oculis semper observat ydemptitatem partis, unde apparet unitas ymaginis, sicut dictum est in visu directo. Quod forme, licet in diversa cadant loca, propter tamen distantiam eorum insensiblem non diversificant apparentiam nisi diversificent partem. Similiter hic, quando remotio puncti ab uno visu modicum maior quam ab alio, erunt loca ymaginum imperceptibiliter remota, unde apparent simul, et ex eis una compacta, que quidem ymaginum loca aliquando non totaliter distant, sed partialiter.

[2.222] In speculis columpnaribus exterioribus, aliquando linea communis superficiei reflexionis et superficiei speculi est linea recta, aliquando circulus, aliquando sectio columpnaris.

[2.223] Cum fuerit linea communis linea recta, erit locus ymaginis in perpendiculari a puncto viso ducta super superficiem speculi tantum distans a linea communi quantum punctum visum ab eadem. Et eadem probatio que dicta est in speculo plano.

[2.224] Cum autem communis linea fuerit circulus, erit aliquando ymaginis locus intra circulum, aliquando extra, aliquando in ipso circulo. Eius rei eadem penitus assignatio que in speculo exteriori sperico.

[2.225] Si vero communis linea fuerit sectio columpnaris, dico quod ymaginum loca quedam intra speculum, quedam in superficie speculi, quedam extra speculum, que in singulari explanabuntur.

[2.226] **[PROPOSITIO 26]** Sit ABG [FIGURE 5.2.26, p. 582] sectio columpnaris, B sit punctus reflexionis, E punctus visus, D centrum visus. Et ducatur a puncto B perpendicularis super superficiem contingentem

83 quotcumque: quocumque *SC1E*; *corr. ex* quocumque *L3* 84 oculis: oculus *F*; circulus *S*; *alter. ex* oculus *in* oculorum *a. m. C1*/observat: observabit *FP1* 85 in[2] *om. P1* 86 loca *corr. ex* lata *F*/tamen: in *scr. et del. C1*; *om. O*/eorum: earum *R* 87 diversificant *corr. ex* diversificet *C1*/diversificant . . . nisi *om. P1*/nisi *alter. ex* etc *in* et *O*; neque *C1*/diversificent *corr. ex* diversificant *P1*; *corr. ex* diversificentur *L3*; *alter. in* diversificant *C1*/*ante* partem *add.* speciem *P1*/similiter hic *transp. O* 88 quando: quanto *SL3*; quare *O*/remotio puncti *OC1*/visu: nusquam *L3*/*post* visu *add.* fuerit *R*/modicum: modico *R*/quam: quoniam *S*/ab *om. P1* 89 loca *inter. O*/simul: similis *SL3*/et *inter. P1* 90 *post* una *add.* imago *R*/que: quando *R*/loca *corr. ex* locu *F* 91 partialiter: perpendicularis *O* 93 superficiei[1] *om. FP1SL3E* 94 circulus: cicirculus *F*; circulis *C1*/circulus aliquando *om. L3* 95 cum fuerit *inter. O* 96 super *om. F*; ad *P1* 97 distans a linea: a linea distans *E*/quantum *mg. a. m. F*; quamvis *S*; *corr. ex* quamvis *L3*/et eadem (98) *om. FP1* 98 *ante* probatio *add.* est *R* 100 intra: intus *S*; inter *L3*/*post* extra *scr. et del.* circulum *F*/ipso circulo (101): ipsa circumferentia *R* 101 eius: cuius *O*/rei eidem *transp. C1*/*post* penitus *add.* est *FP1*/que in speculo: in speculo que *S*/exteriori: exteriore *R* 103 si . . . explanabuntur (105) *om. S*/communis linea *transp. R*/columpnaris *mg. a. m. C1*/quod: quoniam *OL3C1E* 104 loca *om. FP1OL3ER*/*post* quedam[1] *add.* sunt *R* 105 in singulari: singula *OC1* 106 ABG: ABC *R* 107 punctus[1,2]: punctum *R*/E *inter. O*/visus[1]: visum *R* 108 a . . . perpendicularis: perpendicularis a puncto B *L3*/B perpendicularis *transp. S*

speculum in puncto B, que sit TBQ, et ducatur a puncto E perpendicularis super superficiem contingentem speculum in puncto K, que sit EKQ. Et linea contingens speculum in puncto B sit CU; linea contingens speculum in puncto K sit KM. Dico quoniam due perpendiculares TB, EQ concurrent.

[2.227] Ducantur linee EB, DB, et ducatur linea KB. Palam quoniam KM cadet in figura EKB, et linea BC in figura eadem. Igitur BC secabit EK. Secet in puncto C. Palam quoniam angulus TBK est maior recto, et angulus EKB similiter maior recto, quare TB, EK concurrent. Sit concursus punctus Q. Similiter, DBK maior recto. Igitur DB, EK concurrent. Sit concursus punctus H. Igitur H est locus ymaginis. Dico etiam quod proportio EQ ad QH sicut EC ad CH, et etiam quod QH est maior HB.

[2.228] Ducatur HF equidistans EB. Palam quoniam angulus EBC equalis est angulo DBU. Est igitur equalis angulo CBH. Restat EBT equalis angulo HBQ, cum sit TBC rectus, et QBC rectus. Cum igitur CB dividat angulum EBH per equalia, erit proportio EC ad CH sicut EB ad BH.

[2.229] Sed angulus EBT est equalis angulo HFB, quare HF, HB sunt equalia. Sed proportio EB ad HF sicut EQ ad QH. Erit ergo EC ad CH sicut EQ ad QH, quod est propositum. Et ex hoc, cum sit proportio EQ ad QH sicut EB ad BH, et EQ sit maior EB, erit QH maior HB, quod est propositum.

[2.230] Palam ex hoc quod, si supra sectionem GB ducatur perpendicularis super superficiem contingentem sectionem, concurret cum TB. Similiter, quecumque ducatur supra sectionem AB concurret cum TB. Et hec quidem patent cum punctus visus non fuerit in perpendiculari visuali. Palam enim ex superioribus quod unius solius puncti

109 *post* speculum *scr. et del.* que sit *F*/in puncto B *om. L3ER*/in . . . K (110) *om. S*/que . . . sit[1] (111) *mg. a. m. F*/TBQ: QBT *FP1*; GBQ *R*/TBQ . . . sit[1] (111) *mg. a. m. L3E*/ducatur . . . E: a puncto K ducatur *FP1* 110 in puncto K *om. OL3ER* 111 EKQ: GKQ *F*/*post* EKQ *add.* et linea contingens speculum que sit EKQ *L3*/CU *corr. ex* QCU *O* 112 *post* sit *rep. et del.* sit *F*/*post* dico *scr. et del.* q *C1*/quoniam: quod *R* 113 TB *alter. in* TBQ *F*; TBQ *P1*; GBQ *R*/EQ: EKQ *FP1SL3C1ER*; *alter. ex* EB *in* EK *O* 114 KB *corr. ex* EB *E*/quoniam: quod *R* 115 KM *corr. ex* EM *E*/cadet *corr. ex* cadent *C1*/figura[1,2]: figuram *R*/eadem: eandem *R* 116 quoniam: quod *R*/TBK *corr. ex* TBE *E*; GBK *R*/*post* TBK *scr. et del.* DCN *P1* 117 quare: quando *P1*/TB: GB *R*/concurrent *mg. a. m. C1* 118 punctus: punctum *R*/Q: H *FP1*/*ante* similiter *scr. et del.* igitur H est *F*/DBK *corr. ex* DBE *E*/*post* recto *scr. et del.* igitur DBK maior recto *O* 119 punctus: punctum *R* 121 HB *corr. ex* IHB *S*; HD *E* 122 quoniam: quod *R* 123 equalis est *transp. R*/EBT: EBG *R* 124 sit *om. R*/TBC *corr. ex* THC *S*; GBT *R*/rectus et *om. R*/et *om. FP1S*; *inter. OL3C1* (*a. m. C1*)/QBC: TBQ *R*/rectus[2]: sint recti *R* 125 CB: OB *SO*; BC *C1*/proportio *om. R* 127 EBT: EBG *R*/est *om. SOC1* 128 proportio *om. R*/*post* sicut *scr. et del.* EB *F*; *add. P1*/ad[2] *om. FP1* 130 BH: HF *R* 132 *post* palam *inter.* ergo *O*/quod *corr. ex* quo *F*/supra: super *R*/GB: ABC *R* 133 concurret: concurrit *FP1* 134 TB: GB *R*/similiter . . . TB (135) *om. R*/quecumque: quocumque *L3*/supra: super *E* 135 punctus visus: punctum visum *R*/fuerit: fuit *L3* 136 puncti forma (137) *transp. E*

forma per perpendicularem accedit ad speculum et secundum eandem reflectitur, et est punctus perpendicularis existens in superficie visus, punctus enim ultra visum sumptus non potest reflecti super hanc perpendicularem quia non potest accedere ad speculum super perpendicularem propter predictam rationem. Et similiter non poterit reflecti ab alio puncto speculi quam a puncto perpendicularis, quia accideret duas perpendiculares concurrere et effici triangulus cuius duo anguli recti, sicut supra patuit.

[2.231] **[PROPOSITIO 27]** Amplius, sumatur sectio columpnaris [FIGURE 5.2.27, p. 583], et sumatur in ea punctus A, et ducatur contingens sectionem, que sit AT, et sumatur perpendicularis super AT intra speculum, que sit DA.

[2.232] Palam quod AD dividit sectionem in duas partes in quarum utraque est punctus unicus cuius puncti contingens erit equidistans AD. Sit ergo G cuius contingens concurrat cum AD in puncto H, et ducatur perpendicularis super hanc contingentem, que sit QG, et hec quidem necessario concurret cum HD, sicut ostensum est in precedenti figura. Sit concursus in puncto D, et ducatur linea GA usque ad P, et ducatur linea QA. Igitur angulus QAH aut est equalis angulo HAP, aut maior, aut minor.

[2.233] Sit equalis. Procedet igitur forma puncti Q ad A et reflectetur ad P, qui sit visus, et locus ymaginis erit punctus sectionis columpnaris, scilicet G.

[2.234] Si vero supra punctum Q sumatur aliquod punctum, ut punctum F, erit quidem angulus FAH minor angulo HAP. Fiat ei equalis NAH. Concurret quidem NA cum GQ intra columpnam. Sit in puncto

137 per *om. FP1SC1*; *inter. OL3*/perpendicularem: perpendicularis *FP1*/accedit: accedet *L3*; accedat *C1*/eandem: eundem *R* 138 reflectitur: reflectetur *L3*/et est: si cum *O*/punctus: punctum *R*/perpendicularis: particularis *S* 139 punctus: punctum *R*/sumptus: sumptum *R*/reflecti: reflecte *S* 140 quia: et *FP1*; quam et *R*/accedere: accidere *L3* 141 *post* propter *scr. et del.* punctum *P1*/*post* predictam *add.* ibidem *ER* 142 *post* perpendicularis *add.* huius *C1ER* 143 perpendiculares: partes *S*/*post* perpendicularis *scr. et del.* a *P1*/effici *corr. ex* efficit *L3*; efficere *ER*/triangulus *alter. in* triangulum *L3*; triangulum *ER*/*post* anguli *scr. et del.* sicut *F* 144 recti *inter. L3* 146 punctus: punctum *R* 147 sectionem: sectioni *SO*/AT[1]: ET *SO*; *corr. ex* ET *L3*; *alter. in* EAT *a. m. C1*; EAT *R*/et *om. L3*/AT[2]: ET *SOL3*; *alter. in* EAT *a. m. C1* 148 speculum: sectionem *SOC1* 149 AD: A *FP1SOL3E*; DA *R*/in[2] *inter. L3* 150 punctus unicus: punctum unicum *R*/unicus: unius *O*/*post* puncti *add.* linea *R* 151 sit *corr. ex* si *C1*/*post* ergo *add.* aliud punctum *R*/*post* G *inter.* punctus *OL3*/*post* cum *add.* linea *R* 152 perpendicularis: perpendiculariter *FP1L3*/*post* contingentem *rep. et del.* erit (150) . . . contingens (151) *S* 153 precedenti: precedente *R* 154 figura *inter. O* 155 QAH: QDH *P1*; *corr. ex* QHA *L3* 157 procedet: procedit *FP1L3E*/Q *inter. O* 158 qui: quod *R*/sit visus *transp. R*/punctus: punctum *R*/columpnaris: corporis *O* 159 scilicet: S *S*/*post* G *scr. et del.* si *F* 160 aliquod: aliud *SOC1* 161 FAH *corr. ex* FHA *FL3* 162 quidem *inter. O*/NA: NAM *S*/columpnam: columpna *F*/sit *om. FP1*

K. Palam ergo quod ymago puncti F erit in puncto K, et ymagines omnium punctorum ultra punctum Q intra columpnam.

[2.235] Si vero intra Q et T sumatur punctum aliquod, ut punctum C, erit angulus CAH maior angulo HAP. Fiat ei equalis HAM. Palam quod MA cadet supra GQ, et extra sectionem. Sit in puncto O. Erit igitur ymago C in puncto O, et omnium punctorum T et Q interiacentium ymagines erunt extra sectionem inter T et G.

[2.236] Si autem angulus QAH fuerit minor angulo HAP, secetur ex eo equalis, et sit HAN. Palam quod ymago Q erit in puncto K, et omnium punctorum superiorum ymagines erunt infra sectionem. Si vero inferius sumatur C punctum, ut angulus CAH sit equalis angulo HAP, erit ymago C in sectione, et omnes inter C et Q intra, omnes inter C et T extra.

[2.237] Si vero angulus QAH fuerit maior angulo HAP, fiat ei equalis HAM [FIGURE 5.2.27a, p. 583]. Palam quod MA secabit sectionem. Et secet in puncto B, et ducatur contingens super punctum B, que concurrat cum DH in puncto L. Erit autem angulus DLB acutus, et angulus HLB obtusus, et LB concurrens cum HG faciet cum ea acutum. Ducatur perpendicularis a puncto B super LB, que sit SB. Secabit quidem HG, et faciet angulum acutum cum ea, cui angulus contrapositus similiter erit acutus. Et HG secat QA. Sit punctus sectionis U, et facit acutum angulum cum ea super punctum U, quare SB et QU concurrunt. Sit concursus in puncto Z. Palam ergo quod forma puncti Z movebitur ad speculum per ZA, et refertur per AM, et locus ymaginis B. Et ymagines punctorum linee ZS ultra Z erunt intra sectionem, et punctorum citra Z extra sectionem, quod fuit propositum.

[2.238] **[PROPOSITIO 28]** Amplius, ab uno solo puncto speculi columpnaris fit reflexio ad centrum visus, utpote punctus B [FIGURE 5.2.28, p. 584] reflectitur ad A a puncto G. Dico quod non refertur ad ipsum ab alio puncto speculi quam a puncto G.

163 K *inter. a. m. C1* 164 *post* punctorum *add.* linee QF *R* 165 intra: inter *SOR*/T: D *FP1* 166 C: R *R*/CAH *corr. ex* CHA *C1*; RAH *R* 167 MA *corr. ex* MAS *F*; *inter. O*/*post* supra *add.* lineam *ER*/et *om. L3* 168 C: R *R*/*post* punctorum *add.* inter *R*/et *om. R*/*post* et *scr. et del.* que *F* 169 T: D *FP1*; *inter. O*; O *R* 170 autem: vero *R*/minor *corr. ex* maior *mg. a. m. F* 172 ymagines: ymaginetur *F*/infra: intra *R* 173 C: R *R*/CAH: RAH *R*/HAP *corr. ex* AHP *FL3*; AHP *S*; HPA *P1* 174 C^{1} *om. L3C1*; R *R*/sectione *corr. ex* sectionem *F*; sectionem *P1*/et^{1} *inter. O*/*post* inter^{1} *scr. et del.* se *F*/C^{2,3}: R *R*/intra: inter *S* 177 et *om. SOC1* 178 concurrat: concurret *R* 179 *post* DH *add.* ut *R*/erit autem: eritque *R* 180 obtusus et LB *mg. F*/HG: HD *C1*/cum^{2} . . . faciet (182) *mg. F* 181 a *corr. ex* ad *P1*/*post* HG *add.* ut in puncto X *R* 182 cui: cum *FP1*; que *C1*; quoniam *R*/angulus: angulo *SOL3E*; oppositus triangulo *C1*/contrapositus: circa positus *S* 183 *post* acutus *scr. et del.* acutum *C1*/*post* HG *scr. et del.* et faciet angulum *S*/punctus: punctum *R* 184 sit concursus (185) *om. P1* 185 puncto *om. R*/Z *corr. ex* E *L3* 186 refertur: referetur *E*; reflectetur *R* 187 Z *corr. ex* S *a. m. L3* 188 Z extra *P1* 190 punctus: punctum *R*/B: D *S*; *corr. ex* D *L3*; *inter. O* 191 reflectitur: reflectatur *R*/refertur: reflectetur *R*

[2.239] Quoniam si in superficie reflexionis que est ABG sit totus axis speculi, erit linea communis superficiei speculi et superficiei reflexionis linea longitudinis speculi. Et cum in superficie reflexionis sit centrum visus, punctus visus, punctus reflexionis, et punctus axis in quem cadit perpendicularis, una sola superficies sumi potest in qua sit linea illa longitudinis, sive axis, et puncta A, B, quare non potest fieri reflexio ad A nisi ab aliquo puncto linee longitudinis. Sed iam probatum est quod non potest fieri reflexio ad A ab alio puncto linee longitudinis quam a puncto G, quare in hoc situ ab uno solo puncto speculi fit ad A reflexio.

[2.240] Si vero superficies ABG sit equidistans basi columpne, erit linea communis circulus equidistans basi. Et iam patuit quod ab alio puncto illius circuli non potest fieri ad A reflexio. Et si ab alio puncto speculi fiat reflexio, perpendicularis ducta a puncto illo cadet ortogonaliter super axem, et secabit lineam AB in puncto aliquo. A puncto illo ducatur linea ad axem in superficie equidistanti basi columpne. Erit quidem ortogonalis super axem, et ita due perpendiculares efficient cum axe triangulum cuius duo anguli sunt recti, quod est impossibile. Palam ergo quod in hoc situ non refertur B ad A nisi a puncto G.

[2.241] Si vero superficies ABG secet speculum sectione columpnari, dico quod a solo puncto G fit reflexio.

[2.242] Ducatur a puncto A [FIGURE 5.2.28a, p. 584] superficies equidistans basi columpne, que sit EZI, et a puncto G similiter superficies equidistans basi speculi in qua ducatur ab axe linea ad punctum G, que sit TG. Erit quidem perpendicularis super superficiem contingentem speculum in puncto G. Et concurrat cum AB in puncto K, et ducatur a puncto G linea longitudinis speculi, que sit GZ, et sit axis TQ. Et a puncto B perpendicularis ducatur ad superficiem EZI, que sit BH, et ducantur linee AZ, HZ. Et ducatur a puncto Z in superficie illa ad

193 ABG: AGB *S* 194 *post* communis *add.* super superficiem *P1*/speculi[2] *mg. a. m. O* 195 linea *corr. ex* line *P1*/et . . . superficie *inter. O*/*post* reflexionis[2] *rep. et del.* que (193) . . . reflexionis[1] (195); ABG: AGH *S* 196 sit: sicut *SO*; sint *C1*/*post* centrum *add.* A *S*/punctus[1,2,3]: punctum *R*/visus: visum *R*/*post* visus *add.* et *FP1*/et *om. S* 197 quem: quod *R*/sumi: ? *S* 198 sit *mg. a. m. F*; *inter. OL3*; *om. SE*; erit *inter. a. m. C1*/sive *om. R*/A *corr. ex* AA *P1*/*post* B *add.* G *R*/non potest *om. L3* 199 nisi *inter. a. m. E*/aliquo: alio *SO*/sed . . . longitudinis (201) *mg. a. m. E* 200 probatum: propositum *SL3C1*/est *om. FL3*/alio: aliquo *FP1L3*/linee longitudinis *om. R* 201 solo puncto *transp. S* 202 ad *mg. a. m. L3E* 204 linea communis: KG omnis *O* 205 illius . . . puncto *mg. a. m. L3*/ad A *inter. O*/*post* si *scr. et del.* ab *F* 206 *ante* speculi *scr. et del.* illius *L3*/*post* ducta *scr. et del.* a *P1*/cadet: cadent *FP1* 207 *post* aliquo *add.* et *S* 208 *post* in *scr. et del.* perpendiculari *P1*/equidistanti: equidistante *R* 211 recti: recta *FP1*/refertur: reflectetur *R* 216 que: qui *FP1L3C1* 217 equidistans: equidem *F* 218 *post* quidem *add.* TG *E* 219 concurrat: concurret *O*/K *corr. ex* E *a. m. E*/*post* ducatur *rep. et del.* et ducatur *C1* 220 G *inter. F*/GZ *corr. ex* GZG *F*/axis *corr. ex* axit *L3* 221 perpendicularis ducatur *transp. P1*/*post* ducatur *add.* EZ *S*/EZI *corr. ex* ZEI *F*; ZEI *P1* 222 AZ HZ *corr. ex* AS HS *L3*/ducatur: ducantur *S*; *corr. ex* ducantur *O*/Z *corr. ex* S *L3*

axem linea que sit ZQ. Erit quidem perpendicularis super axem, cum axis sit perpendicularis super hanc superficiem, et erit perpendicularis super superficiem contingentem speculum in puncto Z. Et concurrat cum linea AH in puncto L. Dico quod forma puncti H refertur ad A a puncto Z.

[2.243] Ducatur a puncto A equidistans linee KG, que sit AM, que quidem concurret cum BG. Sit concursus in puncto M. Palam quoniam GZ equidistans linee BH, cum utraque sit ortogonalis super superfices equidistantes, quare linea BGM est in superficie harum linearum. Igitur tria puncta M, Z, H sunt in hac superficie. Sed iterum AM est equidistans KG, et LZ equidistans KG, quoniam GZ equidistans TQ et inter superficies equidistantes. Igitur LZ equidistans AM, quare sunt in eadem superficie, et in ea est linea AH. Igitur in hac superficie sunt tria puncta M, Z, H. Et iam patuit quod sunt in superficie MBH. Igitur in linea communi sunt hiis duabus superficiebus. Igitur HZM est linea recta.

[2.244] Palam, cum G sit punctus reflexionis, erit angulus AGK equalis angulo KGB, et ita equalis angulo AMG. Sed est equalis MAG, quia coalternus. Igitur AG, MG sunt equales. Sed quoniam GZ est ortogonalis super quamlibet lineam superficiei AZH, erit quadratum MG equale quadratis MZ, GZ. Et similiter quadratum AG equale quadratis AZ, GZ. Erit igitur AZ equalis MZ, quare angulus AMZ est equalis angulo ZAM. Sed est equalis angulo LZH, et angulus ZAM est equalis angulo LZA, quia coalternus. Igitur angulus AZL est equalis angulo LZH, quare forma puncti H accedens ad punctum Z refertur ad punctum A.

223 ZQ *corr. ex* Z *F*; *corr. ex* SQ *L3*/super *om. S* 225 Z *corr. ex* S *L3* 226 AH: HA *C1*; AK *R*/ quod: quoniam *SOC1*/refertur: reflectetur *R* 227 Z *corr. ex* S *L3* 229 quoniam: quod *R* 230 *post* GZ *add.* est *C1ER*/equidistans: equidistantem *F*; equidistanter *P1*/*post* equidistans *add.* est *SO* (*inter. O*); *scr. et del.* est *C1*/superficies *corr. ex* superficiem *L3*; superficiem *R* 231 equidistantes: equidistantem *R*/*ante* quare *add.* basibus columne *R*/quare: quia *E* 233 KG[1] *corr. ex* KQ *L3*; *corr. ex* EG *a. m. E*/LZ *corr. ex* LS *L3*/KG[2] *corr. ex* EG *a. m. E*/*post* GZ *add.* est *SO* (*inter. O*)/equidistans[3] *corr. ex* equidistantes *L3*/*post* equidistans[3] *scr. et del.* axi *F*; *scr. et del.* KG *S* 234 LZ *corr. ex* LS *L3* 235 *post* superficie[1] *scr. et del.* sunt tria *C1* 236 Z *corr. ex* S *L3*/ patuit *corr. ex* planum *C1*/sunt *om. FP1*; sint *R*/MBH: BMH *R* 237 in . . . sunt: sunt in linea communi *C1R*/igitur *inter. F*/HZM: GHZM *FP1*; HMZ *C1*; *alter. ex* ASM *in* AZM *L3* 239 palam *om. O*/*post* palam *add.* igitur *R*/punctus: punctum *R* 240 KGB *corr. ex* EGB *a. m. E* 241 AG: SLG *S* 242 super: supra *C1*/AZH: ZAH *R*/quadratum *inter. a. m. F* 243 equale *corr. ex* equalis *C1*/quadratis *corr. ex* quadrato *a. m. F*/MZ: AZ *E*/MZ GZ *transp. O*; *corr. ex* MS GS *L3*/et . . . GZ (244) *om. ER*/*post* AG *inter.* est *O* 244 *post* quadratis *scr. et del.* erit igitur *C1*/GZ *corr. ex* GS *L3*/*post* GZ *add.* ? *F*/MZ *corr. ex* MS *L3*/AMZ *corr. ex* AMS *L3* 245 ZAM[1] *corr. ex* SAM *L3*; MAZ *R*/*post* sed *add.* angulus AMZ *R*/est: ? *O*/LZH: KH *S*; *corr. ex* LSH *L3*/ZAM[2] *corr. ex* SAM *L3* 246 angulo *om. R*/LZA *corr. ex* LSA *L3*; *corr. ex* LZH *E*/*post* LZA *rep. et del.* et (245) . . . LZA (246) *E*/igitur angulus *inter. L3*/AZL *corr. ex* ASL *L3* 247 angulo *corr. ex* angulu *F*/ LZH *corr. ex* LSH *L3*/Z *corr. ex* S *L3*/refertur: reflectetur *R*

[2.245] Si ergo dicatur quod ab alio puncto quam a puncto G potest forma B reflecti ad A, illud aliud punctum aut erit in linea longitudinis, que est GZ, aut in alia. Si est in ea, ducatur ab eo perpendicularis, que necessario secabit lineam AK, et erit equidistans linee AM. Et linea ducta a puncto B ad illud punctum necessario concurret cum AM, et erit punctus ille et punctus M in eadem superficie.

[2.246] Et linea illa aut cadet super punctum M aut super aliud. Si super punctum M, erit ducere a puncto B ad punctum M duas lineas rectas. Si autem ad aliud punctum linee AM, ducatur a puncto illo linea ad punctum Z, et probatur quod hec linea cum HZ facit lineam rectam, sicut probatum est de linea ZM. Et ita a puncto H erit ducere duas lineas rectas per punctum Z transeuntes in diversa puncta AM cadentes, quod est impossibile.

[2.247] Palam ergo quod a nullo puncto linee GZ nisi a G potest B reflecti ad A. Si dicatur quod a puncto extra hanc lineam sumpto, ducatur super punctum illud linea longitudinis speculi, et a puncto circuli EZI in quem cadit hec linea, probatur H reflecti ad A secundum supradictam probationem. Sed iam probatum est quod H a puncto Z reflectitur ad A, et ita impossibile. Restat ergo ut a solo puncto speculi reflectitur B ad A, quod est propositum.

[2.248] **[PROPOSITIO 29]** Amplius, dato puncto B, quod reflectitur ad A, erit invenire punctum reflexionis, et hoc patebit per revolutionem probationis.

[2.249] Ducatur a puncto A [FIGURE 5.2.28a, p. 584] superficies equidistans basi columpne, que quidem secabit columpnam super circulum, qui sit EZI. Et ducatur a puncto B perpendicularis super hanc superficiem, que est BH, et inveniatur in hac superficie punctus a quo fit reflexio H ad A, qui sit Z. Et a puncto Z ducatur linea longitudinis, que

249 a *om. S*; *inter. OL3* (*a. m. L3*) 250 aliud *corr. ex* ad *L3*; autem *E* 251 GZ aut *corr. ex* GS ut *L3*/ea: linea GZ *R*/*post* ea *add.* GZ *L3E* (*inter. L3*)/*post* eo *scr. et del.* que est *C1* 252 equidistans: equidem *SO*/linee AM: lineam *S*; *alter. in* lineam *O* 253 illud: illum *FP1*/AM: autem *S* 254 punctus[1,2]: punctum *R*/ille: illud *R*/et: etiam *E* 255 super[1]: supra *S*/M: AY *O*/super[2]: supra *C1*/aliud: alium *FP1*; *corr. ex* illud *mg. C1* 256 super: supra *SC1* 257 *post* rectas *add.* quod est impossibile *ER*/ad *om. S*/aliud: alium *FP1* 258 Z *corr. ex* S *L3*/et *om. P1*/probatur: probabitur *R*/*post* probatur *scr. et del.* cum *P1*/HZ *corr. ex* HS *L3* 259 ZM *corr. ex* SM *L3* 260 Z: ZS *FP1*; *corr. ex* S *L3*/transeuntes: transeuntem *FP1SL3*/*ante* in *add.* per *FP1*/*post* puncta *add.* linee *R* 261 cadentes: cadens *C1* 262 a[1] *om. S*/GZ: GS *L3* 264 illud: illum *FP1*/*post* linea *scr. et del.* LZ *F* 265 EZI *corr. ex* EZ *F*; EZ *S*; *alter. ex* ES *in* EZ *L3*/quem: quod *R*/probatur: probabitur *R*/H *inter. O* 266 sed: licet *L3* 267 reflectitur: refertur *SOL3*/ad A *inter. a. m. E*/*post* impossibile *add.* est *E*/*post* puncto *rep. et del.* puncto *F* 268 reflectitur: reflectatur *SOR* 269 reflectitur: refertur *SO*; reflectatur *R* 270 revolutionem: reflexionem *FP1*; *corr. ex* reflexionem *L3*/*post* revolutionem *add.* predicte *R* 272 *ante* ducatur *add.* verbi gratia *SOC1* 274 qui: que *E*/EZI *alter. ex* SI *in* ESI *S*; *corr. ex* ESI *L3* 275 est: sit *ER*/punctus: punctum *R*/fit *inter. a. m. F*; *om. SL3E*/fit reflexio (276): refertur *OC1* 276 qui: quod *R*/Z[1]: S *L3*/et . . . Z[2] *om. S*/Z[2] *corr. ex* S *L3*

sit ZG, et a puncto Z perpendicularis ZL et huic equidistans a puncto A, que sit AM. Et linea HZ producatur usque concurrat cum ea, et sit concursus in puncto M. Et a puncto M ducatur linea ad B, que necessario secabit lineam ZG, cum sit in eadem superficie cum ea. Quoniam cum BH sit equidistans GZ, erit HZM in superficie illarum, et ita MB in eadem, que, si secaverit ZG in puncto G, erit G punctus reflexionis, quod quidem si revolvas probationem predictam videre poteris.

[2.250] In speculis exterioribus piramidalibus, si linea communis superficiei reflexionis et superficiei speculi fuerit linea longitudinis speculi, erit locus ymaginis sicut assignatus est in speculis planis, et eadem probatio.

[2.251] Quod autem non possit esse linea communis circulus palam per hoc quod perpendicularis ortogonaliter cadit super superficiem contingentem speculum in puncto reflexionis, et circulus necessario erit equidistans basi. Superficies vero equidistans basi non erit ortogonalis super superficiem contingentem speculum.

[2.252] Si vero communis linea fuerit sectio piramidalis, ymagines quedam erunt in superficie speculi, quedam intra speculum, quedam extra. Et idem assignationis modus qui est in speculo columpnari exteriori, et eadem probatio. Et sicut ostensum est in columpnari exteriori, per perpendicularem visualem non reflectitur forma ad oculum nisi puncti superficiei oculi tantum, et hoc ab uno solo speculi puncto, et locus ymaginis eius continuus locis aliarum ymaginum, sicut patuit superius.

[2.253] Restat in hiis speculis declarare quod ab uno solo puncto eius fiat reflexio, quod sic patebit.

[2.254] **[PROPOSITIO 30]** Sit A [FIGURE 5.2.30, p. 585] visus, B punctus visus, G punctus reflexionis, et ducatur super punctum G superficies equidistans basi, que quidem secabit piramidem super circul-

277 ZG *corr. ex* SG *L3*/Z *corr. ex* S *FL3*/ZL *corr. ex* L *F*; L *P1*; SL *L3*/huic *corr. ex* hec *L3C1* 278 *post* et[1] *add.* etiam *ER*/usque: quousque *R* 279 et . . . M[2] *inter. a. m. L3* 280 lineam *om. O*/ZG *corr. ex* SG *L3* 281 equidistans *corr. ex* equidem *S*/GZ: BG *O*/HZM *corr. ex* HSM *L3*/MB: BM *R* 282 secaverit: secuerit *R*/*post* secaverit *rep. et del.* ZG (280) . . . eadem (280) *S*/ZG *corr. ex* LS *L3*/punctus: punctum *R* 283 revolvas: revolueris *F*; volueris *P1*/predictam *rep. P1*/videre poteris: videbis *O*/poteris: possis *L3* 285 superficiei[2] *om. R*/longitudinis *corr. ex* lineis *L3* 287 *post* eadem *add.* est *R*/*post* probatio *add.* C *E* 288 autem *om. S*/linea *corr. ex* lineam *S* 289 perpendicularis: superficies reflexionis *R*/*post* perpendicularis *add.* perpendicularem *S*/ortogonaliter cadit: ortogonalis est *R* 291 erit[1]: est *R*/vero: ergo hec *R* 293 ymagines: ymaginum *O* 294 erunt: erant *S*; erit *O* 295 *post* idem *add.* est *R*/qui: que *FP1L3E*/est: fuit *R* 296 exteriori: exteriore *R*/ostensum *om. R*/exteriori (297): exteriore *R* 297 per *om. S*; *inter. OL3E* (*a. m. E*)/reflectitur: refertur *SO*; reflectetur *R*/oculum *corr. ex* speculum *P1* 299 *post* eius *add.* erit *R* 1 puncto eius (2) *transp. SOC1* 3 A visus *transp. R* 4 punctus[1,2]: punctum *R*/visus: visum *R*

um, qui sit PG. Et ducantur linee AG, BG, AB, et a puncto G ducatur ad centrum circuli linea, que sit GT. Et conus piramidis sit E, a quo ducatur axis, qui erit ET. Et ducatur perpendicularis super superficiem contingentem speculum in puncto G, que sit HG, que, cum dividat angulum AGB per equalia, cadet super AB. Punctus casus sit Z.

[2.255] Et a cono ducatur linea longitudinis speculi ad punctum G, que sit EG, cui linee ducatur equidistans a puncto A, que necessario secabit superficiem circuli GP. Secet in puncto N, et sit AN. Similiter, a puncto B ducatur equidistans eidem EG, scilicet BM, que secet superficiem PG in puncto M. Et a puncto N ducatur equidistans GT, que sit NF, et ducantur linee NG, MG, NM.

[2.256] Palam quod TG secabit NM. Secet in puncto Q. Palam etiam quod MG secabit NF, cum secet ei equidistantem. Sit punctus sectionis F. Et a puncto A ducatur equidistans HZ, que sit AL. Palam quod BG concurret cum AL. Sit concursus L. Deinceps ducatur linea communis superficiei contingenti speculum in puncto G et superficiei circuli PG, que sit GC. Palam quod erit ortogonalis super GT, et similiter super NF.

[2.257] Sumatur etiam linea communis superficiei contingenti et superficiei reflexionis, que sit GD, que quidem, cum secet GH, secabit AL. Sit punctus sectionis D, et erit ortogonaliter super AL.

[2.258] Palam ex predictis quoniam NF est equidistans GT, et AL equidistans GH. Igitur superficies in qua sunt NF, AL est equidistans superficiei GTH. Sed EG equidistans BM, quare sunt in eadem superficie, que superficies secat predictas equidistantes, unam super lineam EG, aliam super lineam FL, quare FL equidistans EG. Sed AN equidistans eidem. Igitur FL equidistans AN.

[2.259] Verum superficies contingens speculum in puncto G secat superficies easdem equidistantes, unam in linea EG, aliam in linea CD. Igitur CD est equidistans EG. Igitur est equidistans AN et LF, quare erit proportio AD ad DL sicut NC ad CF.

6 qui: que *C1*/PG *corr. ex* BG *P1*/*post* BG *inter.* et *L3*/AB *inter. L3*/ducatur: ducantur *FP1* 7 conus: vertex *R* 9 G *mg. C1*/HG *corr. ex* DH *P1* 10 punctus: punctum *R*/Z *corr. ex* S *L3* 11 cono: vertice piramidis *R* 12 sit *inter. O*/equidistans *corr. ex* equidem *S* 13 circuli: speculi *P1* 14 equidistans: equidem *S* 15 N *om. FP1*/ducatur *corr. ex* due *a. m. L3*/equidistans: equidem *F*/*post* equidistans *add.* ipsi *R* 18 punctus: punctum *R* 20 sit: et *O*/deinceps: deinde *R* 21 contingenti: continti *F* 22 GC: GO *R*/ortogonalis: ortogonaliter *FP1SOL3E* 23 *post* NF *scr. et del.* S quidem *C1* 24 *post* contingenti *add.* speculum *R* 26 *post* AL[1] *scr. et del.* equidem *C1*/punctus: punctum *R*/ortogonaliter: ortogonalis *C1R* 27 *post* palam *scr. et del.* quod *L3*/equidistans: equidem *S* 28 equidistans[1]: equidem *S*/superficies: in superficie *P1* 29 sed: ei *O*/*post* sed *add.* linea *R*/EG *corr. ex* G *C1*/equidistans: equidistat *ER* 31 *post* FL[2] *add.* est *ER*/*post* AN *add.* per *L3* 32 equidistans[1] *inter. O*; equidistat *ER*/*post* FL *add.* est *R* 33 secat: secet *S* 34 easdem: eadem *FP1*/unam: una *P1*/aliam: alia *S*/CD: OD *R*/CD igitur CD (35) *inter. L3* 35 CD: OD *R*/*post* CD *scr. et del.* EG *L3*/*post* LF *add.* et a puncto F ducatur linea equidistans LA secans DO in K et AN in I ergo FK equalis LD et KI equalis DA *R* 36 DL: LD *FL3E*/NC: NO *R*/CF: OF *R*

[2.260] Palam etiam quod angulus BGZ equalis est angulo ZGA, et etiam angulo GLA, et etiam angulo GAL, quare GAL, GLA equales. Et GA et GL equales, et GD perpendicularis super AL. Erit AD equalis DL. Erit igitur NC equalis CF, et GC perpendicularis. Erit angulus CFG equalis angulo CNG, Erit igitur angulus NGQ equalis angulo MGQ. Igitur a puncto circuli PG, quod est G, potest punctus M reflecti ad N, non impediente piramide.

[2.261] Dico igitur quod punctum B a solo G refertur ad A. Si enim dicatur quod a puncto alio potest reflecti illud, aut erit in linea longitudinis, que est EG, aut non.

[2.262] Sit in ea, et sit X [FIGURE 5.2.30a, p. 586], et ab eo ducatur perpendicularis super superficiem contingentem speculum in puncto illo, que quidem perpendicularis erit equidistans ZG, et ita equidistans AL. Erit igitur AL in superficie reflexionis huius perpendicularis, et erit similiter in superficie reflexionis perpendicularis ZG. Igitur ille due superficies reflexionis secant se super lineam AL. Sed secant se super punctum B, quod est impossibile, quoniam B non est in linea AL, quod patet per hoc quoniam FL equidistans BM. Restat ergo ut a nullo puncto linee EG preter quam a G possit reflecti B ad A.

[2.263] Si autem ab alio puncto, sit illud U, et ducatur linea longitudinis EUO, et sumatur superficies equidistans basi transiens per punctum U. Palam quoniam AN secabit hanc superficiem. Sit punctus sectionis Y. Similiter BM secabit eandem. Sit punctus sectionis K, et ducantur linee KU, YU, YK. Et cum superficies illa secet piramidem super circulum transeuntem per U, ducatur a puncto U linea ad centrum huius circuli, que sit RU. Et ducantur linee EK, EY, que quidem secabunt superficiem circuli PG, et sint puncta sectionum I, S. Et ducantur linee IO, SO.

37 ZGA *corr. ex* SGA *L3* 38 angulo[1] *alter. in* angulos *a. m. E*/angulo[1] . . . etiam *om. L3*/quare *om. P1*/quare . . . GLA[2] *inter. a. m.* (*post* GAL *add.* et) *L3*/*post* GLA[2] *add.* sunt *R*/et[2] . . . equales (39): et GA equalis GL *mg. a. m. O* 39 et[1] *om. SC1ER*/equales . . . DL (40) *mg. a. m. C1*/erit . . . DL (40) *om. P1* 40 NC: NO *R*/CF: OF *R*/GC: GO *R*/*post* perpendicularis *add.* super NF *R* 41 CFG: OFG *R*/CNG: ONG *R* 42 punctus: penes *P1*; punctum *R* 44 refertur: fertur *L3E*; reflectitur *C1R* 45 a: ab *R*/puncto alio *transp. R*/illud: illum *F* 49 erit *om. FP1*/erit equidistans *transp. C1*/ZG *corr. ex* EG *L3*/ita *mg. S* 50 erit *om. R*/*post* AL[2] *add.* est *R* 51 erit: est *R*/perpendicularis . . . reflexionis (52) *om. S* 52 reflexionis *corr. ex* rereflexionis *F*/super . . . se[2] *om. L3*/AL: AB *E*/se *om. S* 53 in *om. FP1*; *inter. L3* 54 equidistans: equidistant *FP1E*; equidistat *C1R*; *alter. ex* equidistant *in* equidistat *L3*/restat: restant *FP1*/a *inter. O*/*post* nullo *add.* a *S* 55 linee EG *transp. C1*/a G: AG *P1*/ad *inter. L3* 56 si: sed *S*/alio: aliquo *FP1ER*; *corr. ex* aliquo *L3*/*post* puncto *add.* extra lineam EG *R*/ducatur: dicatur *E* 58 quoniam: quod *R*/AN: autem *S*/punctus: punctum *R* 59 sit . . . piramidem (60) *mg. a. m. O*/punctus: punctum *R* 60 YK: AK *S*; *corr. ex* AK *L3* 61 U[2] *om. P1* 62 *post* que[1] *add.* extra circulum producta *R*/RU: YU *S*; *corr. ex* YU *a. m. E* 63 sectionum: sectionis *E*/I S: E S *S*; *corr. ex* Y S *a. m. E*; S I *R* 64 linee *inter. O*/IO SO: IC SC *R*

[2.264] Sicut ergo probatum est de puncto M quod, non impediente piramide, potest reflecti ad N a puncto G, ita probatur de puncto K quod potest reflecti ad punctum Y a puncto U, et est eadem probatio. Et ita angulus RUY est equalis angulo RUK.

[2.265] Palam ergo quoniam BK est equidistans EG, et linea communis superficiei BGEK et superficiei circuli PG est linea MG. Igitur linea EK, cum sit in hac superficie et secet superficiem circuli PG, cadet super lineam communem, que est MG. Erit igitur SMG linea recta.

[2.266] Eodem modo, cum superficies NYEG secet superficiem circuli PG super lineam NG, linea EY concurret cum linea NG. Igitur ING linea est recta. Palam etiam quoniam superficies IOE secat superficiem circuli PG super lineam IO, et secat superficiem huic equidistantem que transit per U super lineam YU. Ergo YU equidistat IO. Similiter superficies SOEK secat superficies illas equidistantes super duas lineas SO, KU. Igitur SO equidistat KU.

[2.267] Similiter, si sumatur superficies secans speculum super lineam longitudinis EO in qua sunt R, U, O, M, secabit illas superficies equidistantes super duas lineas MO, RU. Igitur hee due linee sunt equidistantes. Igitur angulus SOM equalis angulo KUR, et angulus MOI equalis angulo RUY. Sed iam patuit quod angulus KUR equalis est RUY. Igitur angulus SOM equalis est angulo MOI, quare punctus S potest reflecti ad I a puncto O, non impediente piramide.

[2.268] Sed iam probatum est quod punctum M reflecti potest ad I a puncto G, et ita punctum S, quod est in linea SMG, potest reflecti ad I a puncto G. Igitur punctus S reflectitur ad I a duobus punctis circuli PG, quod est impossibile. Restat ergo ut primum sit impossibile, scilicet

66 *post* G *add.* etiam *O*/probatur: probabiliter *P1*; probabitur *FR* 67 ad . . . U: a puncto U ad punctum Y *R*/U *inter. O*/est eadem *transp. R*/probatio: propobatio *O* 68 ita: in *L3*/*post* ita *rep. et del.* ita *F*/RUY: RUI *O*; *corr. ex* UY *a. m. E*/est *om. FP1L3E*; erit *R*/equalis *corr. ex* equalem *L3*/RUK *alter. ex* IUE *in* RUE *a. m. E* 69 ergo *om. SOL3C1ER*/quoniam *om. F*; quod *P1*/BK *alter. in* BE *a. m. E* 71 linea *corr. ex* lineam *L3*/*post* linea *scr. et del.* tamen fit *P1*/et *inter. O* 73 NYEG: NYES *E* 74 lineam: linea *O* 75 ING: IQG *S*/linea est *transp. P1C1*/est *inter. O*/quoniam: quod *R*/IOE: IDE *S*; *corr. ex* IEO *L3*; IEC *R* 76 IO: IC *R*/equidistantem (77): equidistanter *S* 77 ergo YU *inter. L3*/equidistat *corr. ex* equidistant *C1*/IO: YO *S*; IC *R* 78 SOEK: IOEK *L3*; SOE *E*; SEC *R* 79 SO[1,2]: SC *R*/KU igitur SO *om. FP1*/equidistat *corr. ex* equidistant *C1* 80 si *om. FP1*/*post* speculum *rep. et del.* secans speculum *E*/super: et *O*/super . . . longitudinis (81) *mg. P1* 81 lineam: linea *O*/EO: EC *R*/*post* qua *add.* superficie *R*/R *corr. ex* C *L3*; *corr. ex* Y *a. m. E*/*post* U *add.* et *FP1*/O: C *R* 82 equidistantes *corr. ex* equidistans *C1*/RU *corr. ex* CU *L3*; *corr. ex* YU *a. m. E*/hee due linee: due linee hee *C1* 83 SOM: SCM *R*/*post* equalis *add.* est *R* 84 MOI: MCI *R*/RUY *corr. ex* RUT *F*/*post* quod *scr. et del.* L *F* 85 est *om. C1*/RUY *corr. ex* CUY *L3*/SOM: SCM *R*/MOI: MOY *S*; MCI *R*/punctus: punctum *R* 86 O: C *R* 87 est *om. S*; *inter. OC1* (*a. m. C1*) 88 et . . . G (89) *om. R* 89 punctus: punctum *FP1R*/S: F *L3*/reflectitur: reflectetur *SOC1*/*post* circuli *scr. et del.* puncto *C1* 90 sit impossibile *transp. C1*

quod punctus B reflectatur ad A ab aliquo alio puncto speculi quam a G, quod est propositum.

[2.269] **[PROPOSITIO 31]** Amplius, dato speculo piramidali, est invenire punctum reflexionis.

[2.270] Verbi gratia, sit G [FIGURE 5.2.31, p. 587] conus piramidis, et super ipsum fiat superficies equidistans basi piramidis, que sit MNG. A sit punctus visus, B centrum visus. A et B aut erunt citra illam superficiem; aut ultra; aut in ipsa superficie; aut unum citra, aliud ultra; aut unum in superficie, aliud citra vel ultra.

[2.271] Sint ultra superficiem, et a puncto A ducatur superficies secans piramidem equidistans basi, et ducatur a puncto G linea ad punctum B, que producta cadet in superficiem ab A ductam, cum sit inter superficies equidistantes. Punctus in quo cadit hec linea sit H.

[2.272] Probatur autem modo supradicto quoniam A refertur ad H ab aliquo puncto circuli piramidis quem efficit superficies secans ducta a punctis A, H. Et inveniatur in circulo illo punctus reflexionis, et sit E. Et ducatur linea AB, et linea longitudinis piramidis GE, axis piramidis GT.

[2.273] Et ducatur a puncto E linea ad centrum circuli, que quidem cadet super axem, et sit ET, et erit ortogonalis super superficiem contingentem circulum illum in puncto E. Et ductis lineis AE HE, secabit angulum earum per equalia, et dividet lineam AH. Sit punctus divisionis R.

[2.274] Palam quoniam GE, ET efficiunt superficiem secantem lineam AB. Sit punctus sectionis F, et a puncto F ducatur perpendicularis super lineam GE et sit FC, que quidem erit ortogonalis super superficiem contingentem piramidem super lineam GE. Deinde a puncto A

91 quod punctus: ut punctum *R*/reflectatur: reflectitur *E*/A *inter. L3*/alio *inter. O*/alio puncto *transp. R*/*post* puncto *add.* circuli *E* 92 G: AG *P1*/quod *inter. a. m. C1* 95 conus: vertex *R*/ piramidis: piramidalis *FP1SER*; *corr. ex* piramidalis *C1* 96 *ante* et *add.* speculi *R*/ipsum *corr. ex* ipsam *S*/fiat *corr. ex* fias *P1*/piramidis: piramidali *FP1*; piramidalis *SC1*/MNG A sit (97) *om. FP1* 97 punctus visus: punctum visum *R* 98 aut^2 . . . ultra *mg. a. m. F*/*post* aut^2 *scr. et del.* unum *L3*/ipsa *om. L3*; ipsam *E*/*post* superficie *scr. et del.* aliud in ipsa superficie *L3*/aut^3 . . . superficie (99) *om. S*/*post* citra *add.* et *E*/aut^4 *inter. O* 99 unum . . . aliud *inter. a. m. L3*/*post* superficie *add.* et *FP1*/aliud: alium *F*/ultra *corr. ex* extra *P1* 100 sint: sicut *S*; sit *E*/ultra: citra *R* 101 equidistans: equidistanter *FR* 102 ab: ad *P1*/A: ab *S*; *inter. O* 103 equidistantes *corr. ex* equidem *S*/punctus in quo: punctum in quod *R* 104 probatur autem *rep. S*/supradicto: predicto *FP1*/quoniam: quod *R*/refertur: reflectitur *R* 105 aliquo: alio *FP1SOL3E*; *corr. ex* alio *C1*/piramidis: piramidalis *SOL3C1E*; *om. R*/*post* secans *add.* piramidem *R* 106 punctus: punctum *R* 107 GE: G *S*/*post* GE *add.* et *ER*/axis . . . GT (108) *mg. a. m. L3* 110 et^2 *om. S*/ ortogonalis: ortogonaliter *O*/superficiem: lineam *R* 112 punctus: punctum *R* 113 R: P *S*; *corr. ex* Y *L3*; *alter. in* Y *E* 114 ET *corr. ex* et *L3*; ER *C1* 115 punctus: punctum *R* 116 FC: FQ *R*/ortogonalis: ortogonaliter *FP1O*

ducatur equidistans linee FC, et sit AL. FC autem concurret cum axe in punto K. Et a puncto A ducatur equidistans linee RT, que sit AS, et ducatur a puncto E linea communis superficiei AEH et superficiei contingenti piramidem in linea GE, que sit EO. Cadet quidem ortogonaliter super AS, cum sit ortogonalis super ER.

[2.275] Et ducatur linea BC, que producta necessario concurrat cum linea AL. Sit concursus in puncto L, et ducatur a puncto C linea communis superficiei contingenti et superficiei ABL, que sit CP. Et ducantur linee LS, PO.

[2.276] Palam quoniam superficies ALS est equidistans superficiei GEK, et linee CE, PO sunt in superficie contingenti, que superficies secat illas superficies equidistantes super duas lineas CE, PO. Igitur CE equidistans PO.

[2.277] Ducatur autem linea HE donec concurrat cum AS in puncto S. Palam quod linea ES est in superficie HEG, et in eadem est linea BL, et hec superficies secat predictas superficies equidistantes in duabus lineis EC, LS. Igitur EC est equidistans LS. Erit igitur PO equidistans LS, quare proportio AO ad OS sicut AP ad PL.

[2.278] Sed palam quod angulus HER est equalis angulo REA. Erit angulus ESA equalis angulo EAS, et EO perpendicularis. Erit AO equalis OS. Erit igitur AP equalis PL. Et CP perpendicularis super AL, cum sit perpendicularis super FCK. Ergo CL equalis CA, et angulus CLA equalis angulo LAC. Erit ergo angulus BCF equalis angulo ACF. Igitur A refertur ad B a puncto C, quod est propositum.

[2.279] Si vero centrum visus et punctus visus fuerit in superficie MGN [FIGURE 5.2.31a, p. 588], sit unum in puncto M, aliud in puncto N, et ducantur linee MG, NG, MN, et dividatur MGN per equalia per

118 FC[1] *corr. ex* FT *a. m. E*; FQ *R*/et . . . FC[2] *inter. a. m.* (AL *rep.*) *L3*/FC[2]: FQ *R*/*post* FC[2] *add.* A *S*/autem: aut *S*/concurret: concurrat *ER* 119 equidistans *corr. ex* equidem *S* 120 superficiei[1] *om. SC1*; *inter. L3*/*post* superficiei[1] *add.* reflexionis *R*/AEH et superficiei *om. P1* 121 quidem: quod *FP1* 122 ortogonalis: ortogonaliter *FP1L3E*/ER: ET *R* 123 BC: BO *S*; BQ *R*/concurrat: concurret *C1ER* 124 *post* sit *add.* punctus *E*; *add.* punctum *R*/in puncto *om. ER*/C: Q *R* 125 CP: OP *R* 126 linee *om. ER* 127 ALS: AL *FP1* 128 GEK: GER *C1*; GEF *E*/CE: QE *R*/*post* CE *add.* donec (131) . . . linea[2] (132) *SL3* (concurrat: concurrant *S*; quod: quoniam *L3*)/*ante* PO *add.* BL *FP1SL3*/PO: BPO *S*/*post* in *scr. et del.* super *O*/contingenti: contingente *R* 129 CE: QE *R* 130 CE: QE *R*/equidistans: equidistet *C1ER* 131 HE . . . linea[2] (132) *om. FP1SL3*; *mg. O*/AS: HS *R* 132 quod: quoniam *C1E*/BL: HE *F*; HC *P1*; H *S*; HL *C1E*; *alter. ex* H *in* HC *L3* 134 EC[1,2]: EQ *R*/EC[2]: ES *P1*/equidistans[1]: equidem *S* 135 proportio *om. R*/AO: AC *S*/PL *corr. ex* PA *F* 136 quod: quoniam *C1*/HER *corr. ex* HEL *S*/est *om. E*/est equalis *transp. R*/*post* erit *add.* ergo *FP1* 137 angulo . . . CA (139) *mg. O*/EO: EA *E*/*post* EO *add.* est *R*/*post* perpendicularis *add.* super AS *R*/*post* erit *add.* ergo *R*/AO: AC *S* 138 OS *corr. ex* OF *F*/AP *mg. L3*/PL: PB *FP1*/CP: QP *R*/*post* perpendicularis *add.* est *R* 139 FCK: FK *SC1R*; FCFT *O*; FC *E*; *corr. ex* FK *L3*/CL: QL *R*/CA: AL *S*; AC *L3C1E*; AQ *R* 140 CLA: QLA *R*/LAC: LAQ *R*/BCF: BEL *FP1*; HCF *S*; BQF *R*/ACF: AQF *R* 141 refertur: reflectetur *R*/a *inter. O*/C: Q *R*/quod est: et ita *SC1*; *om. O* 142 punctus visus: punctum visum *R*/fuerit: sint *C1*; fuerint *R* 143 aliud: alium *F* 144 *post* dividatur *add.* angulus *R*/MGN: MNG *S*

lineam UG. Palam quoniam N a puncto G refertur ad M. Palam etiam quod linea UG et axis piramidis sunt in superficie secante piramidem super lineam longitudinis.

[2.280] A puncto U ducatur ortogonalis super hanc lineam longitudinis, que sit UE. Et super punctum E ducatur superficies equidistans basi, que secabit piramidem super circulum. Linea communis superficiei UEG et huic circulo sit ET. Palam quoniam cadet super axem et super centrum circuli.

[2.281] Deinde a puncto M ducatur equidistans linee GE, que quidem in superficie illius circuli cadat in punctum H. Similiter, a puncto N ducatur equidistans GE, que cadat in punctum A. Et ducatur AH, et ET secet eam in puncto R.

[2.282] Palam quoniam MH equidistans GE est in eadem superficie cum ea, que superficies secat superficiem MGN et superficiem HEA super duas lineas MG, HE. Igitur MG est equidistans HE. Similiter, AN, GE sunt in superficie secante illas equidistantes super NG, AE. Igitur NG equidistans AE. Similiter, superficies UGE secat easdem superficies super duas lineas RE, UG. Igitur UG, MG equidistantes HE, RE, quare angulus MGU equalis angulo HER, et angulus UGN equalis angulo REA, et angulus HER equalis angulo REA. Et ita punctus A potest reflecti ad H a puncto E.

[2.283] Si ergo a puncto A ducatur equidistans UE et alia equidistans RE, et ducatur ME donec concurrat cum linea equidistante UE, et ducantur linee communes, ut prius, et iteretur probatio predicta, patebit quoniam N potest reflecti ad M a puncto E. Erit igitur E punctus reflexionis, quod est propositum.

145 UG: QG *R*/quoniam: quod *R*/refertur: reflectitur *R* 146 quod *corr. ex* quoniam *P1*/UG: QG *R*/piramidis: piramidalis *S* 148 U: Q *R*/*post* ducatur *scr. et del.* linea *C1*/lineam *om. P1* 149 *ante* que *add.* GE *R*/UE: QE *R*/E *inter. a. m. C1*/ducatur: fiat *R*/equidistans: equidistanter *F*; equidem *SOL3* 151 UEG: QEG *R*/ET: et *S* 153 a puncto: ab *inter. O*/ducatur *corr. ex* ducantur *L3*/equidistans: equidem *SO*/GE: EG *R*/*post* GE *scr. et del*. quoniam *P1* 154 cadat: cadet *E* 155 equidistans: equidem *S*/cadat: cadit *C1*; cadet *E*/AH *corr. ex* KH *S*; AB *R* 156 secet: secat *O*/R: A *FP1* 157 MH: in H *S*; MB *R*/equidistans: equidem *S*; equidistat *R* 158 ea: ipsa *R*/superficiem^{2}: superficies *S*; *corr. ex* superficies *L3C1*/HEA: BEA *R* 159 *ante* super *add.* equidistans *R*/*post* MG *add.* et *C1*/HE^{1}: HG *E*; BE *R*/est equidistans *transp. R*/HE^{2} *corr. ex* EH *S*; BE *R* 160 AN: NA *R*/*post* illas *add.* superficies *R*/equidistantes *alter. ex* quidem *in* equidem *F*; equidem *OL3*/equidistantes . . . NG (161) *om. S* 161 equidistans *alter. ex* equidem *in* equidistat *mg. a. m. C1*; equidistat *ER*/UGE: QGE *R*/*post* easdem *scr. et del.* superficies *F* 162 RE: R *FP1*/UG^{1,2}: QG *R*/*ante* igitur *add.* igitur RE QG equidistant *R*/*ante* MG *add.* et *ER*/equidistantes: equidistant *C1ER*/HE *corr. ex* EH *S*; BE *R* 163 MGU: MGQ *R*/HER *corr. ex* HEY *a. m. F*; BER *R*/UGN: UGA *E*; QGN *R* 164 HER: BER *R*/punctus: punctum *R* 165 *post* ad *add.* punctum B *R* 166 ergo *corr. ex* vero *P1*/UE: QE *R*/equidistans: equidem *S* 167 ME: BE *R*/equidistante: equidistanter *F*; equidem *SOL3*; equidistanti *E*/*post* equidistante *add.* ipsi *R*/UE: QE *R* 168 *post* prius *add.* et ME NE *R* 169 quoniam: quod *R*/N *corr. ex* non *L3*/punctus: punctum *R*

[2.284] Si vero ambo fuerint citra MGN [FIGURE 5.2.31b, p. 588], fiat piramis huic opposita. Et est ut protrahantur linee longitudinis piramidis iam facte, et a puncto A ducatur superficies secans hanc ultimam piramidem, que sit equidistans basi, que quidem secabit piramidem super circulum, que sit YZ.

[2.285] B aut erit in hac superficie, aut non. Si fuerit, fiat operatio a puncto B. Si non, ducatur linea GB usque dum concurrat cum hac superficie. Et sit concursus in puncto D. Palam quoniam A refertur ad D ab aliquo puncto circuli YZ interiori. Inveniatur punctus ille (sicut deinceps probabimus et docebimus, non ex anterioribus), et sit Z. Et ducentur linee DZ, AZ, et linea PZ dividat illum angulum per equalia.

[2.286] Et producatur linea ZG ad aliam piramidem, que quidem perveniet ad superficiem eius, et erit linea longitudinis. Et sit linea ZGE. Palam quoniam superficies PZE secabit lineam AB. Secet in puncto Q, et ducatur a puncto Q perpendicularis super lineam GE, et cadat in punctum E. Et erit perpendicularis super superficiem contingentem piramidem super lineam GE. Et super punctum E fiat superficies equidistans basi, que sit AEH, et ducatur a puncto D linea equidistans ZE, que sit DH, concurrens cum superficie illa in puncto H. Et eidem linee sit equidistans AA.

[2.287] Palam quoniam DH est equidistans ZE, et sunt in eadem superficie, que superficies secat superficies equidistantes super duas lineas DZ, HE. Igitur HE DZ sunt equidistantes. Similiter, AZ, AE equidistantes. Et palam quoniam PZ transit per centrum circuli YZ, similiter RET per centrum alterius circuli super quem superficies AEH secat piramidem. Igitur superficies PZER secat duas superficies equidistantes super duas lineas PZ, RE. Igitur PZ equidistans RE, quare

171 citra: ultra *R* 173 a puncto *mg. F*/hanc ultimam (174) *transp. L3* 174 que[1] . . . piramidem (175) *om. ER*/*post* sit *scr. et del.* QM sit *C1*/que[2] *om. FP1*; *inter. L3* 175 que: qui *OE*/que sit *om. R*/YZ *corr. ex* YS *L3* 176 aut[1] *corr. ex* autem *P1*; autem *R*/hac: hanc *S*/operatio: comparatio *C1* 177 GB: DH *P1*/concurrat: currat *S* 178 quoniam: quod *R*/refertur: reflectitur *R* 179 aliquo: alio *SO*/interiori: interiore *R*/punctus ille: punctum illud *R* 180 *post* probabimus *add.* et dicemus *FP1OL3*/et docebimus *om. O*/*post* docebimus *scr. et del.* vel dicemus *C1* 181 ducentur: ducantur *R*/DZ AZ *corr. ex* DS AS *L3*; AZ DZ AD *R*/et *om. O*/PZ *corr. ex* PS *L3*/illum *inter. O*/illum angulum *transp. R* 182 *post* et *add.* a puncto G ducatur GZ linea longitudinis et ducatur AB et *R*/que *corr. ex* quasi *P1* 183 superficiem: superficies *O*/linea[2] *om. R* 184 ZGE *corr. ex* GE *O*/palam . . . GE (185) *mg. O*/quoniam: quod *OR*/PZE: PEZ *P1* 185 perpendicularis: perpendiculariter *FP1L3*/cadat: cadet *FP1* 187 *post* GE *rep. et del.* et[2] (185) . . . GE (187) *E* 188 basi *om. L3*/AEH: HAE *O*; FEH *R*/D: B *SO* 189 ZE *corr. ex* SE *L3*/que . . . ZE (191) *om. S*/DH: BH *O*; *corr. ex* ACB *C1* 190 AA *corr. ex* A *a. m. L3*; *corr. ex* AL *a. m. C1*; AF *R* 191 ZE *corr. ex* SE *L3*/et: quod *R* 193 DZ[1,2] *corr. ex* DS *L3*/AZ *corr. ex* AS *L3*/AE: FE *R* 194 *ante* equidistantes *add.* sunt *R*/et *om. FP1*/et palam: similiter *R*/PZ *alter. ex* PS *in* PZS *F*; PZS *P1*; *corr. ex* PS *L3*/centrum: centra *E*/YZ: YS *L3* 196 secat[1] *corr. ex* secet *E*/PZER: per Z et *P1*; *corr. ex* PSER *L3*/*post* superficies[2] *scr. et del.* duas *E* 197 PZ[1]: PSZ *FP1*; *corr. ex* PS *L3*/PZ[2]: PS *L3*/equidistans: equidistat *C1ER*

angulus AZP equalis angulo AER. Et ita erit angulus AER equalis angulo REH, quare A refertur ad H a puncto E.

[2.288] Igitur, si a puncto A protraxerimus equidistantem QE, et aliam equidistantem RE, et lineas communes, sicut supra, et iteraverimus modum probandi predictum, patebit quoniam punctus A refertur ad B a puncto E, quod est propositum.

[2.289] Si vero centrum visus fuerit in superficie equidistante basi que est supra conum, scilicet G, et punctum visus ultra hanc superficiem, erit invenire punctum reflexionis hoc modo.

[2.290] Sit enim centrum visus M [FIGURE 5.2.31c, p. 589], punctus visus A, et sit MGN superficies equidistans basi piramidis. Et a puncto A ducatur superficies equidistans basi piramidis, que secabit piramidem super circulum DEK cuius centrum T. Et a puncto M ducatur perpendicularis super hanc superficiem, que sit MH, et ducatur linea HT. Et a puncto A ducatur ad lineam HT intra circulum linea AEQ ut EQ sit equalis QT, secundum supradicta. Et ducatur linea TEI, et a puncto H ducatur equidistans TE et equalis, que sit HB . Et ducantur linee MB, BE. Palam quoniam superfices GTE secabit lineam AM. Sit punctus sectionis F, et ducatur a puncto F perpendicularis super lineam GE cadens in puncto O, que sit FOC. Et ducantur linee MO, AO. Dico quoniam O est punctus reflexionis.

[2.291] Palam quoniam HB equidistans et equalis TE. Igitur HT equidistans et equalis BE. Sed MH equidistans et equalis GT, cum utraque perpendicularis. Igitur HT equidistans et equalis MG. Igitur MG equidistans et equalis BE, quare MB equidistans et equalis GE.

198 AZP: AP *FP1*; *corr. ex* ASP *L3*/AER[1]: AES *FP1*; AEZ *E*/AER[1,2]: FER *R*/*ante* et *add.* et angulus DZF angulo HER *R* 199 quare . . . H *om. S*/A: F *R*/refertur: referetur *E*; reflectetur *R*/E . . . puncto (200) *mg. O* 200 A: F *R*/equidistantem *corr. ex* equidistans *a. m. C1* 201 equidistantem: equidem *F*/lineas: linea *FP1*/*post* supra *add.* EC *FP1*/iteraverimus (202): iteravimus *FP1L3* 202 quoniam: quod *R*/punctus: punctum *SOC1R* 203 refertur: reflectetur *R*/B: D *SOC1*/*ante* a *add.* et *S* 204 centrum: punctus *FP1SOC1*; *corr. ex* punctus *L3*/equidistante: equidistanti *E*/basi *om. ER* 205 conum: verticem *R*/et *om. FP1*; *scr. et del. L3*/punctum: centrum *FP1SOC1*; *corr. ex* centrum *L3*; punctus *E*/visus: visum *R*/ultra: citra *R* 206 *post* hoc *scr. et del.* i *C1* 207 punctus visus (208): punctum visum *R* 208 A: ? *S*; *inter. OL3*/et[1] . . . superficies *om. FP1* 209 que . . . piramidem *om. P1* 210 *post* circulum *add.* que sit *R*/DEK: DEH *O*/centrum *corr. ex* centri *a. m. L3*/*post* centrum *scr. et del.* visus *S* 211 *post* ducatur *add.* axis GT et *R*/*post* HT *add.* et ducatur ab M ad A linea recta MA *R* 212 AEQ *corr. ex* EAQ *L3*/ut: et *R* 213 supradicta: supradictam *FP1L3* 214 TE: DE *FP1*/*post* equalis *add.* ei *FP1*/et[2] *inter. O* 215 *post* BE *add.* GE *R*/quoniam: quod *R*/secabit: secat *SOC1*/*post* AM *add.* et *C1*/punctus: punctum *R* 216 perpendicularis: perpendiculariter *FP1L3*/GE: EG *R* 217 *ante* cadens *add.* et producatur ad axem *R*/puncto: punctum *R*/sit *om. FP1*; *inter. L3*/FOC: FOP *R* 218 quoniam: quod *R*/punctus: punctum *R* 220 equidistans[1]: equidem *FP1L3*/et[1] *om. S*/BE: EB *R*/equidistans et equalis: equalis et equidistans *R* 221 *post* utraque *add.* sit *R*/perpendicularis: perpendiculari *E*/*post* HT *add.* est *OER*/equidistans et *om. C1*/igitur MG (222) *mg. O* 222 *post* equidistans[1] *scr. et del.* M *L3*/BE: DE *S*

[2.292] Palam etiam quod angulus QTE equalis angulo QET, et ita equalis angulo AEI. Sed est equalis angulo IEB. Igitur, IEB equalis angulo IEA, quare A refertur ad B a puncto E. Et cum MB equidistans sit GE, si a puncto A ducatur equidistans FOC et equidistans TE, et iteretur figura supradicta et probatio, palam quoniam A refertur ad M a puncto O, et ita propositum.

[2.293] Si vero M sit in superficie, et A citra superficiem, fiet piramis alia huic opposita. Et fiat super A superficies equidistans basi huius piramidis, et invenitur in circulo huius superficiei punctus reflexionis ex punctis interioribus. Et ducatur a puncto illo linea ADG, et producatur. Et invenietur punctus, secundum superiora, et idem probandi modus.

[2.294] Si autem puncta, scilicet centrum visus et punctus visus, ita disponantur ut unum sit citra superficiem coni, aliud ultra, sit unum L [FIGURE 5.2.31e, p. 589], aliud A, superficies coni MGN.

[2.295] Et ducatur a puncto A superficies equidistans basi secans piramidem super circulum DE cuius centrum T, et ducatur linea LG. Concurret quidem cum superficie AED. Sit concursus K, et in circulo DE inveniatur punctus, qui sit E, ita ut contingens ducta a puncto illo, que sit SE, dividat per equalia angulum quem continent linee KE, AE.

[2.296] Et a puncto L ducatur linea equidistans GE, que necessario concurret cum linea KE. Sit concursus B. Palam quod L est in superficie GEK, et LB in eadem superficie equidistans GE. Et ducatur linea TEI. Palam quoniam superficies GTE secat lineam LA. Secet in puncto U, a quo ducatur perpendicularis super superficiem contingentem, que sit UOC. Et ducantur linee AO, LO.

223 QTE: QDE *FP1*/*post* equalis *add* est *R*/QET . . . angulo[2] (224) *mg. a. m. L3* 224 *post* sed *add.* QTE *R*/IEB[1]: ZEB *FP1*/igitur IEB *om. FP1* 225 angulo *om. FP1L3E*; est *R*/refertur: reflectitur *R*/MB: linea BM *R* 226 *post* sit *add.* linee *R*/si: sed *C1*/FOC: FOP *R*/equidistans *inter. O*/TE: IT *P1SOL3C1ER* 227 quoniam: quod *R*/refertur: reflectetur *R*/M: AN *S* 228 O *om. P1*/*post* ita *add.* est *R* 229 superficie *corr. ex* superbie *F*/et . . . superficiem *mg. O*/citra: ultra *R*/superficiem: superficies *S*; *corr. ex* superficies *L3C1* 230 equidistans: equidem *FSOL3* 231 et *om. S*/invenitur: invenietur *SOC1*; inveniatur *R*/punctus: punctum *R* 232 punctis: puncti *S*/illo *om. P1*/ADG *corr. ex* ADADG *F*; *corr. ex* DG *L3*/et[2] *om. FP1* 233 punctus: punctum *R*/*post* idem *add.* est *R* 235 scilicet *inter. L3*/et punctus visus *om. FP1*; *mg. a. m. E*/punctus visus: punctum visum *R*/*post* visum *add.* et *L3* 236 disponantur: disponatur *L3*/coni: verticis *R*/ultra *corr. ex* extra *E*/sit[2] *rep. L3*/L: B *R* 237 aliud: ad *FP1*/coni: verticis *R*/MGN *corr. ex* MG *O* 238 secans: secabit *R* 239 piramidem: sperem *SOC1*; *corr. ex* sperem *L3*/super *rep. S*; *corr. ex* circa *L3*/*post* circulum *add.* que sit *R*/cuius centrum: centrum eius sit *R*/centrum *om. FP1*/T *rep. P1*/*post* et *add.* ducatur axis GT et *R*/LG: BG *R* 240 concursus: circulus *S*/et *om. O* 241 inveniatur *corr. ex* invenietur *a. m. C1*/punctus qui: punctum quod *R*/E: extra *R* 242 KE AE *om. P1*/*post* AE *add.* et ducatur linea longitudinis GE *R* 243 L: B *R*/linea *inter. E* 244 cum *inter. L3*/B: H *R*/L: B *OC1*; H *R* 245 LB in: BHM *R*/*post* superficie *inter.* sit *a. m. O*; *add.* quia *R*/equidistans: equidistanter *FP1*/*post* equidistans *add.* est *R*/GE: GK *E*/TEI: DEI *F*; DCI *P1* 246 quoniam: quod *R*/LA: BA *R* 247 quo *inter. a. m. S* 248 UOC *corr. ex* UIC *L3*; UOP *R*/AO: AC *S*/LO: LC *S*; *corr. ex* LG *L3*; BO *R*

[2.297] Palam quoniam AES equalis est angulo SEK, et cum angulus IES est rectus, et SET rectus, erit IEA equalis angulo TEK. Et ita angulus AEI equalis est angulo IEB, quare A refertur ad B a puncto E. Si ergo a puncto A ducantur equidistans UO et equidistans IT, et iteretur probatio, patebit quoniam refertur A a puncto O ad L, et ita propositum.

[2.298] Palam ergo quomodo sit invenire punctum reflexionis, et hec que dicta sunt in unico visu intelligenda sunt. In duplici autem visu, idem accidit, quoniam eadem forma et idem locus forme comprehendetur ab utroque oculo, et sicut dictum est in speculo sperico exteriori, forme a duobus oculis comprehense in hiis speculis propter contiguitatem videntur eadem, et aliquando simul sunt in loco, aliquando commiscentur earum loca in parte, aliquando separantur, sed modicum.

[2.299] Forma autem que secundum perpendicularem in hiis speculis descendit secundum eandem regreditur, sicut supra patuit, et forma illa ab uno oculo super perpendicularem percipitur ab alio oculo secundum lineam reflexionis. Sed loca formarum continua, unde eadem apparet utrique visui forma.

[2.300] **[PROPOSITIO 32]** In speculis spericis concavis, aliquando perpendicularis a puncto viso ducta secat lineam reflexionis, aliquando est equidistans ei. Quando secat, erit locus forme aliquando in speculo, aliquando ultra speculum, aliquando citra. Et cum fuerit locus forme citra speculum, aliquando erit inter visum et speculum, aliquando in centro visus, aliquando citra centrum visus. Et nos hoc demonstrabimus.

[2.301] Sit A [FIGURE 5.2.32, p. 590] centrum visus, D centrum speculi, et fiat superficies super hec puncta, que secabit speculum super circulum, qui circulus sit HBFG. Erit quidem hec superficies superficies reflexionis, quoniam est ortogonalis super quamlibet superficiem

249 quoniam: quod *R*/*ante* AES *scr. et del.* superficies *P1*; *add.* angulus *R*/est *inter. O* 250 IES: LES *P1*/est: sit *R*/et SET *inter. L3*/SET *corr. ex* ET *O*/rectus *om. L3* 251 est *om. R*/IEB: IEH *R*/refertur: reflectetur *R*/B: H *R* 252 ducantur: ducatur *OC1R*/UO: UC *SO* 253 quoniam: quod *R*/refertur: reflectetur *R*/refertur A *transp. SOC1* (refertur *inter. O*)/L: B *R*/*post* ita *add.* patet *R* 254 quomodo: quoniam *R*/sit: sic *P1*/invenire: invenitur *FP1*; *corr. ex* invenitur *L3* 255 in: de *R* 257 comprehendetur: comprehenditur *R*/oculo: visu *R*/est *om. SOL3*/speculo sperico *corr. ex* sperico speculo *P1* 258 exteriori: exteriore *R* 259 eadem: una *R*/simul: similis *SO*/in *om. SO*/*post* loco *add.* et *ER* 262 que secundum: que per *mg. a. m. O*/secundum: per *FP1ER*; per *inter. L3*/in hiis *inter. L3* 263 speculis *om. E* 264 oculo[1]: circulo *C1*/super . . . alio *mg. a. m. L3*/percipitur: percuritur *FP1O*; ? *S*; percurrit et *L3*/oculo[2] *om. SOL3* 265 lineam: lineas *O*/*post* continua *add.* sunt *ER* 266 utrique: utique *FP1*; *om. ER*/*post* forma *add.* de spericis concavis *L3E* 269 ei: ti *C1* 271 *post* aliquando *add.* autem *L3* 272 hoc: hec *ER* 276 hec superficies *transp. R*/superficies (277) *inter. L3* 277 quamlibet: quelibet *S*; *corr. ex* quam *L3*

contingentem circulum. Et ducatur linea AD, et a puncto A ducatur linea ad circulum maior AD, que sit AE. Et a puncto D ducatur ad circulum equidistans linee AE, que sit DH, et producatur AD usque in puncta B, I, et ducatur linea DE.

[2.302] Palam quoniam angulus AED est minor recto, quoniam ED dyameter, et quelibet linea in circulo cum dyametro facit angulum acutum. Et super punctum E fiat angulus equalis angulo AED, qui sit DET. Palam quoniam ET cadet intra circulum, et secabit lineam DH. Sit punctus sectionis T. Palam etiam quod angulus ADE maior angulo DET, quia AE maior AD, et ita ET secabit AB. Secet in puncto Z.

[2.303] Deinde a puncto A ducatur ad arcum EH linea que sit AN, et ducatur linea DN, et supra punctum N fiat angulus equalis angulo DNA per lineam NM, que necessario cadet intra circulum et secabit DH. Secet in puncto M. Palam etiam quod AN concurret cum DH extra circulum. Sit concursus L.

[2.304] Ducatur etiam a puncto A linea ad arcum EF, que sit AG, et ducatur DG, et sit angulus AGD equalis angulo DGQ. Palam quod QG secabit DH. Sit punctus sectionis Q. Palam etiam quod AG concurret cum DH ex parte F. Sit concursus O. Quod autem GQ cadat inter D et H palam cum arcus quem secat GO ex circulo sit maior arcu GH. Si enim ducatur linea GH, angulus HGD respiciet maiorem arcum angulo DGA.

[2.305] Item, a puncto A ducatur ad arcum FB linea AC secans DH in puncto S ut sit CS maior SD, et ducatur DC. Palam quod angulus DCA est acutus. Fiat ei equalis, qui sit DCK. Palam, cum angulus CDS sit maior angulo DCS, CK concurret cum DH. Sit concursus in puncto K.

[2.306] Palam secundum supradicta quod punctus T movetur ad E, et refertur ad A. Et perpendicularis a puncto T ducta est TD, que

278 et[2] . . . ad[1] (279) *om. L3* 279 maior *corr. ex* maiorem *S*/*post* maior *add.* quam *R* 280 linee *om. R* 282 quoniam: quod *R*/*post* ED *add.* est *O* 283 dyameter: semidiameter *R* 284 AED *corr. ex* ED *O*/*post* AED *scr. et del.* maior angulo *S*/qui: que *R* 285 quoniam: quod *R*/ET *corr. ex* et *FL3* 286 punctus: punctum *R*/*post* maior *inter.* est *O*; *add.* est *R* 287 quia: qua *S*/quia . . . AD *om. R*/ET *mg. L3*/*ante* secabit *add.* FS *L3*/*post* secabit *add.* lineam *C1* 288 *post* sit *scr. et del.* communis *P1* 289 et[2] *inter. a. m. C1*/supra: super *R* 290 intra: inter *FP1SL3* 291 etiam: est *S*/AN *corr. ex* angulus *L3*/concurret *corr. ex* concurrens *a. m. E* 292 L: ML *R* 293 *post* etiam *scr. et del.* linea *C1*/linea *om. O*/EF: EIF *R* 294 sit: fiat *R*/angulus *om. O*/AGD: QGD *SO*; DGQ *R*/angulo *om. O*/DGQ: DGA *SO*; AGD *R* 295 *post* DH *add.* ut patuit *R*/punctus: punctum *R*/etiam quod *transp. E*/*post* quod *scr. et del.* angulus *P1* 296 cum *inter. a. m. E*/GQ *corr. ex* GA *L3*/cadat: cadit *L3* 298 HGD: BGD *R*/respiciet maiorem *transp. R*/*post* maiorem *scr. et del.* angulum *C1* 299 DGA: AGD *R* 300 item: iterum *FP1E*/AC: AK *R* 1 CS: KS *R*/SD: LD *S*/DC: KD *R* 2 DCA: dicta *P1*; DKA *R*/qui: que *R*/DCK: *alter. in* DCE *deinde corr. ex* DCE *a. m. E*; DKU *R*/*post* palam *add.* quod *R*/CDS: KDS *R* 3 angulo *om. O*/DCS: CDCS *S*; DKS *R*/CK: KU *R* 4 K *alter. in* E *a. m. E*; U *R* 5 *post* palam *add.* ergo *SO* (*inter. O*)/supradicta: predicta *FP1*/punctus: punctum *R* 6 et[1] *inter. a. m. L3*/refertur: reflectitur *R*/TD . . . est (10) *mg. a. m. FL3*

perpendicularis est super superficiem contingentem circulum, et est equidistans linee reflexionis, que est AE, unde non concurret cum ea.

[2.307] Punctum autem Z movetur ad E, et refertur ad A. Et perpendicularis a puncto Z ducta est AZ, que concurrit cum AE in puncto A, unde locus forme puncti Z erit A.

[2.308] Punctum vero M movetur ad N, et refertur ad A. Et perpendicularis ducta a puncto M, que est MD, concurrit cum AN in puncto L, quod est ultra speculum, et locus forme puncti M erit L.

[2.309] Forma vero puncti Q movetur ad G, et refertur ad A, et locus eius erit O, qui est ultra visum.

[2.310] Et forma puncti K movetur ad C, et refertur ad A, et perpendicularis ab eo est KD, et locus ymaginis S.

[2.311] Palam igitur ex predictis quod ymaginum quedam ultra speculum, quedam inter visum et speculum, quedem in ipso visu, quedam citra visum, quod est propositum.

[2.312] Amplius, palam quoniam visus adquirit formas sibi oppositas, unde, cum locus ymaginis fuerit ultra speculum aut inter visum et speculum, comprehenditur veritas illius ymaginis. Cum autem perpendicularis a puncto viso ducta fuerit equidistans linee reflexionis, apparebit quidem ymago in puncto reflexionis. Quoniam, cum punctus ille sit sensualis, sumpto puncto eius intellectuali medio, ymago cuiuscumque partis illius puncti sensualis ultra medium sumpte erit ultra speculum; ymago partis citra medium erit inter visum et speculum, et cum totalis forma ex ulterioribus et citerioribus videatur una continua, necessario forma illius puncti sensualis videbitur in ipso speculo in loco reflexionis.

[2.313] Verum in ymaginibus quarum locus fuerit in centro visus, non comprehenditur veritas earum, unde sepius error accidit in

7 circulum: speculum *R* 8 equidistans: equidem *S*/que: quem *P1*/unde: unum *S*/concurret: concurrit *FP1E*; *corr. ex* curret *S*/ea: illa *FP1* 9 autem: A *P1*/*post* Z *add.* vero *FP1*/refertur: reflectitur *R* 10 a . . . ducta: ducta a puncto Z *R*/*post* ducta *rep. et del.* est (6) . . . ducta (10) *O*/concurrit: concurret *C1* 11 puncti: in puncto *E*/A[2] *inter. a. m. C1* 12 punctum *corr. ex* punctis *L3*; *corr. ex* productum *a. m. E*/movetur: videtur *L3*/refertur: reflectitur *R* 13 ducta *om. O*/AN: autem *S* 14 quod: que *E* 15 vero *inter. a. m. C1*/et[1] *mg. a. m. C1*/refertur: reflectitur *R*/ad[2] *rep. P1*/A *om. S* 17 K: U *R*/C: K *R*/refertur: reflectitur *R* 18 S: C *P1* 19 *post* predictis *scr. et del.* secundum *L3C1*/quedam . . . speculum[1] (20) *om. R*/*post* quedam *add.* sunt *OC1* (*inter. O*) 21 *post* visum *add.* quedam ultra visum apparent *R*/est *om. L3* 22 *post* visus *add.* perfectius *R* 23 unde: unum *S* 24 illius ymaginis *transp. C1* 26 quidem *om. R*/punctus ille (27): punctum illud *R* 27 sensualis: sensuale *R* 28 cuiuscumque: cuiuslibet *L3*/*post* medium *scr. et del.* illius *P1*/erit ultra speculum (29): ultra speculum erit *C1* 29 *post* speculum *add.* et *R*/partis *om. P1* 30 citerioribus *corr. ex* ceterioribus *L3*/*post* citerioribus *add.* partibus *R*/*post* una *add.* et *R* 31 sensualis *om. O* 33 in[1] *inter. a. m. E*/quarum *corr. ex* qualiter *a. m. F*; *corr. ex* quare *L3C1* (*a. m. C1*)/in centro *corr. ex* intro *L3*/*post* visus *add.* eo *S* 34 unde: unum *S*

huiusmodi speculis. Quod autem hoc pateat, erigatur super superficiem speculi lignum perpendicularem minus medietate semidyametri speculi. Et citra caput huius ligni sit centrum visus, et dirigatur visus ad punctum speculi cuius longitudo a ligno maior quam longitudo centri visus a dyametro per lignum transeunte. Videbitur quidem ymago illius ligni ultra visum, nec erit certa comprehensio eius, immo apparebit quasi arcuata, cum non sit. In hiis ergo speculis non comprehenditur veritas ymaginis nisi cuius locus fuerit ultra speculum aut inter visum et speculum. Cum autem fuerit centrum visus in perpendiculari per lignum transeunte, non plene comprehendit formam illius ligni.

[2.314] Si vero visus fuerit in dyametro spere et in centro eius, cum quelibet linea ab eo ducta ad speculum sit perpendicularis super speculum, non comprehendetur forma alicuius puncti nisi puncti portionis circuli interiacentis latera piramidis visualis que a centro circuli intelligitur protendi. Quoniam forma cuiuslibet alterius puncti cadet in speculum super lineam declinatam, et necessario refertur super declinatam, quare linea reflexionis non transibit per centrum, et ita non continget centrum visus.

[2.315] Si vero fuerit visus in dyametro sed non in centro, non comprehendet formam alicuius puncti semidyametri in quo est. Quoniam angulus quem efficient due linee a puncto sumpto in semidyametro et a centro visus in idem speculi punctum non dividetur per perpendicularem ab illo puncto speculi ductam, cum illa perpendicularis tendat ad centrum speculi. Sed formam alicuius puncti alterius semidyametri percipere poterit.

[2.316] **[PROPOSITIO 33]** Amplius, viso puncto in huiusmodi speculo, cum non fuerit perpendicularis equidistans linee reflexionis,

35 huiusmodi *corr. ex* huius *L3*; hiis *R*/quod: ut *R*/pateat: appareat *C1* 36 perpendicularem *alter. ex* perpendiculari *in* perpendiculariter *L3*; perpendiculariter *C1ER* 37 citra: contra *FP1*; circa *OL3ER*/huius: huiusmodi *C1*/ligni *corr. ex* lignus *C1* 38 ad: in *O*/punctum *corr. ex* speculum *L3*/*post* speculi *scr. et del.* u *L3*/longitudo[1] . . . quam *om. S*/a ligno *om. P1*/a . . . longitudo[2] *inter. a. m. L3*/*post* ligno *add.* est *L3*; *add.* sit *R* 39 transeunte *corr. ex* transeuntem *L3C1*; transeuntem *E* 40 *post* visum *scr. et del* verum *F*/nec erit *mg. a. m. F* 41 quasi *om. R*/arcuata *corr. ex* arcuta *O* 42 *post* comprehenditur *scr. et del.* y *F*/*post* veritas *rep. et del.* veritas *E*/ymaginis: yma *L3*/fuerit: fuit *L3*/ultra *inter. E* 43 aut: ut *S*/fuerit . . . visus: centrum visus fuerit *R* 46 visus *om. FP1*/et *om. P1*/*post* centro *scr. et del.* si *S* 47 quelibet: qualibet *S*/ducta ad speculum: ad speculum ducta *R*/perpendicularis: perpendiculariter *FP1L3* 48 nisi puncti: nisi forma *mg. F*/puncti: forma *P1* 49 circuli[1]: oculi *OR*; *alter. in* oculi *L3E* (*a. m. E*)/*post* latera *scr. et del.* oi c *P1*/circuli[2]: speculi *R* 50 alterius puncti *transp. P1* 51 refertur: reflectetur *R* 52 et . . . centrum (53) *om. O* 53 *post* centrum *scr. et del.* vestrum *E* 54 sed . . . centro *om. ER* 55 alicuius: alterius *R*/quo: qua *R* 57 semidyametro: dyametro *P1*; *corr. ex* diametro *a. m. L3*/dividetur: videtur *P1* 58 per *om. FP1*; *inter. O*/perpendicularem: partem *L3*; pendicularem *E*/illo: alio *FP1SL3C1E*; *corr. ex* alio *O* 61 huiusmodi *corr. ex* huiusiusmodi *F*; huius *P1* 62 non *om. P1*/equidistans: equidem *FC1*

linea a centro speculi ad punctum visum ducta sic se habebit ad lineam ab eodem centro ad locum ymaginis ductam sicut linea a puncto viso ad punctum quem diximus contingentie se habet ad lineam a puncto contingentie ad locum ymaginis ductam.

[2.317] Verbi gratia, sit E [FIGURE 5.2.33, p. 590] centrum speculi, B punctus visus, A centrum visus, G punctus reflexionis, linea contingentie ZG. ZG aut concurret cum EB, aut erit equidistans ei.

[2.318] Concurrat in puncto T. Linea EB concurret cum AG, et non in puncto G, cum EB, BG sint due linee. Igitur aut concurrent ultra G, aut inter G et A, aut in A, aut citra A. Sit ultra G, et in puncto H. Dico ergo quoniam proportio EB ad EH sicut BT ad TH.

[2.319] Ducatur perpendicularis EG, et a puncto H ducatur equidistans linee BG, que concurret cum EG. Sit concursus L, et a puncto B ducatur equidistans GH, que necessario concurret cum ZT. Sit concursus Q.

[2.320] Palam quoniam angulus BGE est equalis angulo AGE. Sed angulo BGE equalis est angulo GLH, et angulus AGE equalis angulo LGH. Igitur LH equalis est GH. Similiter, angulus BGQ equalis est AGZ, et angulus AGZ equalis angulo GQB, et ita BQ equalis BG, quare proportio BG ad HL sicut BQ ad HG.

[2.321] Sed quoniam angulus GHT equalis est angulo TBQ, erit triangulus TBQ similis triangulo GHT. Igitur proportio QB ad HG sicut BT ad TH, et ita BG ad HL sicut BT ad TH. Sed cum triangulus BGE sit similis triangulo HEL, erit proportio BG ad HL sicut EB ad EH, et ita EB ad EH sicut BT ad TH, quod est propositum.

[2.322] Eadem erit probatio si locus ymaginis fuerit inter A et G [FIGURE 5.2.33a, p. 591], aut in A [FIGURE 5.2.33b, p. 591], aut ultra A [FIGURE 5.2.33c, p. 591].

63 sic *om. SOC1R*; *corr. ex* S *L3* 64 linea a *corr. ex* linea *a. m. C1* 65 quem: quod *R*/*post* lineam *scr. et del.* a punctum *P1* 68 punctus^{1,2}: punctum *R*/visus *inter. a. m. O*; visum *R* 69 *post* ZG^1 *add.* que *FP1* (*mg. a. m. F*)/ZG^2: GZ *L3*/*post* ZG^2 *scr. et del.* arcus *P1*; *add.* autem *R*/concurret *corr. ex* curret *F* 70 *post* linea *add.* vero *R*/concurret: concurrat *C1*; concurrit *R*/*post* cum *add.* linea *SO*/et: sed *R* 71 EB BG: BE AG *R*/*post* sint *scr. et del.* linee *P1*/concurrent: concurret *E*; concurrit *R* 72 aut^2 . . . A^3 *om. P1*/aut citra A *mg. O*/citra: ultra *R* 73 quoniam: quod est *R* 74 ducatur^1: producatur *R* 75 sit: sicut *L3*/et . . . B *om. C1*/B: H *P1*/*post* B *scr. et del.* d *C1* 78 quoniam: quod *R*/est equalis *transp. C1*/est . . . BGE (79) *om. FP1*/angulo *om. R* 79 angulo^1: angulus *R*/BGE: AGE *C1*/*post* BGE *rep. et del.* equalis (78) . . . BGE (79) *E*/equalis est *transp. R*/est *om. P1*/GLH: HLG *O*/et . . . LGH (80) *scr. et del. E*/angulus *om. O*/*post* AGE *add.* est *O*/angulo^3 *om. O* 80 *post* LGH *add.* ergo angulus GLH equalis est angulo LGH *R*/est^2 *om. P1*/*post* est^2 *rep.* GH . . . est *S*; *add.* angulo *R* 81 et^1 *corr. ex* equalis *S*/AGZ *corr. ex* AGS *L3*/*post* equalis^1 *add.* est *OC1ER*/*post* equalis^2 *add.* est *R*/quare *corr. ex* qualiter *F*/quare . . . BG (82) *mg. O* 82 HG: HD *S* 83 *post* quoniam *add.* G *FP1L3*/GHT: GHTT *S*/equalis est *transp. R*/TBQ: TQB *P1*/erit . . . TBQ (84) *rep. S*; *mg. a. m. L3* 84 triangulus: triangulum *R*/similis: simile *R* 85 triangulus: triangulum *R*/BGE . . . triangulo (86) *inter. a. m. L3*/sit: est *L3* 86 similis: simile *R*/BG *corr. ex* EG *P1*/et . . . EH (87) *inter. a. m. L3* 87 est *om. O* 88 erit probatio *transp. C1* 89 ultra: intra *P1*/A^2 *om. R*

[2.323] Si vero linea contingentie ZG sit equidistans perpendiculari, que est BH [FIGURE 5.2.33d, p. 591], ducatur perpendicularis GE, que, cum sit perpendicularis super ZG, erit perpendicularis super BH. Et erit angulus BEG equalis angulo HEG, et angulus BGE equalis est angulo EGH. Restat triangulus BGE similis triangulo EGH. Igitur proportio BE ad EH sicut BG ad GH, quod est propositum, quia in hoc casu non potest sumi aliud punctum contingentie quam punctus G, eo modo quo punctum contingentie supra appellavimus.

[2.324] **[PROPOSITIO 34]** Amplius, sit circulus DGT [FIGURE 5.2.34, p. 592], et A centrum visus intra speculum, E centrum speculi, B punctus visus. Et ducatur dyameter DAG.

[2.325] Si fuerit B in semidymetro EG, poterit esse reflexio ab aliquo puncto semicirculi GTD et ab aliquo puncto semicirculi ei oppositi. Quoniam quocumque puncto semidyametri EG sumpto, si ab eo ducatur linea ad aliquod punctum semicirculi GTD, et a puncto A ad idem punctum ducatur alia linea, ille due linee efficient angulum quem dividet semidyameter ductus a puncto E in illud punctum. Similiter in semicirculo opposito.

[2.326] Si vero B fuerit extra dyametrum DAG [FIGURE 5.2.34a, p. 592], ducatur dyameter transiens per B, qui sit TBQ. Dico quoniam B potest reflecti ad A per arcum interiacentem semidyametros in quibus sunt A et B, et similiter per eius oppositum, id est per arcum TD et per arcum GQ, et non poterit reflecti ab aliquo puncto arcus GT vel arcus QD.

[2.327] Verbi gratia, sumatur punctum in arcu GT, quod sit K, et ducantur linee AK, KB donec KB cadat super dyametrum DG in puncto O. Cum O et A sint ex eadem parte centri circuli quod est E, perpendicularis ducta a puncto K ad E non dividet angulum OKA, et ita B non refertur ad A a puncto K. Similiter, sumpto alio puncto quod sit F,

91 *post* contingentie *add.* scilicet *FP1*/ZG *alter. ex* GC *in* GZ *F*; GZ *P1*/*post* sit *scr. et del.* EG *L3*/perpendiculari *mg. L3* 92 BH: BEH *R*/*post* BH *add.* et *E*/GE: G *S* 93 ZG: GZ *R*/et *inter. O* 94 BEG: BEH *P1*/HEG: HEB *S* 95 triangulus: triangulum *R*/similis: simile *R* 96 EH: HE *R*/quia *om. FP1*; quare *R* 97 punctus: punctum *R* 99 DGT: ABGD *R* 100 A: H *R*/B: Z *R* 101 punctus visus: punctum visum *R*/DAG: BED *R* 102 B: Z *R*/EG: BE *R*/aliquo: alio *SO* 103 GTD *corr. ex* TGD *L3*; DTG *E*; BAD *R*/aliquo: alio *SOL3E* 104 *post* puncto *scr. et del.* semicirculi *C1*/EG: ?G *F*; BE *R* 105 GTD *om. ER*/A: H *R* 106 efficient: efficiunt *C1* 107 dividet: divident *E*/*post* dividet *add.* per equalia *R*/ductus: ducta *R*/E *inter. F*/in[1]: ad *R*/illud: illum *FP1*/*post* punctum *rep. et del.* punctum *E* 109 *post* B *add.* punctum visum *R* 110 qui: que *C1R*/TBQ: TQ *R*/*post* TBQ *add.* et *FP1*/quoniam: quod *R* 111 *post* ad *add.* visum *R*/semidyametros: diametros *R* 112 *post* est *scr. et del.* arcum et *F*/TD . . . arcum (113) *om. FP1* 113 et: sed *O*/GT: GR *R*/QD: GQ *L3* 114 GT: prope T *R* 115 KB cadat *transp. R*/cadat: cadet *P1* 116 *post* cum *add.* igitur *R*/centri circuli *transp. L3*; *corr. ex* circuli centri *E*/E *om. L3*/perpendicularis (117): perpendiculariter *L3* 117 K *corr. ex* KY *F*; *corr. ex* KS *L3*/et . . . AFB (119) *om. E*/B non *transp.*; B *inter. O* 118 refertur: reflectetur *R*/alio: aliquo *FP1*/*ante* puncto[2] *scr. et del.* al *F*

patebit quoniam perpendicularis EF non dividet angulum AFB, et ita non refertur B ad A a puncto F.

[2.328] Quod autem a puncto arcus TD vel arcus GQ possit fieri reflexio palam per hoc. Sit M punctum arcus DT, et ducantur linee AM, MB; fiet quadrangulum AMBE. Igitur perpendicularis EM dividet angulum AMB.

[2.329] Pari modo, sit H punctus arcus GQ. Linea AH secabit dyametrum TQ in puncto C, et linea HB eundem in puncto B. Et sunt hec duo puncta ex diversis partibus centri, quare linea EH dividet illum angulum.

[2.330] Pari modo, si fuerit B in superficie speculi aut extra speculum, dum A sit intra speculum, idem erit modus probandi qui prius. Similiter, si A fuerit in superficie speculi, B intus aut exterius.

[2.331] Verum, si a puncto A ducatur equidistans TE, que sit AP [FIGURE 5.2.34b, p. 593], loca ymaginum reflexarum a punctis arcus TP erunt extra speculum; loca autem ymaginum arcus PD ultra centrum visus, quod est A; loca autem ymaginum arcus QG sunt inter centrum visus et speculum. Et quod supradictum est de locis ymaginum idem intelligendum, ducta AM equidistans linee TQ.

[2.332] Si vero A fuerit extra speculum, B intra [FIGURE 5.2.34c, p. 593], patebit quod diximus. Ducantur a puncto A linee contingentes circulum GTD, que sint AH, AZ, et ducantur duo dyametri AEG, TEQ, et B in dyametro TEQ. Refertur B ad A ab aliquo puncto arcus TD, sed palam quod non ab aliquo puncto arcus ZD. Igitur ab aliquo puncto arcus TZ, et similiter ab aliquo puncto arcus oppositi TD, scilicet arcus GQ. Sed ab arcu TG vel DQ non fiet reflexio secundum supradictum modum.

[2.333] Si vero B sit extra hunc dyametrum et supra alium, qui similiter sit TEQ, fiet reflexio ab arcu TD, et a sola parte eius TZ, et ab arcu opposito, qui est GQ. Sed ab arcu TG vel DQ non fiet reflexio.

119 quoniam: quod *R*/AFB: OKA *S*/*post* ita *add.* B *S* 120 non . . . B: B non refertur *E*/refertur: reflectetur *R* 121 arcus *om. S* 122 *post* hoc *scr. et del.* quod *L3*/DT: TD *R* 123 MB: AB *S*/*ante* fiet *add.* et *S*/AMBE: ABE *O*/dividet: dividit *FP1*; dividat *E*/angulum (124) *om. S* 125 pari: simili *R*/*post* sit *scr. et del.* modo *S*/punctus: punctum *R*/GQ: GA *S*/AH: AD *S* 126 TQ *corr. ex* DQ *F*; QT *C1*/puncto[2] *corr. ex* punctum *P1*/*post* hec *add.* etiam *FP1L3ER* 129 in: ut *P1* 130 dum . . . speculum *inter. a. m. L3*/modus probandi *transp. R* 131 B *alter. in* D *E*/intus: interius *R* 132 verum . . . TQ (137) *transp. ad* 147 *post* reflexio (equidistans [137]: equidistante) *R*/que: qui *E* 133 punctis: punctus *O*/arcus *inter. O* 134 erunt: esset *FP1*/*post* speculum *add.* que ab ipso P erit perpendicularis equidistans linee reflexionis *O*/loca . . . A (135) *mg. a. m. C1* (autem *om.*; *post* PD *add.* sunt) 135 quod . . . visus (136) *om. S*/autem *inter. P1*/sunt *om. C1* 137 *post* idem *add.* erit *L3* 139 *post* ducantur *add.* enim *R*/a . . . linee: linee a puncto A *R*/A *inter. a. m. C1* 140 GTD: DCG *E*; DTG *R*/sint: sit *FP1L3*; sunt *C1*/AZ: DZ *O*/duo: due *R* 141 TEQ *corr. ex* DEQ *F*/refertur: reflectetur *R*/refertur B *transp. L3*/aliquo: alio *SO*; *corr. ex* alio *L3*/TD: DT *S* 142 ab[1] *om. O*/aliquo: liquo *F*/TZ: CG *S*/et *om. O* 143 ab[1] *rep. S*; *inter. C1*/oppositi: opposito *E*/*post* oppositi *add.* ipsi *R*/TD *om. S*/scilicet: id est *inter. O*/GQ: QG *C1*/*post* GQ *add.* reflexio fiet *R* 144 DQ *corr. ex* D *F*; *corr. ex* Q *P1* 145 sit: fuerit *R*/hunc: hanc *R*/supra: super *FP1R*; *corr. ex* super *S*/qui: que *R* 147 *post* arcu *inter.* ei *a. m. L3*

[2.334] **[PROPOSITIO 35]** Amplius, sumpto dyametro circuli in sperico speculo concavo, quilibet punctus illius dyametri quantumcumque producti potest esse locus ymaginum.

[2.335] Verbi gratia, sit AG [FIGURE 5.2.35, p. 594] dyameter circuli AMG, cuius D centrum. Sumatur in hoc dyametro punctus Z, E centrum visus. Dico quod Z potest esse locus ymaginis.

[2.336] Verbi gratia, ducatur linea ETZ, T punctus circuli. Ducatur linea DT. Erit angulus ETD acutus. Fiat ei equalis qui sit DTL. Palam quod L refertur ad E a puncto T, et eius ymago erit Z.

[2.337] Similiter, sumpto puncto L, patebit quod est locus ymaginis. Ducatur linea EL usque in B punctum circuli, et ducatur linea DB. Erit angulus EBD acutus. Fiat ei equalis, qui sit DBC. Refertur quidem punctus C ad E a puncto B, et locus ymaginis eius erit L, et ita sumpto quocumque alio puncto, eadem erit probatio.

[2.338] Amplius, punctorum qui comprehenduntur in hiis speculis quorumdam ymagines quatuor loca sortiuntur, quorumdam tria, quorumdam dua, quorumdam unum. Punctus cuius ymago in quatuor ceciderit loca a quatuor punctis determinatis refertur, non ab aliis, vel pluribus. Punctus cuius ymago tria sibi usurpat loca a tribus punctis speculi refertur, non a pluribus; cuius duo a duobus punctis; cuius autem ymago in unicum cadit locum poterit esse quod ab uno tantum puncto fit eius reflexio, et poterit esse quod a quolibet circuli determinati puncto, non ab alio.

[2.339] **[PROPOSITIO 36]** Verbi gratia sit E [FIGURE 5.2.36, p. 594] centrum visus; H sit punctus visus in eodem dyametro; D sit centrum circuli. Ducatur dyameter ZEHA. Aut ED est equalis DH, aut non.

148 sumpto: sumpta *R* 149 sperico speculo *transp. FP1*/quilibet punctus: quodlibet punctum *R* 150 producti *corr. ex* produci *L3*; producte *R*/ymaginum: ymaginis *O* 152 hoc: hac *ER*/punctus: punctum *R* 154 verbi gratia *om. R*/ETZ: EZT *P1*; GTZ *E*/ETZ T: et ZT *S*/*post* EZT *add.* per *R*/T *om. P1*; *inter. L3*/punctus: punctum *R*/*post* circuli *add.* et *SOR* 155 ETD *corr. ex* et D *F*/*ante* acutus *scr. et del.* et *L3*/*post* fiat *add.* autem *R*/ei *inter. a. m. C1*/qui *corr. ex* que *F*; que *L3ER*/DTL . . . sit (159) *om. S* 156 refertur: reflectetur *R*/Z *corr. ex* S *L3* 157 sumpto puncto *transp. L3*; *corr. ex* puncto sumpto *O*/puncto L *transp. R* 158 *post* ducatur[1] *add.* enim *R*/EL: EB *L3*; LE *R*/*post* usque *add.* ad *L3*; *scr. et del.* ad *C1*/DB: BD *R* 159 EBD: EDB *E*/DBC: DBP *R*/refertur: reflectetur *R* 160 punctus: punctum *R*/C: P *R*/eius *om. R*/sumpto quocumque *transp. C1* 161 eadem *mg. a. m. C1*/eadem erit *transp. R* 162 qui *corr. ex* quidem *C1*; que *R* 164 quorumdam . . . unum: quorumdam unum quorumdam dua *L3*/dua: duo *R*/punctus: punctum *R*/ymago: hymago *O* 165 *post* a *scr. et del.* G *F*/refertur: reflectitur *R* 166 punctus: punctum *R*/ymago: hymago *O*/*post* sibi *scr. et del.* suscip *F* 167 refertur: reflectitur *R*/punctis: punctus *OL3E*; *om. R*/*ante* cuius[2] *add.* puncti autem *R*/cuius autem (168) *corr. ex* autem cuius *F* 168 autem: enim *L3*/poterit: ponit *FP1*; possit *L3E*; *alter. ex* possit *in* potest *C1* 169 eius *om. R* 172 centrum[1] *rep. L3*/punctus visus: punctum visum *R*/eodem: eadem *R* 173 ZEHA *corr. ex* EHZA *F*; HEZA *P1*; *corr. ex* SEHA *a. m. L3*/ED *mg. a. m. E*/est *corr. ex* quod *E*

[2.340] Sit equalis, et super EH ducatur a puncto D perpendicularis dyameter GDB, et ducantur linee HG, GE, HB, BE. Palam quoniam triangulus HGD equalis triangulo EGD, et equalis triangulo HBD et triangulo EBD. Palam quod, cum angulus HGE divisus sit per equalia, H a puncto G refertur ad E, et locus ymaginis eius E. Similiter, H a puncto B refertur ad E, et locus ymaginis eius E.

[2.341] Si igitur dyametro ZEHA immoto moveatur semicirculus AGZ per speram, aut solus triangulus HGE, describet quidem punctus G motu suo circulum, et a quolibet puncto illius circuli refertur H ad E, et locus ymaginis eius semper erit punctus E, et ita propositum.

[2.342] Quod ab alio puncto quam illius circuli non possit fieri reflexio puncti H ad E palam per hoc. Sumatur punctum C. Erit quidem linea EC maior linea EG, et linea HC minor linea HG, quare non erit proportio EC ad HC sicut ED ad DH. Igitur linea DC non dividet angulum ECH per equalia, quare H a puncto C non potest reflecti ad E. Eadem erit improbatio si sumatur C inter G et Z.

[2.343] Si vero ED fuerit maior DH, mutetur figura, et addatur linee EH linea HQ [FIGURE 5.2.36a, p. 595] ut productum ex EQ in QH sit equale quadrato DQ. Erit igitur proportio EQ ad DQ sicut DQ ad HQ, unde EQ ad DQ sicut ED ad DH, sicut probat Euclides.

[2.344] Fiat circulus ad quantitatem semidyametri QD, cuius Q centrum, G, B loca sectionis duorum circulorum, et ducantur linee EG, EB, QG, QB, DG, DB, HG, HB. Palam ergo quod erit proportio EQ ad QG sicut QG ad QH, et angulus GQH communis utrique triangulo EQG,

174 perpendicularis: perpendicularem *SOL3*; perpendiculariter *ER* 175 dyameter: dyametrum *O*; *alter. in* dyametrum *L3*/quoniam: quod *R* 176 triangulus *corr. ex* angulus *P1*; triangulum *R*/equalis[1,2]: equale *R*/*post* equalis[1] *inter.* est *O*/EGD *corr. ex* GD *L3*; EDG *ER*/et[1] *rep. L3* 178 refertur: reflectetur *R*/E[1] *corr. ex* EE *F*/*post* locus *add.* est *E*/*post* eius *add.* est *R*/similiter . . . E[2] (179) *inter. a. m. L3*; *mg. a. m. C1E* (B: D *C1*); *scr. et del. S* (eius: est; *post* E[2] *rep. et del.* similiter H a puncto B) 179 refertur: reflectetur *R* 180 si *scr. et del. S*/ZEHA *corr. ex* SEHA *L3*/immoto: immota *R*/moveatur: moveantur *P1* 181 AGZ *corr. ex* AGS *L3*/*post* speram *add.* speculi *R*/aut: autem *P1*/solus *corr. ex* solut *L3*; solum *R*/triangulus: triangulum *R*/describet *corr. ex* describit *F*/punctus: punctum *R* 182 G *om. S*; *inter. O*/illius *om. R*/*post* illius *scr. et del.* diametri *P1*/refertur: reflectetur *R* 183 punctus: punctum *R*/et ita: quod est *C1*/*post* ita *add.* patet *R*/*post* propositum *scr. et del.* et est polus illus circuli *O* 184 *post* quod *add.* autem *FP1R*/*post* quam *add.* aliquo *R*/*post* illius *inter.* scilicet *a. m. E*/circuli *inter. a. m. E* 185 *post* reflexio *scr. et del.* et est polus illus circuli *O*/C: O *S*/*post* C *add.* et ducatur EC CH *R* 186 linea[1] *om. R*/linea HC minor *om. S* 187 HC: CH *R*/DH *corr. ex* DHC *L3* 188 non *inter. O*/E *corr. ex* ea *F* 189 improbatio: probatio *FP1R*/sumatur *corr. ex* sumiatur *F* 190 *post* figura *scr. et del.* ada *S* 191 EH: DH *R*/productum *corr. ex* punctum *O* 192 DQ[2]: HQ *O*/sicut . . . HQ *mg. O*/*post* DQ[3] *rep. et del.* sicut DQ *E*/ad[2] . . . Q (194) *om. S* 193 unde . . . DH *om. R*/ad DQ *mg. a. m. F* 195 *post* loca *scr. et del.* ymaginum *P1*/EB: EH *S* 196 DG *rep. P1*/HG HB *transp. C1*/HB *inter. O*

HQG. Igitur illi duo trianguli sunt similes. Erit ergo proportio EQ ad QG sicut EG ad GH. Erit igitur proportio ED ad DH sicut EG ad GH, quare linea DG dividet angulum EGH per equalia.

[2.345] Unde punctus H a puncto G refertur ad E, et locus ymaginis eius punctus E. Similiter, H a puncto B refertur ad E, et locus ymaginis est E.

[2.346] Si ergo moveatur triangulus EGH, punctis E, H immotis, punctus G describet in spera circulum a quolibet puncto cuius refertur H ad E, et semper erit locus ymaginis E.

[2.347] Et quod ab alio puncto quam illius circuli non possit H reflecti ad E palam, ut prius. Si enim sumatur C inter G et A, erit EC maior EG, et HC minor HG, nec erit proportio EC ad HC sicut ED ad DH, et ita DC non dividit angulum ECH per equalia. Similiter, si C sumatur inter G et Z poterit improbari.

[2.348] Et ita propositum, notandum tamen quod E est punctus intellectualis, et circulus ille cuius E est polus est circulus intellectualis, et H punctus intellectualis. Unde quod dictum est secundum geometricam demonstrationem est intelligendum non secundum visus probationem, cum intellectualia visum lateant. Sed quoniam forma H continua videtur formis aliorum punctorum, videbitur quidem a visu forma cuius punctus medius H, et locus puncti medii illius forme erit E, et reflectetur hec forma a loco speculi circularis cuius medium erit circulus predictus, et E polus eius.

[2.349] Cum autem ED sit maior DH, et in tantum poterit esse maior quod non refertur H ad E a puncto G, sciendum quod nisi fuerit

198 HQG *om. P1*/illi: illa *R*/trianguli: triangula *R*/similes: similia *R*/erit ergo *transp. C1*/proportio: portio *S* 199 erit . . . GH[2] *om. S*/proportio: portio *O*; probatio *L3*; *om. R*/DH *corr. ex* AH *F* 200 linea *om. P1*/dividet: dividit *SC1*; dividat *L3*; *alter. ex* dividit *in* dividat *O* 201 unde: unum *S*/punctus: punctum *R*/refertur: reflectetur *R* 202 *post* eius *add.* est *FP1O*/punctus: punctum *R*/refertur: reflectetur *R*/ad *om. FP1* 203 est: cum *P1*; eius *E*; *om. SO*/est E *transp. L3*/*post* est *add.* punctus *E*; *add.* punctum *R*/*post* E *rep. et del.* similiter (202) . . . est E (203) (est E *transp.*) *E* 204 triangulus: triangulum *R*/punctis *corr. ex* punctus *L3*/H: Q *P1*/immotis: immotu *P1*; *om. O* 205 punctus: motus *FP1*; *corr. ex* punctis *L3*; punctum *R*/G *mg. a. m. L3*; E *S*/describet: describit *FP1*/quolibet . . . cuius: cuius quolibet puncto *R*/refertur: reflectetur *R* 206 E[1] *om. P1* 207 quam *inter. a. m. E*/*post* quam *add.* aliquo *R* 208 *post* A *add.* et *L3*/EC *corr. ex* hec *S*; *corr. ex* EA *L3*/EG: ED *E* 209 nec: non *L3ER*/*ante* erit *add.* ergo *R* 210 dividit: dividet *R*/equalia: equa *C1*/si C *corr. ex* sic *P1L3* 211 Z *corr. ex* S *L3* 212 *post* ita *add.* est *P1*; *add.* patet *R*/E *inter.* S/est *om. L3*/punctus intellectualis (213): punctum intellectuale *R* 213 *post* intellectualis[1] *add.* est *L3* 214 punctus intellectualis: punctum intellectuale *R*/*post* secundum *add.* quod *P1* 215 geometricam *corr. ex* geometriam *O*/*post* intelligendum *add.* si *L3E*; *add.* sed *C1* 216 intellectualia: intellectu alia *E* 217 continua videtur *inter. O* 218 forma: formam *S*/punctus medius: punctum medium *R* 219 hec: H *R*/hec forma *transp. L3*/*post* speculi *scr. et del.* s *S*/circularis: circuli *FP1*; circulari *OR* 220 predictus: precedens *FP1* 221 sit: fuerit *R*/et *om. R*/in tantum *corr. ex* iterum *FOL3* (*mg. a. m. F*); item *S* 222 quod[1]: ut *R*/refertur: reflectatur *R*/E a: ea *P1*/quod *inter. O*/nisi *alter. in* si *mg. a. m.F*; si *P1*

proportio EA ad AH maior quam ED ad DH, non poterit H reflecti ad E.

[2.350] Si enim potest reflecti, reflectatur a puncto quod sit G. Erit quidem angulus GDH minor recto, cum respiciat sectionem minorem quarta. Ducatur a puncto G contingens, que necessario concurret cum EA. Sit concursus Q. Erit quidem proportio EQ ad QH sicut ED ad DH (ex [33]), sed maior est proportio EA ad AH quam EQ ad QH. Igitur maior est EA ad AH quam ED ad EH, et ita necessario, si H refertur ad E, erit proportio EA ad AH maior ED ad DH. Palam ergo que dicta sunt cum centrum visus et punctus visus fuerint in eodem dyametro.

[2.351] **[PROPOSITION 37]** Amplius, cum punctum visum et centrum visus non fuerint in eodem dyametro, et fuerint extra speculum, non refertur punctus visus ad centrum visus nisi ab uno tantum speculi puncto.

[2.352] Verbi gratia, sit T [FIGURE 5.2.37, p. 595] punctus visus, H centrum visus, D centrum spere, et ducantur linee HD, TD. Superficies quidem HDT secat speram super circulum EBQG.

[2.353] Palam quoniam T non refertur ad H nisi ab aliquo puncto huius circuli. Palam etiam quod non refertur ab arcu QG vel BA, secundum modum predictum. Refertur ergo aut ab arcu GB aut AQ.

[2.354] Dividatur angulus TDH per equalia per lineam LEDZ, et a puncto E ducatur contingens, que sit KEF. Si puncta T, H fuerint super illam contingentem, non reflectetur T ad H ab aliquo puncto arcus BG. Cum enim a puncto T ducetur linea ad aliquem interiorem punctum huius arcus, linea a puncto H ad idem punctum ducta cadet super ipsum exterius, non interius, et ideo non erit reflexio.

223 ad[1] *inter. L3* 226 quidem: quidam *SL3*/angulus *om. R*/*post* GDH *scr. et del.* non poterit H *S* 228 sit . . . ad[2] *mg. C1*/Q: F *R*/EQ: EF *R*/QH: FH *R* 229 *ante* sed *inter.* figura *a. m. O*/est *om. C1*/EQ . . . quam (230) *inter. a. m. L3*/EQ: EF *R*/QH: FH *R* 230 EH: DH *L3C1ER*/refertur: reflectitur *R* 231 *post* maior *add.* quam *R*/*post* DH *add.* p *F*/palam: patent *R* 232 et *inter. C1*/et . . . visus[2] *inter. a. m. O*/punctus visus: punctum visum *R*/fuerint: fuerit *O*/eodem: eadem *R* 234 eodem: eadem *R*/fuerint: fuerit *OL3* 235 refertur . . . visus[1]: reflectetur punctum visum *R*/*post* visus[1] *add.* H *L3*/visus[2] *om. O* 237 T: Z *FP1*/punctus: punctum *ER*/visus: visum *R* 238 *post* TD *add.* DT HT *R*/superficies (239): ses *P1* 239 *post* circulum *add.* qui sit *R*/EBQG: EB *P1* 240 quoniam: quod *R*/refertur: reflectetur *R*/ad: a *O*/aliquo: alio *SO*; *corr. ex* alio *L3* 241 *post* circuli *add.* producantur ergo HD TD usque ad circumferentiam circuli *R*/etiam *om. ER*/refertur: reflectetur *R*/*post* refertur *scr. et del.* ad H *S*/BA *corr. ex* AB *C1* 242 refertur: reflectetur *R*/aut: autem *P1*/GB: BG *L3* 243 LEDZ: Z *P1*; *corr. ex* Z *mg. F*; LED *O*; LEZD *SC1*; *alter. ex* LESD *in* LEZD *L3* 244 fuerint: fuerit *O* 245 aliquo *corr. ex* alio *L3* 246 aliquem: aliquod *R*/*post* aliquem *add.* minorem *F*/interiorem *mg. a. m. F*; interius *R*/*post* interiorem *add.* minorem *P1* 247 H *corr. ex* D *C1*/*post* ipsum *add.* arcum *C1* 248 non interius *transp. R*

[2.355] Et quod ab uno puncto arcus AQ tantum fiat reflexio palam erit ex hoc. Ducantur linee TZ, HZ. Cum angulus TDH divisus sit per equalia, TDZ equalis angulo HDZ.

]2.356] Linee TD HD aut sunt equales, aut non sunt equales. Si sint equales, et DZ communis, erit triangulus TZD equalis triangulo HZD, et angulus TZH divisus per equalia per lineam DZ, et ita T refertur ad H a puncto Z.

[2.357] Quod ab alio puncto non possit sic constabit. Sumatur punctus O, et ducantur linee TO, HO, et linea ODM dividat angulum illum per equalia. Planum quod TZ minor TO, et HO minor HZ, et proportio TZ ad HZ sicut TL ad LH, et erit proportio TO ad HO sicut TM ad MH. Sed minor est proportio HO ad TO quam HZ ad TZ. Ergo minor HM ad MT quam HL ad LT, quod est impossibile.

[2.358] Palam igitur quod, si T et H equaliter distant a centro et fuerint supra contingentem, non refertur T ad H nisi ab uno speculi puncto tantum, et unicus erit ei ymaginis locus.

[2.359] Amplius, BDQ, ADG [FIGURE 5.2.37b, p. 596] sint duo dyametri spere, et dyameter EDZ dividat angulum BDG per equalia, et a puncto E ducantur due perpendiculares super duos dyametros BD, GD, que sunt ET, EH.

[2.360] Palam quod triangulus ETD equalis est triangulo EHD, cum ED sit communis utrique. T igitur refertur ad H a puncto E. Eodem modo, a puncto Z. Et palam quod non refertur ad E ab aliquo puncto

249 arcus AQ tantum: tantum arcus AQ *R* 250 erit ex hoc *corr. ex* ex hoc erit *P1*/*post* ducantur *add.* enim *R*/TZ HZ *corr. ex* TS HS *L3* 251 *post* equalia *add.* erit *OR* (*inter. O*)/TDZ: TDS *F*; *alter. ex* DS *in* TDS *P1*; *corr. ex* TDS *L3*/*post* TDZ *add.* erit *E*/HDZ *corr. ex* HD *a. m. L3* 252 *post* linee *add.* igitur *R*/sunt[1]: sint *E*/aut . . . equales[2] *om. FP1*/sunt equales[2] *om. O*/sint: sunt *SOC1R* 253 DZ: DS *FP1*; *corr. ex* DS *L3*/communis erit *corr. ex* erit communis *C1*/triangulus: triangulum *R*/TZD: TSD *FP1*; *corr. ex* TSD *L3*/equalis: equale *R* 254 TZH *alter. ex* DIH *in* DZH *F*; DZH *P1*; *corr. ex* TSH *L3*/per[1] *om. F*/DZ: DS *F*; *corr. ex* DS *L3*/refertur: reflectetur *R* 255 Z: S *FP1*; *corr. ex* S *L3* 256 *post* quod *add.* autem *R*/*post* possit *scr. et del.* fieri *P1*/sic *inter. O*/punctus (257): punctum *R* 257 ducantur: reducantur *FP1*/*post* ODM *add.* per centrum D *R*/*post* angulum *scr. et del.* illum *L3* 258 *post* equalia *scr. et del.* per lineam *S*/TZ: TS *FP1*; *corr. ex* TS *L3*/*post* TZ *add.* sit *P1*; *inter.* est *O*/*post* minor[1] *add.* est *R*/HZ: HS *FP1*; *corr. ex* HS *L3* 259 TZ: TH *FP1*; *corr. ex* TS *L3*/HZ: HS *FP1*; *corr. ex* HS *L3*/TL: DL *FP1*; D *L3*/proportio *om. FP1*/TO: DO *P1*/HO: O *S*/MH: H *FP1*; *corr. ex* IMH *C1* 260 HO: HZ *SOC1*; *corr. ex* HZ *L3*/TO: TZ *SOC1*; *alter. in* TZ *L3*/quam: quod *P1*; *inter. a. m. E*/HZ *corr. ex* BZ *S*; *corr. ex* HOS *L3*; HOZ *C1*/TZ: TOZ *SC1*; TO *O*; *alter. ex* TOS *in* TOZ *L3*/*post* minor[2] *add.* est proportio *R* 262 si T *om. S*/distant: distent *R*/*post* et[2] *inter.* si *a. m. O* 263 supra: super *R*/refertur: reflectetur *R*/T: D *FP1*/ 264 ei: eius *P1R*/locus *om. P1*/*post* locus *add.* si vero TD HD sunt inequales secentur ad equalitatem et fiat demonstratio ut antea *R* 265 BDQ: DBQ *C1*/sint: sunt *P1E*/duo: due *R* 266 *ante* spere *scr. et del.* EDZ *P1*/dyameter *corr. ex* dyametri *L3*/angulum *om. L3* 267 duos: duas *R*/BD: HD *O*/GD *alter. in* GED *deinde corr. ex* GED *F*; DG *R* 268 que: qui *FP1*/sunt: sint *SOR*/ET EH *transp. C1* 269 triangulus: triangulum *R*/equalis: equale *R*/*post* EHD *add.* et angulus TED angulo HED latusque TD lateri HD et latus ET lateri EH *R* 270 communis *corr. ex* cos *C1*/T *corr. ex* TT *C1*/refertur: reflectetur *R* 271 et *inter. O*/*post* quod *add.* T *ER*/*post* non *add.* T *C1*/refertur: reflectetur *R*/ad . . . refertur (272) *mg. a. m. C1*/E *alter. in* B *L3*; H *C1ER*

arcus AB vel arcus GQ, nec refertur ab alio puncto arcus AQ quam a puncto Z, secundum supradictam probationem. Verum quod ab alio puncto arcus BG quam a puncto E non possit reflecti patebit sic.

[2.361] Detur O punctum, et ducantur linee TO, HO DO. Fiat circulus ad quantitatem linee DE transiens per tria puncta T, D, H, cuius quidem circuli linea DE erit dyameter, cum angulus ETD quem respicit sit rectus. Igitur circulus ille transibit per punctum E.

[2.362] Cum igitur E sit communis utrique circulo, et sit super eundem dyametrum, continget circulus minor maiorem in puncto E, sicut probat Euclides. Igitur circulus iste secabit lineam DO.

[2.363] Secet in puncto I, et ducantur linee TI, HI. Iam patet quod TD equalis est DH. Igitur angulus TID equalis angulo DIH, quia super equales arcus. Restat angulus TIO equalis angulo HIO; et angulus IOT est equalis angulo IOH, ex ypotesi, et IO commune. Erit triangulus TIO equalis triangulo HIO, et erit TO equalis HO, quod est impossibile, quoniam HO maior HE, TO minor TE, et TE, sicut prius probatum est, equalis est HE. Restat ergo ut T non reflectatur ad H ab alio puncto quam ab E vel a Z.

[2.364] Iterum, a puncto E ducatur linea super dyametrum TD, qui sit EM, et secetur a linea HD pars equalis MD, que sit ND, et ducantur EN, EM. Palam quod angulus EMD est maior recto. Secetur ex eo equalis recto per lineam CM, que concurret cum DE. Sit concursus punctus C. et ducatur NC, et fiat circulus ad quantitatem CD transiens per tria puncta M, D, N. Cum CMD sit rectus, erit CD dyameter, et transibit circulus per C. Palam ergo quod M refertur ad N a puncto E, et similiter a puncto Z, et non ab aliquo puncto arcus AB vel QG; et palam quod non ab alio puncto arcus AQ quam a puncto Z.

272 nec: non *L3*/refertur: reflectetur *R*/alio: aliquo *FP1*; illo *L3* 273 Z: S *L3*/supradictam: predictam *C1* 275 TO: TD *S*; ED *L3*; OT *C1*/TO HO DO: DO HO TO *FP1ER* (DO: OD *ER*)/*post* fiat *add.* que *R* 277 circuli . . . angulus *mg. O*/quem: quoniam *FP1*/respicit *alter. in* recipit *L3* 279 sit[2]: sint *OC1*/eundem (280): eandem *R* 280 continget: contingens *O*/maiorem *inter. O*/ E . . . I (282) *mg. a. m. C1* 281 Euclides: Euclidis *R*/iste: illo *L3*/DO: TO *FP1* 282 I: L *R*/ TI *corr. ex* TH *S*; TL *R*/HI: HL *R*/patet: patebit *FP1*; *corr. ex* patebit *L3* 283 TD: TG *FP1*; *corr. ex* TG *L3*/equalis est *transp. R*/DH: HD *R*/TID: TLD *R*/DIH: DH *FP1*; DLH *R* 284 angulus[1] *om. R*/TIO: TLO *R*/angulo *corr. ex* triangulo *C1*/HIO *corr. ex* HYO *S*; *corr. ex* TLO *L3*; HLO *R*/et . . . HIO (286) *mg. a. m. C1*/IOT: LOT *R* 285 est *om. FP1L3ER*/IOH: LOH *R*/IO: LO *R*/*post* commune *add.* latus *R*/triangulus: triangulo *FP1*; triangulum *R* 286 TIO equalis: TLO equale *R*/*post* equalis *add.* tri *O*/HIO: HLO *R*/*post* et *scr. et del.* I *P1*/TO: DO *FP1*/HO: HA *FP1*; OH *O* 287 *post* HE *add.* et *R*/TE[1]: DE *FP1*/et TE *inter. a. m. C1* 288 T: TE *S*/ad *inter. a. m. C1*/alio: ali *L3*; *scr. et del.* p *C1* 289 ab E: alie *F*; alio *P1*/a Z: A *S* 290 iterum: item *SOC1R*/TD *inter. C1*/qui: que *R* 291 et[1] *om. L3*; *corr. ex* item *E*/a *inter. OL3*/HD: HO *F*/que *om. O*/ducantur: ducatur *SOC1* 292 EN EM *transp. R*/EM *om. SO*; *inter. L3*/angulus *om. R*/est maior *transp. R* 293 *post* equalis *scr. et del.* angulo *S*/CM: PM *R* 294 punctus C: punctum P *R*/NC: NP *R*/CD: PD *R* 295 CMD *corr. ex* CM *L3*; PMD *R*/*post* erit *scr. et del.* rectus *S*/CD: PD *R*/et *inter. O* 296 C: P *R*/*post* ergo *add.* per *S*/refertur: reflectetur *R* 297 et[1] *om. SO*/*post* puncto *scr. et del.* to *C1*/Z: E *FP1*; *corr. ex* S *L3*/aliquo: alio *FP1S*; *corr. ex* alio *OL3*/QG: Q *FP1*; GQ *SOL3ER* 298 arcus *mg. a. m. C1*/AQ . . . arcus (299) *mg. a. m. L3*/AQ: BG *FP1*/Z: S *FP1*

[2.365] Et quod non ab alio puncto arcus BG quam a puncto E palam secundum modum predictum. Sumpto enim puncto, et ductis lineis ad punctum illud a punctis T, D, H, et sumpto puncto in quo circulus ultimus secabit dyametrum, et a puncto sectionis ductis lineis ad puncta T, H, eadem erit improbatio que prius.

[2.366] Palam ergo ex predictis quod si angulum contentum a duobus dyametris per equalia dividat tertius dyameter, et a termino illius dyametri ducantur perpendiculares ad illos dyametros, puncta dyametrorum in que cadunt ad se invicem reflectuntur a duobus punctis speculi tantum. Puncta etiam dyametrorum citra hos terminos perpendicularium sumpta, id est, versus centrum, reflectitur quodlibet a duobus punctis tantum, et unum refertur ad illud quod equaliter distat a centro, et omnium talium duplex est ymaginis locus.

[2.367] Amplius, sumptis duobus dyametris BQ, AG, et EZ [FIGURE 5.2.37c, p. 597] dividente angulum eorum per equalia, et sumatur in BD punctus T supra punctum in quem cadit perpendicularis ducta a puncto E, et in DG sumatur DH equalis DT, et ducantur TE, HE. Refertur quidem T ad H a puncto E, et similiter a puncto Z, non ab alio puncto arcus AQ, nec ab aliquo puncto arcus AB vel GQ.

[2.368] Deinceps, a puncto T ducatur perpendicularis super TD, que quidem concurret cum DE extra circulum spere, cum angulus DTE sit acutus. Concurrat ergo in puncto O, et ducantur linee TO, HO. Et fiat circulus transiens per tria puncta T, D, H, qui necessario transibit per punctum O, et erit DO dyameter eius. Et ducantur linee TO, HO, et ducatur linea contingens circulum BZG in puncto E, que sit KE. Palam quoniam ultimus circulus secabit primum, scilicet BZG, in duobus punctis. Sint puncta illa L, M, et ducantur linee TL, HL, LD, TM, DM, HM.

299 BG *corr. ex* AB *L3*/palam *om. R* 300 *post* enim *scr. et del.* su *F* 1 ad . . . illud *R*/ad . . . lineis (2) *om. S*/illud: illum *FP1*/D: B *FP1* 2 puncto *corr. ex* punctis *L3*; punctis *R* 3 erit *om. P1*/improbatio *corr. ex* probatio *F* 4 si *mg. F*/a *om. R* 5 duobus: duabus *R*/tertius: certius *S*; tertia *R*/termino: tertio *S*; *corr. ex* tertio *L3* 6 illos: illas *R* 7 cadunt: cadent *C1* 8 *post* tantum *add.* et *O*/puncta: punctorum *R*/etiam: autem *FP1R*; *om. O*/terminos *corr. ex* diametros *L3*/perpendicularium (9): perpendicularis *O* 9 sumpta: sumptorum *R*/id est: L *FP1*; I *SOL3C1*/quodlibet: quod si *FP1* 10 unum *corr. ex* unde *L3*/refertur: reflectitur *C1R*/ illud *alter. in* aliud *L3* 12 duobus: duabus *R*/BQ: BC *FP1*; *corr. ex* BG *L3* 13 eorum: earum *R*/et *om. R* 14 punctus: punctis *FP1*; punctum *R*/T *corr. ex* Z *a. m. E*/quem: quod *R*/cadit: cadet *SC1* 15 TE: DE *F*; DC *P1*/refertur: ref *FP1*; reflectetur *R* 16 E: HE *S*/Z *corr. ex* S *L3*/ *post* Z *add.* et *O*/ab . . . nec (17) *om. C1* 17 aliquo *corr. ex* alio *L3*/GQ *corr. ex* GA *L3* 18 deinceps: deinde *R*/perpendicularis *om. S*/TD: TAD *L3* 19 DTE: BDE *R* 20 concurrat: concurret *FP1* 21 qui: que *FP1L3E*/transibit . . . O (22): per punctum O transibit *C1* 22 et[2] . . . HO *om. R* 23 in . . . BZG (24) *mg. a. m.* (scilicet *om.*) *L3*/que . . . BZG (24) *om. FP1*/sit *inter. a. m. E* 24 quoniam: quod *R* 25 puncta illa *transp. ER*/TL: A *S*/HL: YL *E*/LD: DL *SOC1* (*inter. O*)

[2.369] Cum ergo arcus TD sit equalis arcui HD, erit angulus TLD equalis angulo DLH, et ita T refertur ad H a puncto L. Similiter, angulus TMD equalis angulo DMH, et ita T refertur ad H a puncto M. Palam igitur quod T refertur ad H a quatuor punctis, scilicet E, Z, L, M, et quadruplex erit locus ymaginis eius.

[2.370] Et non potest T reflecti ad H ab alio puncto quam aliquo istorum. Detur enim punctus F, et ducantur linee TF, HF, DF, et producatur DF usque concurrat cum contingenti KE, et sit concursus K. Et ducantur linee TK, HK. Igitur angulus TFD equalis angulo DFH, ex ypotesi; restat angulus TFK equalis angulo KFH. Sed angulus TKF equalis est angulo FKH, quia super equales arcus, et FK communis. Erit triangulus equalis triangulo, et ita TK equalis KH, quod est impossibile, quoniam HK maior HO, et TK minor TO, et TO equalis HO.

[2.371] Palam igitur quod non fit reflexio ab aliquo puncto quam a punctis quatuor circuli.

[2.372] Igitur, si in diversis dyametris sumantur duo puncta, scilicet T, H, equaliter a centro distantia, si fuerint super puncta dyametrorum in que cadunt perpendiculares ducte a termino diametri dividentis per equalia angulum duorum dyametrorum, aut fuerint inter centrum et puncta illa, id est citra perpendiculares, dum equaliter distent a centro, reflectetur quidem T ad H a duobus punctis tantum.

[2.373] Si vero T et H fuerint a locis perpendicularium usque in circulum, reflectetur quidem T ad H a quatuor punctis. Si vero fuerint in circulo vel extra, tamen citra contingentem KE, reflectetur quidem T ad H a duobus punctis tantum. Si vero supra contingentem fuerint, reflectetur quidem T ad H ab uno puncto tantum. Et hec quidem accidunt, dum T equaliter distet a centro cum puncto H.

27 TLD: TLA *S* 28 T: D *L3*/refertur: reflectetur *R*/*post* refertur *rep. et del.* ergo (27) . . . equalis (27) *L3*/ad *rep. L3*/angulus . . . M (29) *om. FP1* 29 DMH *corr. ex* TMH *L3*/refertur: reflectetur *R*/a puncto M *om. E*/*post* a *scr. et del.* quatuor punctis scilicet EZLM (scilicet *inter. a. m.*) *C1* 30 refertur: reflectitur *R*/ad . . . punctis: a quatuor punctis ad H *R*/a *inter. E*/punctis: punctus *S*/scilicet *om. L3* 32 *post* quam *add.* ab *C1R* 33 punctus F: F punctum *R*/TF HF *corr. ex* TK HK *S*/producatur (34) *corr. ex* producantur *S* 34 usque: quousque *R*/contingenti: contingente *ER* 37 equalis est *transp. R* 38 triangulus equalis: triangulum equale *R*/*post* triangulus *scr. et del.* eri *F*/equalis triangulo *mg. O*/TK: KT *L3*/*ante* equalis[2] *add.* est *L3* 39 HK: HE *E*/HO: HC *E* 41 fit *inter. a. m. E*; est *inter. L3*; est *R*/reflexio: refertur *SOC1*/aliquo *alter. in* alio *L3C1* 42 circuli *om. SOL3C1ER* 43 igitur si *transp. FP1* (igitur *mg. F*)/sumantur: sumatur *O* 44 fuerint: fiunt *L3*/*ante* super *add.* a puncto *L3*/puncta: punctis *R* 45 perpendiculares . . . termino *om. FP1*/a termino: ? *C1*/diametri: perpendicularis *FP1S*; *corr. ex* perpendicularis *L3*/dividentis: dens *FP1* 46 duorum: duarum *R*/fuerint: fuerit *L3* 47 *post* et *scr. et del.* a *L3*/dum *corr. ex* cum *L3*/distent: distant *L3* 48 reflectetur: reflectitur *C1*/quidem *om. C1* 49 T *om. S*/T . . . fuerint: fuerint T et H *R*/in: ad *R* 50 a . . . H (52) *om. S*/fuerint: fuerit *O* 51 contingentem *corr. ex* contingentie *L3* 52 si . . . tantum (53) *mg. a. m.* (uno puncto *transp.*) *L3*/*post* vero *rep. et del.* et (49) . . . locis (49) *S* 53 accidunt (54) *corr. ex* cadunt *S* 54 distet: distat *R*

[2.374] **[PROPOSITIO 38]** Amplius, si fuerint T, H in diversis dyametris, et longitudo eorum a centro fuerit inequalis, reflectentur quidem ad se ab uno puncto.

[2.375] Verbi gratia, ducantur dyametri ADG, BDQ [FIGURE 5.2.38, p. 598], et EZ dividat angulum eorum per equalia. Et T propinquior sit centro D quam H. Et sumatur linea LQ, et dividatur in puncto M ut sit proportio QM ad ML sicut HD ad DT. Et dividatur LQ per equalia in punctum N, et a puncto N ducatur perpendicularis NK, et super punctum L fiat angulus equalis medietati anguli ADT per lineam FL. Erit quidem angulus FLQ acutus, quare FL concurret cum NK. Concurrat in puncto F, et a puncto M ducatur linea ad latus FL concurrens cum latere NK in puncto quod sit K. Et secet linea illa latus FL in puncto C ut sit proportio KC ad CL sicut HD ad DZ.

[2.376] Deinceps super punctum D fiat angulus equalis angulo LCM, qui sit IDA, et sit I punctus circuli supra Z, aut infra. Et supra I punctum fiat angulus equalis CLM, qui sit OID, et super hanc lineam OI ducatur perpendicularis a puncto H, que sit HC, et producatur linea CF equalis linee CI, et ducantur linee HF, HI.

[2.377] Palam secundum predicta quod a puncto M non potest linea duci ad latus FL dividens ipsum eo modo quo dividit linea MCK preter hanc solam lineam MCK. Si enim possit, sit MPO. Palam quoniam PO minor erit CK, quod quidem patebit, ducta linea PY equidistans CK, que erit minor CK et maior PO. Et PL maior CL. Igitur non erit proportio PO ad PL sicut KC ad CL, quare non erit PO ad PL sicut HD ad DT. Restat ergo ut a puncto M non ducatur alia quam MCK similis ei.

[2.378] Verum cum ODI sit equalis angulo LCM, et angulus OID equalis angulo CLM, erit triangulus CLM similis triangulo IOD. Igitur angulus IOD erit equalis angulo LMC. Restat angulus COH equalis

55 si fuerint *om. FP1L3*/si . . . H: T H si fuerint *ER* 56 *ante* et *add.* si fuerint *FP1L3*/reflectentur: reflectetur *SOL3E*; reflexio fiet *R*/quidem ad se (57) *om. R* 59 propinquior: propinquius *R*/sit *om. FP1* 60 LQ: LY *R*/sit proportio (61) *transp. S* 61 proportio: propinquior *FP1*/QM: YM *R*/DT: AT *S*/LQ: LY *R*/per: in *R* 62 punctum *alter. in* puncto *C1*; puncto *R*/NK: NB *S* 63 anguli *om. R*/FL: FK *L3* 64 FLQ: FLY *R*/NK: HK *O*/concurrat *corr. ex* concurret *a. m. C1*; concurrant *R* 65 *post* concurrens *add.* igitur *E* 66 NK *corr. ex* NM *P1*/linea illa: lineam illam *O*/illa *om. P1*/FL: FB *SL3* 67 CL *corr. ex* KCL *F*/DZ: DT *FP1SL3*; DA *O* 68 deinceps: deinde *R* 69 I: A *SO*/I punctus *alter. ex* punctus A *in* A punctus *C1*/punctus: punctum *R*/circuli: cui *S*/et[2] *om. L3*/supra: super *R* 70 qui *inter. O*/OID: CID *S*/super: supra *FE*/hanc *om. O* 71 que sit HC *inter. O*/HC: HR *R*/linea *om. ER* 72 CF: RX *R*/CI: RI *R*/HF: HX *R* 73 predicta *corr. ex* predictam *L3*; supradictam *C1* 74 linea: lineam *R*/preter . . . MCK (75) *inter. E* 75 quoniam *om. FP1*; quod *R* 76 CK[1]: T *FP1*/ducta *om. P1*/PY: PQ *R*/equidistans: equidistanter *C1*; equidistanti *E* 77 que: cum *C1*/que . . . CK *mg. O*/maior[2] . . . PL[1] (78) *om. S* 78 CL: KL *C1*/*post* erit *add.* proportio *R*/DT: DA *O* 79 ut *om. S*/puncto *corr. ex* punctum *L3* 80 LCM . . . angulo (81) *om. P1* 81 CLM[1]: DM *S*/triangulus: triangulum *R*/similis: simile *R* 82 angulus[1] *corr. ex* triangulus *C1*/*post* IOD *scr. et del.* igitur *F*/equalis angulo *corr. ex* triangulo equalis *C1*/*post* restat *add.* ergo *S*/COH: ROH *R*

angulo KMN, et angulus HCO rectus equalis angulo KNM. Restat angulus NKM equalis angulo CHO.

[2.379] Ducta autem linea DI donec concurrat cum HC in puncto R, erit angulus RDH equalis angulo KCF. Erit triangulus RDH similis triangulo CKF. Igitur proportio RD ad DH sicut FC ad KC. Sed HD ad DI sicut KC ad CL. Igitur RD ad DI sicut FC ad CL. Igitur RI ad DI sicut FL ad CL. Sed DI ad IO sicut CL ad LM, cum triangulus DIO sit similis triangulo CLM. Igitur RI ad IO sicut FL ad LM. Sed proportio RI ad IC sicut FL ad LN, quoniam triangulus RIC similis est triangulo FLN. Igitur, proportio IO ad IC sicut LM ad LN. Igitur proportio QM ad LM sicut FO ad IO.

[2.380] Ducta autem a puncto I linea UI equidistante linee HF, et producta linea DA donec concurrat cum UI in puncto U, erit triangulus OUI triangulo HOF similis. Erit igitur proportio HO ad OU sicut QM ad ML, et ita HO ad OU sicut HD ad DT. Sed quoniam triangulus HCI equalis est triangulo HCF, cum HC sit perpendicularis, igitur angulus HFC equalis est angulo CIH, et ita CIH equalis est angulo UIO, quare proportio HO ad OU sicut HI ad UI, et ita HI ad UI sicut HD ad DT.

[2.381] Verum angulus UID maior est angulo DIH. Secetur ab eo equalis, et sit PID, et ducatur linea TP. P sit punctus dyametri DA.

[2.382] Palam quod proportio HI ad UI constat ex proportione HI ad IP et IP ad UI, et proportio HI ad IP sicut HD ad DP, quoniam DI dividit angulum PIH per equalia. Igitur proportio HI ad UI, que est HD ad DT, constat ex proportione HD ad DP et PI ad UI. Sed proportio HD ad DT constat ex proportione HD ad DP et DP ad DT. Igitur proportio DP ad DT sicut PI ad UI.

83 HCO: HCD *P1*; *corr. ex* H *O*; HRO *R*/*post* equalis *add.* erit *ER* (*mg. a. m. E*) 84 *ante* NKM *add.* ergo *FP1*/CHO: RHO *R* 85 *post* autem *scr. et del.* angulo *F*/HC: HR *R*/R: S *R* 86 *post* erit[1] *add.* equalis *L3*/RDH[1,2]: SDH *R*/KCF: HCF *E*/*post* KCF *add.* et *R*/erit: et *C1*/triangulus: triangulum *R*/RDH: SDH *R*/similis: simile *R* 87 RD: SD *R*/KC: CK *R*/ad[3] *rep. S* 88 DI[1] *corr. ex* TK *L3*/KC: KO *S*/RD: SD *R*/DI[2] *corr. ex* CH *L3*/FC: KC *SC1*; *corr. ex* CK *L3*/CL *corr. ex* D *L3*/RI: SI *R* 89 triangulus: triangulum *R* 90 similis *corr. ex* equalis *L3*; simile *R*/RI: SI *R*/FL: LF *C1* 91 RI: SI *R*/IC: IR *R*/FL . . . IO (92) *om. FP1*/LN: LM *E*/triangulus: triangulum *R*/RIC: SIR *R*/similis: simile *R* 92 IC: IR *R*/QM: YM *R* 93 LM: ML *SOL3*/FO: XO *R* 94 I: U *SC1*/linea UI equidistante: equidistante linea UI *R*/HF: HX *R*/UI . . . linea (95) *mg. a. m. C1* 95 concurrat *inter. O*/*post* UI *add.* concurrat *R*/triangulus: triangulum *R* 96 OUI *om. P1*; OIU *O*/triangulo *rep. S*/triangulo HOF similis: similis triangulo HOF *L3*/HOF *corr. ex* OF *O*; HOX *R*/similis: simile *R*/erit igitur *transp. R*/OU *corr. ex* UO *L3*/QM: YM *R* 97 OU: HOU *S*/triangulus: triangulum *R*/HCI: HOI *S*; HRI *R* 98 equalis: equale *R*/HCF: HEH *F*; HEB *P1*; HRX *R*/HC: HR *R*/ 99 HFC: HXR *R*/equalis est *transp. C1*/CIH[1,2]: RIH *R*/et . . . UIO *mg. a. m. E* 100 UI[1,2]: IN *L3*; IU *R* 102 ducatur linea *transp. O*/TP: DP *E*; PT *R*/*ante* P *add.* et *R*/punctus: punctum *R*/dyametri *om. P1* 104 et[1] . . . IP[2] *mg. O*; *inter. a. m. L3*/IP[2]: PI *R*/sicut *mg. a. m. C1*/HD: DH *R* 105 PIH: PIK *P1*/*post* igitur *rep. et del.* proportio (104) . . . equalia (105) *L3*/*ante* proportio *add.* igitur *L3* 106 constat *om. O*/proportione: probatione *FP1*/et . . . DP[1] (107) *mg. a. m. E*; *om. R* 107 et DP *om. F*; *inter. L3* 108 DP: TP *FP1L3E*; quod *S*

[2.383] Verum angulus OIH est medietas anguli UIH, sed angulus DIH est medietas anguli PIH. Restat angulus DIO medietas anguli PIU, sed angulus DIO est medietas anguli TDP, quia est equalis angulo FLM. Igitur angulus PIU est equalis angulo TDP, et proportio DP ad DT sicut PI ad UI. Igitur triangulus UIP similis est triangulo TPD, et angulus UPI equalis TPD. Erit igitur TPI linea recta, quia angulus DPT cum angulo TPO valet duos rectos, et ita angulus OPI cum angulo OPT valet duos rectos. Et ita T refertur ad H a puncto I, et hec quidem erit probatio sive sit T extra circulum, sive intra, et similiter, sumpto puncto H extra vel intra, dum inequaliter a centro.

[2.384] **[PROPOSITIO 39]** Amplius, ductis dyametris BQ, AG, et dyametro EZ dividente angulum BDG per equalia, dico quoniam, quicumque punctus sumatur in arcu AQ preter punctum Z, ab illo poterunt reflecti infinita paria punctorum inequaliter a centro distantium.

[2.385] Verbi gratia, sumatur punctus H [FIGURE 5.2.39, p. 599], et sumatur in dyametro GD punctus L. Et a dyametro BD secetur MD equalis LD, et ducantur linee LM, LH, MH, DH. Punctus in quo EZ dividit LM sit F; erit LF equalis FM.

[2.386] Et ducatur HD usque cadat super LM in puncto N. Erit igitur LN minor NM. Verum, cum angulus MDF sit equalis FDL et angulo QDZ, et angulus MDA equalis angulo LDQ, et angulus ADH equalis angulo NDL, erit angulus LDH maior angulo MDH. Igitur LH erit maior MH, cum MD, DH equalia sint LD, DH. Erit ergo angulus DHL minor angulo DHM, si enim esset equalis, esset proportio LH ad MH sicut LN ad NM, quod est impossibile. Si autem fuerit maior, secetur ex eo equalis, et improbetur hoc modo. Igitur est minor.

[2.387] Secetur igitur ab angulo MHD equalis illi, qui sit THD. Igitur T refertur ad L a puncto H, et TD minor LD.

109 est . . . DIH (110) *om. S* 110 est medietas *transp. R*/PIH: PLH *F* 111 TDP *corr. ex* TPD *C1*/quia: quod *S*/*post* est[2] *scr. et del.* angulus *P1* 112 PIU *om. FP1*/TDP *corr. ex* TPQ *S*; *inter. O*/ et . . . UI (113) *om. O* 113 triangulus: triangulum *R*/similis: simile *R*/est *om. FP1L3ER*/TPD: DPD *L3* 114 angulus[1] *corr. ex* angel *F*/UPI: UIP *L3*/igitur *om. FP1*/quia: quare *E*/DPT: TPD *FP1* 115 angulo[2]: angulus *FP1* 116 duos rectos *inter. O*/refertur: reflectetur *R*/a . . . H (118) *om. S*/hec quidem: eadem *R* 117 similiter: simpliciter *C1* 118 inequaliter: LN equaliter *FP1*/*post* inequaliter *add.* distent *R* 119 BQ: BG *P1*; LQ *S* 120 EZ *corr. ex* DZ *C1*/quoniam . . . punctus (121): quod quodcumque punctum *R* 122 poterunt: potu *FP1*; potuit *S*; *corr. ex* poterit *E*/reflecti *inter. O* 124 punctus H: H punctum *R* 125 dyametro: semidiametro *R*/ GD: DG *S*/punctus: punctum *R*/ dyametro: semidiametro *R* 126 punctus: punctum *R* 127 LF: FL *R* 128 et *om. O*/usque: quousque *R* 129 cum: est *FP1*/MDF: FDM *C1* 130 QDZ *corr. ex* QDS *L3* 131 angulo[1] . . . maior *om. E* 132 *post* erit[1] *add.* M *S*/LD: ID *SO*/ ergo angulus *transp. E* 133 esset: esse *FP1* 134 MH: HM *R* 135 et: ut *L3*/improbetur: probetur *S*; improbabitur *R*/hoc: eodem *R* 136 MHD *corr. ex* MDH *L3*/illi: ei *C1* 137 refertur: reflectetur *R*/*post* et *add.* linea *R*/*post* TD *add.* est *R*

[2.388] Similiter, si sumantur in dyametris HD, GD alia puncta quam L, M equaliter a puncto D distantia, probatur similiter quod a puncto H fit reflexio punctorum adinvicem et inequaliter distantium a centro. Et ita de infinitis punctis in hiis dyametris sumptis similis erit probatio, et a quocumque puncto arcus AQ sumpto preter quam a puncto Z.

[2.389] **[PROPOSITIO 40]** Amplius, sumptis T, L [FIGURE 5.2.40, p. 599] in dyametris quorum inequalis sit longitudo a centro, et reflectantur adinvicem a puncto H, non erit reflecti T ad L ab alio puncto arcus AQ quam a puncto H.

[2.390] Si enim ab alio, sit illud K, et ducantur TK, LK, DK, LT, TH, LH, NDH. Et producatur DK usque cadat in LT in puncto C. Palam quoniam proportio LH ad TH sicut LN ad NT.

[2.391] Et similiter, cum angulus TKC sit equalis angulo LKC, ex ypotesi, erit proportio LK ad TK sicut LC ad CT. Sed LH maior LK, et TH minor TK. Igitur maior est proportio LH ad TH quam LK ad TK, quare maior erit proportio LN ad NT quam LC ad CT, quod plane impossibile. Restat ut ab alio puncto arcus AQ quam a puncto H non possit T reflecti ad L. Palam ergo que accidunt in arcu AQ.

[2.392] **[PROPOSITIO 41]** Amplius, sit A [FIGURE 5.2.41, p. 600] centrum visus, B centrum speculi, et ducatur dyameter DABG. Et sumatur superficies in qua sit AB quocumque modo que secabit speram super circulum qui sit DLG. Dico quod a quolibet puncto semicirculi DLG reflectuntur puncta ad A inequalis longitudinis a centro cum eo.

[2.393] Verbi gratia, sumatur punctus E, et ducantur linee EA, EB. Palam quoniam angulus AEB erit acutus, quia cadet in minorem arcum semicirculo. Fiat ei equalis, et sit OEB, et ducatur linea OE quantumlibet. Palam quod quodlibet punctum illius linee refertur ad A a puncto E.

[2.394] Ducta autem a puncto B ad lineam OE perpendiculari, aut erit perpendicularis illa equalis BA, aut maior, aut minor. Si fuerit

138 dyametris: semidiametris *R*/HD: BD *FP1OL3C1ER* 139 probatur: probabitur *R* 140 et *om. SOL3C1ER*/inequaliter: equaliter *FP1* 141 probatio *corr. ex* proportio *a. m. L3*; proportio *E* 143 *post* sumptis *add.* punctis *R*/L: B *P1E* 144 et *om. R* 145 *ante* adinvicem *add.* ipsi *R*/erit: poterit *R*/T: D *F*; *om. P1O* 147 illud: illum *F*/LT TH *om. S*/TH *om. F* 148 usque cadat in: quousque concurrat cum *R*/LT *corr. ex* LE *L3*; *corr. ex* LZ *a. m. E*/C: P *R* 149 quoniam: quod *R*/TH: HT *R* 150 TKC: TPK *R*/angulo *om. R*/LKC: LKF *R* 151 ad TK *om. P1*/TK: KT *R*/LC: LP *R*/CT: PT *R* 152 *post* igitur *rep.* maior (151) . . . igitur (152) *S* 153 LC: LP *R*/CT: TC *P1*; OC *L3*; PT *R*/*post* plane *inter.* est *O* 154 ab *om. L3*/puncto[1] . . . quam *om. P1* 155 T *inter. L3*/*post* L *add.* quod est impossibile *O*/que: quod *FP1*/in *inter. C1* 157 dyameter: dyametri *F* 159 qui: que *FP1*/DLG: DG *FP1* 160 longitudinis: lineis *FP1* 161 punctus: punctum *R*/E *inter. O* 162 quoniam: quod *R*/cadet: cadit *P1O*; *corr. ex* caderet *E*/in *inter. C1*/arcum *corr. ex* acutum *P1* 163 ei *inter. L3*/OEB: PEB *R*/ducatur: producatur *R*/OE: BE *R* 164 refertur: reflectetur *R*/A *inter. O* 165 OE: CE *O*; *inter. a. m. C1*; PE *R*/perpendiculari: perpendiculariter *L3* 166 *post* equalis *add.* linee *O*/aut minor *mg. O*

equalis, linee omnes ducte a puncto B ad lineam OE, preter illam perpendicularem, erunt maiores linea BA, et ita quodlibet punctum linee OE, uno excepto, inequaliter distabit a centro puncto A.

[2.395] Si vero perpendicularis fuerit maior, omnia puncta linee illius plus distabunt a centro quam A punctum. Si autem perpendicularis fuerit minor, erit ducere a puncto B duas lineas ex diversis partibus perpendicularis equales linee BA, et omnes alie linee aut minores erunt aut maiores. Palam igitur quoniam a puncto E reflectuntur puncta ad A quorum longitudo a centro inequalis est longitudini A ab eodem, quod est propositum.

[3.396] Constat ex hiis quod, si sumatur A extra circulum—et sit H [FIGURE 5.41a, p. 600]—et ducatur dyameter HDBG et due contingentes HT, HQ, a quolibet puncto arcus TG, preter quam a T vel G, poterit fieri reflexio ad H punctorum inequaliter distantium a centro cum puncto H. Et erit eadem probatio.

[2.397] **[PROPOSITIO 42]** Amplius, ex hiis constabit quod, facta reflexione ad A a puncto E vel alio puncto inequaliter distante a centro puncto A, dyameter in quo fuerit punctus reflexus cum dyametro ABG facit duos angulos, unum respicientem angulum reflexionis, alium ei collateralem, qui quidem collateralis aliquando erit maior angulo reflexionis, aliquando minor.

[2.398] Verbi gratia, ducatur perpendicularis FB [FIGURE 5.2.42, p. 600] super EO. BA aut est perpendicularis super ea aut non.

[2.399] Sit perpendicularis. Erit igitur EA equidistans FB, et erunt duo anguli FBA, FEA equales duobus rectis. Ducta autem linea BO, erunt duo anguli OBA, OEA minores duobus rectis.. Igitur erit angulus OBG maior angulo OEA, qui est angulus reflexionis. Et cum

167 OE: OC *P1*; CE *O*; PE *R*/illam: illum *S*; *corr. ex* illum *C1* 168 *post* linea *scr. et del.* fuerit maior omnia puncta linee illius plus distabunt a centro quam ad punctum si A *S*/ita *mg. F*/OE: OC *P1*; PE *R* 169 *post* centro *add.* cum *R* 171 A punctum *transp. O*/punctum: puncto *FP1* 172 minor *corr. ex* maior *L3*/*post* erit *add.* possibile *R* 173 perpendicularis *corr. ex* perpendiculares *L3*; perpendiculares *E*/et *om. O*/alie *inter. O*/alie linee *transp. C1* 174 quoniam: quod *C1R* 175 A[2] *om. FP1*; *inter. O* 177 A: visus *R* 178 HDBG: HBDG *R* 179 TG: TGQ *R*/*post* a[2] *add.* punctis *R*/vel *om. R*/*post* G *add.* Q *R*/poterit: potest *R* 181 erit eadem *transp. R*/probatio: proportio *FP1* 183 vel *corr. ex* a *S*/puncto[2]: puncti *O*/distante: distanti *SE*; distantis *O*/*post* centro *add.* cum *R* 184 puncto *corr. ex* punctus *S*/dyameter *corr. ex* diametri *C1*/quo: qua *R*/punctus reflexus: punctum reflexum *R* 185 angulos *om. S*/unum: unam U *FP1* 186 angulo: alio *L3*/*post* angulo *add.* constante ex angulo incidentie et *R* 188 perpendicularis: per *F*/perpendicularis FB *transp.* (FB: BF) *SOC1*; pendicularis FB *mg. a. m. F*/FB *om. L3*/*post* FB *add.* quod si EC *mg. a. m. F*; *add.* fiet *P1* 189 super[1] . . . non *om. FP1*/EO: EC *O*/BA *corr. ex* KA *S*/est: erit *L3ER* 190 erit . . . et *om. R*/igitur EA *transp.* (EA *inter. a. m.*) *L3*/equidistans: quidem *FP1*; equidem *SO*/FB: FL *S*/*post* erunt *add.* ergo *R* 191 anguli *inter. O* 192 OBA *corr. ex* FBA *F*/rectis *om. FP1SC1*; *inter. OL3* 193 OEA *om. P1*/*post* angulus *add.* constante ex angulo incidentie et *R*

triangulus EBF sit equalis triangulo EBA, erit BF equalis BA, et ita OB maior BA.

[2.400] Ducta autem linea BN, erunt duo anguli NBA, NEA maiores duobus rectis. Erit ergo angulus NBG minor angulo NEA, et NB maior BA, et ita N et O reflectuntur ad A a puncto E. Et inequaliter distant a centro puncto A, et dyameter OB cum dyametro ABG ex parte G facit angulum maiorem angulo reflexionis, et dyameter NB maiorem, et ita propositum.

[2.401] Si vero BA non fuerit perpendicularis super EA, ducatur perpendicularis, que sit BK, que quidem sive cadat supra AB [FIGURE 5.2.42a, p. 600], aut sub [FIGURE 5.2.42b, p. 600]. Eadem erit probatio.

[2.402] Et BF sit perpendicularis super EO, et ducatur FT equalis AK, et ducatur TB. Palam quoniam in triangulo KEB angulus EKB rectus equalis EFB, et angulus KEB equalis angulo FEB. Restat tertius tertio equalis, et cum EB sit latus commune utrique triangulo, erunt trianguli equales, et erit FB equalis KB. Sed AK est equalis FT. Erit AB equalis BT, et angulus ABK equalis angulo FBT.

[2.403] Addito communi angulo FBA, erit KBF equalis TBA. Sed KBF et FEA valent duos rectos, quare TBA, TEA valent duos rectos, et ita TBG equalis est angulo TEA, qui est angulus reflexionis.

[2.404] Si igitur a puncto B ad lineam ET ducatur linea ultra T, faciet angulum cum BG ex parte G minorem angulo reflexionis. Et erit linea illa maior AB, quoniam TB equalis AB.

[2.405] Et quelibet linea a puncto B ducta ad ET et citra T faciet angulum cum BG ex parte G maiorem angulo reflexionis, et erit inequalis AB, et ita propositum.

194 triangulus: triangulum *R*/EBF: EDF *E*/equalis: equale *R*/*post* EBA *add.* et *R*/BF: BG *P1* 196 BN *inter. C1*/*post* maiores *scr. et del.* d *F* 198 et O *om. FP1S*; *inter. L3*/reflectuntur: refertur *FP1SOC1*/inequaliter: equaliter *FP1* 199 *post* centro *add.* cum *R*/OB: O *FP1SO*; *corr. ex* O *L3*; *corr. ex* OBO *E*/G *alter. in* A *OL3* 200 maiorem[1]: minorem *FP1SOL3E*; *corr. ex* minorem *a. m. C1*/angulo reflexionis *transp. C1*/*post* reflexionis *add.* et incidentie *R*/et *om. S*/NB: N *FP1S*; *alter. ex* N *in* GN *O*; *corr. ex* N *a. m. L3*; *corr. ex* NBN *E*/maiorem[2] *alter. in* minorem *a. m. C1*; minorem *R*/et ita: quod est *C1* 201 *ante* propositum *add.* patet *R* 202 super . . . perpendicularis (203) *om. S*/*post* EA *scr. et del.* ducatur perpendicularis super ea *L3* 204 probatio: pro *P1* 205 *post* perpendicularis *add.* et *O*/EO: EC *O* 206 AK: AB *L3*/TB: BT *R*/palam *om. P1*/quoniam: quod *R*/angulus EKB *inter.* (EKB: KEB) *O*/EKB: EB *FP1* 207 *post* equalis[1] *inter.* est *O; add.* est angulo *R*/EFB: OFB *P1O*/EFB . . . angulo[2] *mg. a. m. C1*/*post* angulo[2] *add.* reflexionis *E* 208 EB sit latus: sit latus EB *R*/triangulo *corr. ex* angulo *a. m. C1*/*post* erunt *scr. et del.* commune *C1* 209 trianguli equales: triangula equalia *R*/equales *rep. S*/FT: T *FP1*/*post* erit[2] *add.* ergo *R* 211 *post* addito *add.* igitur *R* 212 quare . . . rectos[2] *om. S*/TBA: HA *FP1*; THA *C1*/TEA . . . rectos[2] *inter. a. m. L3* 213 equalis est *transp. C1*/*post* angulus *add.* constans ex angulo incidentie et *R* 214 ET *corr. ex* TE *F*/ducatur *corr. ex* ducantur *P1*/faciet: facit *C1*/angulum *om. L3ER* 215 BG: B *FP1*/*ante* ex *add.* et *FP1*/G *inter. a. m. E*/*post* G *add.* angulum *R*/*post* angulo *add.* constante ex angulo incidentie et *R* 216 maior: minor *FP1*/*post* TB *add.* est *O*/*post* equalis *add.* est *R* 217 et[1] *om. O*/ducta ad ET: ad ET ducta *R*/ad ET *om. FP1*/et[2] *om. R* 218 cum *om. R*/BG: TBG *R*/*post* angulo *add.* constante ex angulo incidentie et *R*/inequalis (219): equalis *FP1*; minor *R* 219 *post* ita *add.* est *R*

[2.406] **[PROPOSITIO 43]** Amplius, sit B centrum visus, G centrum spere. Ducatur dyameter ZBGD [FIGURE 5.2.43, p. 601], et sumatur superficies in qua sit dyameter secans speram super circulum ZEH. Dico quoniam, si punctus A refertur ad B ab aliquo puncto circuli, et inequalis est distantia puncti A a centro et puncti B ab eodem, dyameter AG cum dyametro GD ex parte D faciet angulum quem impossibile est esse equalem angulo reflexionis.

[2.407] Sit equalis, et T sit punctus reflexionis, et sit AG inequalis BG. Ducantur linee TA, TG, TB, et fiat circulus transiens per tria puncta A, G, B, qui necessario transibit per punctum T. Si enim extra, ductis lineis a punctis A, B, ad idem punctum illius circuli extra fiet angulus minor angulo ATB. Et probabitur esse equalis.

[2.408] Quoniam, cum angulo AGB valebit duos rectos, et angulus ATB, cum sit equalis angulo AGD, ex ypotesi, cum angulo AGB valet duos rectos, et ita impossibile. Similiter, si circulus citra T ceciderit, eadem erit improbatio.

[2.409] Restat ergo ut transeat per punctum T, et cum angulus ATG sit equalis angulo BTG, erit arcus AG equalis arcui BG, et ita AG erit equalis BG. Et positum est esse inequalem, et ita propositum.

[2.410] **[PROPOSITIO 44]** Amplius, sumptis in duobus dyametris EGH, ZGD [FIGURE 5.2.44, p. 602] duobus punctis A, B ut BG sit maior AG, dico quoniam, si punctus A refertur ad B a duobus punctis arcus EZ, non erit uterque angulus reflexionis minor angulo AGD.

[2.411] Sumantur enim duo puncta T, Q in arcu EZ a quibus A refertur ad B, scilicet T, Q, et ducantur linee BT, GT, AT, BQ, GQ, AQ. Et si angulus ATB minor est angulo AGD, dico quoniam angulus AQB non erit minor angulo AGD.

221 ducatur: ducantur *L3* 222 *post* circulum *add.* qui sit *R*/ZEH: TEH *O*; EZH *R* 223 quoniam: quod *R*/si *inter. a. m. C1*/punctus: punctum *R*/refertur: reflectitur *R*/aliquo: alio *SOE*; *corr. ex* alio *L3* 224 A *om. FP1*; *inter. SL3*/eodem: eadem *FP1OL3*/dyameter *corr. ex* diametro *L3* 225 GD *corr. ex* HG *L3*/quem: quoniam *FP1* 226 *post* angulo *add.* constanti ex angulo incidentie et *R* 227 *post* sit^{1} *add.* enim *R*/*post* equalis *scr. et del.* vel *C1*/punctus: punctum *R*/et^{2} *om. FP1*/inequalis: in EK *S* 228 *post* BG *add.* et *R*/TG TB *transp. FP1R* (TG *corr. ex* DG *F*)/circulus transiens *transp. FP1*/transiens *om. O* 229 enim: equalis *FP1*/*post* enim *add.* cadit *R* 230 fiet: fiat *L3* 231 minor: maior *O*/angulo . . . equalis (233) *om. S*/ATB: ATH *FP1* 232 *post* rectos *add.* et anguli AGB et AGD valent duos rectos *R* 233 cum^{1} *om. R*/sit: est *R*/*post* ypotesi *add.* ergo angulus ATB *R*/angulo^{2}: angulus *L3* 234 *post* si *scr. et del.* L *F* 236 et *om. FP1L3ER*/*post* cum *add.* igitur *R* 237 ita *om. P1* 238 positum: propositum *L3*/inequalem: inequales *R*/*post* ita *add.* est *R* 239 duobus: duabus *R* 240 EGH: EGB *O*/ZGD: ZDG *S*/BG: G *FP1* 241 quoniam: quod *R*/punctus: punctum *R*/refertur: reflectitur *L3*; reflectatur *R*/a *inter. L3*/arcus EZ (242) *om. FP1* 242 angulus: angulo *S*/*post* angulus *add.* constans ex angulo incidentie et *R* 244 refertur: reflectitur *L3*; reflectatur *R*/scilicet T Q *scr. et del. O*; *om. C1ER*/BT: DT *F*/BQ GQ: LQ GA *S* 245 si *om. O*/ATB *corr. ex* ABT *F*/dico *mg. a. m. C1*/quoniam: quod *R*/AQB *corr. ex* ABQ *L3* 246 angulo *om. L3ER*/AGD: AGB *S*

[2.412] Sit enim minor, et ducatur linea GN dividens angulum dyametrorum per equalia, et ducatur linea AB quam dividat GN per punctum F. Palam quod proportio BG ad GA sicut BF ad FA. Sed BG maior GA; erit BF maior FA.

[2.413] Dividatur AB per medium in puncto K, et fiat circulus transiens per tria puncta A, B, T, qui quidem circulus non transibit per G, quoniam essent anguli AGB, BTA equales duobus rectis, et palam quod sunt minores, cum angulus BTA sit minor angulo AGD. Igitur transibit supra G.

[2.414] Similiter, non transibit per Q. Quoniam, sumpto puncto circuli in quo linea GQ secat ipsum, scilicet M, esset arcus AM equalis arcui MB, cum respiciant equales angulos supra Q, quod manet impossibile, quoniam, sumpto O puncto in quo linea GT secat hunc circulum, erit arcus AO equalis arcui OB, quia respiciunt equales angulos supra T. Restat ut hic circulus transeat supra Q, si enim infra, eadem erit improbatio.

[2.415] Ducatur autem linea a puncto O ad punctum K, que quidem, cum dividat cordam AB per equalia, et similiter arcum AB, erit perpendicularis super AB. Verum angulus BAG maior angulo ABG, cum BG maior GA. Et angulus BFG valet duos angulos FAG, FGA, et angulus AFG valet duos angulos FBG, FGB.

[2.416] Sed AGF equalis FGB, et FAG maior FBG. Igitur angulus BFG maior est angulo AFG. Igitur AFG minor est recto, quare NFB minor recto. Sed OK supra FB facit angulum rectum. Igitur producta concurret cum GN supra BF, et inferius numquam.

[2.417] Facto autem circulo transeunte per tria puncta A, Q, B, transibit supra G, et GQ dividet arcum eius AB per equalia. Sed K dividit cordam AB per equalia. Ergo KO concurret cum GN infra BF et supra punctum G. Igitur prius concurret cum GN infra FB, et iam improbatum est.

247 sit *corr. ex* si *L3* 249 proportio *corr. ex* probatio *L3*/GA: GD *FP1*/*post* sed *add.* cum *R* 250 *post* maior[1] *add.* sit *R* 251 puncto: punctum *L3* 252 non *inter. L3*/non transibit *inter. a. m.* (transibit: transit) *E*/per *om. FP1* 253 essent . . . BTA: anguli AGB BTA essent *R*/AGB: HGB *L3*/BTA: HTA *S*/equales: equalis *FP1* 254 cum: quam *S*/sit *inter. O* 258 arcui *corr. ex* arcuu *P1*/MB: BM *R*/supra: super *R*/*post* supra *add.* igitur *C1*/manet: manifeste est (est *inter.*) *O* 259 O puncto *transp. R* 260 *post* AO *rep. et del.* erit arcus AO *E*/equalis: equales *L3*/quia *inter. O* 261 supra[1]: super *R*/ut: in *S*/supra[2]: super *FP1SL3* 262 improbatio *corr. ex* probatio *S* 263 *post* K *scr. et del.* Q *F*/que: qui *E* 264 dividat: dividit *C1*/cordam: coram *FP1* 266 *post* BG *add.* sit *R*/GA *corr. ex* BA *P1*/et angulus BFG *om. E*/*post* BFG *scr. et del.* valet duos angulos FAG et angulus AFG *O*/et[2] . . . FGB (267) *om. O* 267 FGB *om. FP1* 268 *post* AGF *add.* est *O*/*post* equalis *add.* est *R*/FBG: FLG *L3* 269 maior[1]: minor *FP1S*/AFG[2]: FG *FP1*; BFG *C1R*/maior: minor *C1*/NFB: GFB *SL3* 270 minor: maior *L3*/*post* minor *add.* est *OR* (*inter. O*)/supra: super *R* 273 et: quia *E*/sed: set *L3*/K *alter. in* KE *E*; KO *R* 274 dividit: dividet *FP1*/ergo: G *S*; igitur *inter. O*/et: id est *F*; L *P1* 275 *post* igitur *add.* KO concurrens cum BA *R*/FB: BF *R* 276 improbatum: probatum *L3*/*post* est *scr. et del.* GN *E*

[2.418] Restat ergo ut angulus AQB non sit minor angulo AGD, aut A non reflectitur ad B a puncto Q. Similis erit improbatio, sumpto quolibet puncto arcus EN.

[2.419] Sumpto autem puncto in arcu NZ, qui sit C , et fiat reflexio puncti A ad B a puncto C, ut angulus reflexionis supra C sit minor angulo AGD, sicut angulus reflexionis supra T minor eodem, improbatur hoc modo.

[2.420] Ducantur AC, BC, GC. Oportet necessario quod GC dividat KO propter arcum AB, quem dividet ex circulo ABT linea GC per equalia, et similiter linea KO. Sit ergo punctus concursus linee GC cum KO punctus L. Ducta linea TC, cum due linee GC, GT sint equales, erunt duo anguli GCT, GTC equales, et uterque acutus.

[2.421] Ducta igitur perpendiculari super GT a puncto T, erit contingens circulo speculi, et producta cadet super terminum dyametri minoris circuli, cum angulus quem efficit cum TG respiciat semicirculum minoris. Et cum TO cadat super KO, et KO producta transeat per centrum minoris circuli, necessario illa perpendicularis cadet super terminum KO producta, et TC est inferior illa perpendiculari, habito respectu ad N.

[2.422] Igitur quecumque linea ducatur ad lineam TC secans dyametrum illius circuli, qui est OK, cadet in punctum linee TC citra illam perpendicularem. Cum igitur GC cadat in C et secet OK, erit C citra perpendicularem et infra arcum illius perpendicularis.

[2.423] Facto igitur circulo transeunte per tria puncta A, B, C, transibit quidem per C, et secabit circulum ABT in duobus punctis A, B. Et cum exeat a puncto B, iterum redeat in punctum C, et cum sit citra illum circulum, necessario secabit illum in tertio puncto, quod est impossibile.

277 *post* aut *add.* quod *R* 278 reflectitur: refertur *L3*; reflectetur *ER* 279 quolibet: quodlibet *S* 280 *post* sit *scr. et del.* se *S*/C: P *R*/et *om. R* 281 C[1,2]: P *R*/*post* angulus *add.* constans ex angulo incidentie et *R* 282 *post* angulus *add.* constans ex angulo incidentie et *R*/*post* minor *add.* est *R*/eodem: eadem *O*/improbatur (283): improbetur *L3*; improbabitur *ER* 283 *ante* hoc *add.* autem *R* 284 AC BC GC: AP BP GP *R*/*post* oportet *add.* ergo *R*/quod: ut *R*/GC: GO *S*; GP *R* 285 dividet: dividit *R* 286 punctus: punctum *R*/GC: GP *R* 287 punctus: punctum *R*/*post* L *add.* et *R*/ducta: ducatur *L3ER*/TC: TP *R*/*post* cum *add.* igitur *R*/GC: GO *S*; GP *R* 288 GCT GTC: GOT GES *L3*; GPT GTP *R*/acutus: arcus *P1* 289 GT: GP *FP1*/erit contingens (290): continget *R* 290 circulo: circulum *R*/super: in *C1*/terminum: tantum *FP1*; tertium *SO* 291 angulus *om. S*/TG: GT *R*/*post* respiciat *add.* arcum *C1ER*/semicirculum (292): semicirculi *C1ER* 292 *post* minoris *add.* circuli *R*/cum: circuli *L3*/super: supra *R*/KO[1]: KC *FP1*/et KO *om. L3*; *mg. a. m. C1*/KO *corr. ex* EO *a. m. E* 293 circuli: cui *L3*/super (294): supra *C1* 294 terminum: tantum *F*/producta: producti *O*/TC: PT *R*/*post* perpendiculari *scr. et del.* et *C1*/habito: habitu *FP1* 296 *post* ducatur *add.* a puncto G *R*/TC: TP *R* 297 dyametrum: dyametros *S*; *corr. ex* diametro *C1*/qui: que *FP1R*; *inter. O*/*post* in *scr. et del.* p *C1*/*post* punctum *add.* aliquod *R*/TC: TP *R*/TC citra: et intra *FP1* 298 illam: illum *C1*/cum . . . perpendicularem (299) *om. L3*/igitur: G *S*/GC: C *FP1*; *alter. in* GH *C1*; GP *R*/*post* in *scr. et del.* se *P1*/C[1,2]: P *R* 299 citra *inter. a. m. C1*/infra: ita *S* 300 igitur: isto *FP1*/C: P *R* 1 C: se *P1*; L *OR* 2 *post* B[2] *add.* et *FP1*/punctum: puncto *L3E*/C *om. L3*; P *R*/et *om. R*/*ante* cum *add.* inferius puncto T *R*/*post* cum *add.* P *R* 3 *post* illum[1] *add.* silum *L3*

[2.424] Restat igitur ut punctus A non reflectatur ad B a duobus punctis arcus interiacentis eorum dyametros, id est arcus EZ, ut uterque angulus reflexionis sit minor angulo AGD, quod est propositum.

[2.425] **[PROPOSITIO 45]** Amplius, dico quoniam est reflecti duo puncta a se inequalis longitudinis a centro a duobus punctis arcus ipsa respicientis, id est dyametros in quibus sunt puncta illa interiacentis.

[2.426] Verbi gratia, sumptis duobus dyametris in circulo spere, scilicet BD, GD [FIGURE 5.2.45, p. 603], dividatur angulus eorum per equalia per dyametrum ED, et in BD sumatur punctus M supra punctum in quem cadet perpendicularis ducta a puncto E super BD. Et sumatur ND equalis MD, et fiat circulus transiens per tria puncta D, N, M. Necessario circulus ille transibit extra E, si enim per E, fieret quadrangulum a quatuor punctis D, N, E, M, et duo anguli illius quadranguli sibi oppositi sunt equales duobus rectis, quod quidem non esset, cum linea EM sit supra perpendicularem, et angulus EMD acutus.

[2.427] Ei similiter oppositus supra N acutus, quia EN supra perpendicularem. Similis erit improbatio si transeat circulus citra E. Transibit igitur extra, et secabit circulum spere in duobus punctis, sicut T, L.

[2.428] Et ducantur linee MT, DT, NT, ML, DL, NL, et ducatur linea MN secans TD in puncto F, lineam ED in puncto P. Palam cum MD sit equalis ND, et PD commune, et angulus equalis angulo, erit triangulus equalis triangulo, et erit angulus FPD rectus. Igitur angulus PFD acutus.

[2.429] Ducatur a puncto F perpendicularis super TD, que sit KF. Palam quoniam aliquis punctus linee NL erit inferior puncto K. Sumpta inferioritate respectu T, sit ille punctus Z, et ducatur linea TZ usque ad circulum cadens in punctum circuli, qui sit C. Arcus NC aut est minor arcu TL, aut non.

[2.430] Si non fuerit minor, sumatur ex eo arcus minor, et ad terminum illius arcus ducatur linea a puncto T, et erit idem.

5 punctus: punctum *R* 6 EZ: ES *L3*/ut: et *FP1* 7 *post* angulus *add.* constans ex angulo incidentie et *R*/AGD: GED *L3*/quod est propositum *om. L3ER*/est *inter. P1* 8 quoniam est: quod possunt *R* 9 a[1] *corr. ex* ad *L3*; ad *C1ER*/a centro *om. P1* 10 id est: et *L3*/puncta illa *transp. L3* 11 duobus dyametris: duabus semidiametris *R* 12 GD *om. FP1* 13 per *om. E*/ dyametrum: semidiametrum *R*/BD: BG *FP1S*/punctus: punctum *R* 14 quem: quod *R*/BD: BG *FP1S* 15 et *om. L3*/D N M *mg. a. m. C1*/N M *transp. OER* 16 E[1] *inter. L3* 18 oppositi: opposita *FP1* 19 *post* et *add.* id eo *R*/*post* acutus *add.* et *L3R* (*inter. L3*) 20 ei similiter *transp. R*/*post* ei *inter.* et *O*/supra[1]: super *R*/acutus quia EN: quia EN acutus *FP1* 21 *ante* similis *add.* et *FP1*; *add.* est *R*/improbatio *corr. ex* improbatione *F*; *corr. ex* probatio *a. m. C1* 23 NT: ut *F* 24 TD *inter. O*/in[1] . . . ED *om. FP1*; *mg. a. m. C1* 25 commune: communis *R*/*post* angulus *add.* NDP *R*/*post* angulo *add.* MDP *R*/triangulus: triangulum *R* 26 equalis: equale *R*/FPD: SPD *FP1SOL3*/*post* acutus *scr. et del.* quidem *L3* 28 quoniam . . . punctus: quod aliquod punctum *R*/ NL *inter. a. m. E*/inferior: inferius *R*/*post* inferior *add.* in *FP1* 29 T: N *R*/ille punctus: illud punctum *R*/linea TZ *transp. R* 30 qui: quod *R*/C: O *R*/NC: N *FP1*; NO *R*/est minor *transp. R* 31 TL *corr. ex* D *L3* 32 ad *mg. a. m. L3*/terminum (33): certum *FP1*; *corr. ex* terminus *L3* 33 *post* idem *add.* ac si arcus NO esset minor arcu TL *R*

[2.431] Sit igitur NC minor TL. Palam quoniam angulus TNL erit maior angulo CTN, quia respicit maiorem arcum. Secetur ex eo equalis, et sit INZ, et super punctum T fiat angulus equalis angulo CTN, qui sit OTM. Cum angulus TML sit maior angulo MTO, concurret linea TO cum linea LM. Concurrat in puncto O.

[2.432] Cum igitur angulus LMT sit equalis duobus angulis MOT, MTO, et angulus LNT sit equalis angulo LMT, quia super eundem arcum, et angulus INZ sit equalis angulo MTO, erit angulus INT equalis angulo MOT, et ita triangulus MOT similis triangulo INT, et similiter triangulus INZ est similis triangulo TNZ. Et ita proportio NT ad TO sicut NI ad MO, et similiter proportio TN ad TZ sicut IN ad NZ.

[2.433] Sed TZ maior TO, quod sic patet. Sit R punctus in quo TZ secat KF. Angulus TFR est rectus, quare angulus FTR acutus. Igitur angulus OTF ei equalis est acutus. Et KF perpendicularis super TD, quare producta concurret cum TO, et linea ducta a puncto T ad punctum concursus, cuius linee pars est TO, erit equalis linee TR. Et ita TO minor TZ, quare maior est proportio NT ad TO quam NT ad TZ.

[2.434] Igitur maior est proportio IN ad MO quam IN ad NZ, quare MO minor NZ. Secetur ergo ex NZ equalis ei que sit NS.

[2.435] Quoniam angulus LND cum angulo LMD valet duos rectos, erit angulus LND equalis angulo OMD, et SN, ND equalia OM, MD. Igitur OD equalis est SD.

[2.436] Sed ZD maior SD, quoniam angulus LND cum angulo LMD valet duos rectos. Sed angulus LMD acutus, cum angulus EMD sit acutus. Igitur angulus LND maior est recto. Igitur ZD maior SD, quare ZD maior OD.

34 sit *om. FP1S*; *inter. O*; sic *inter. L3*/NC: NO *R*/TL: LC *S*; *corr. ex* D *L3*/quoniam *om. R*/angulus *corr. ex* arcus *L3*/TNL: TOL *O* 35 CTN: OTN *R*/arcum *om. P1*/*post* equalis *scr. et del.* linea *C1* 36 *post* T *add.* linee TM *R*/CTN: et N *FP1*; OTN *R*/sit[2]: si *FP1* 37 OTM: ? *O*; OTN *E*; QTM *R*/*post* cum *add.* igitur *R*/angulus *om. S*/TML: CLM *FP1*; LTM *S*/*ante* sit *scr. et del.* sit maior angulo CLM *P1*/MTO: MTQ *R*/TO: TQ *R* 38 O: Q *R* 39 cum . . . et (40) *om. FP1O*/MOT MTO (40): MQT MTQ *R* 40 LNT: LMT *FP1*/angulo *om. L3ER*/LMT: LM *P1*/*post* quia *add.* sunt *R*/*post* super *rep. et del.* quia super *L3* 41 angulo *om. R*/MTO: MTQ *R*/erit . . . MOT[1] (42) *mg. a. m. E*/INT *corr. ex* LMT *L3* 42 MOT[1,2]: MQT *R*/triangulus: triangulum *R*/similis: simile *R* 43 triangulus: triangulum *R*/similis *om. FP1*; simile *R*/TNZ *corr. ex* TNR *P1*/TO: TQ *R* 44 MO: MQ *R*/TN: DN *FP1* 45 TZ[1] *om. P1*/TO: TQ *R*/sic: si *P1*; sit *S*; *corr. ex* sit *F*/punctus: punctum *R* 46 secat *corr. ex* secatur *L3*/TFR: TF et *FP1*; *corr. ex* TF *O* 47 OTF: OIF *FP1*; QTF *R*/*post* est *scr. et del.* et *P1*/perpendicularis: perpendiculares *S* 48 *post* quare *add.* KF *R*/TO: DO *FP1*; TD *E*; TQ *R*/*ante* et *add.* sit concursus S *R*/*post* linea *add.* TS *R*/ducta . . . TO[1] (49) *mg. a. m.* (T *corr. ex* D) *E* 49 TO[1] *rep. P1*/TO[1,2]: TQ *R* 50 TZ: TC *FP1SO*; *corr. ex* TC *L3*/maior *corr. ex* minor *O*/proportio . . . est (51) *om. S*/TO: TQ *R* 51 est *om. P1*/IN *corr. ex* I *O*/MO: MQ *R*/IN: in *P1*/NZ: OZ *FP1* 52 MO: MQ *R*/*post* minor *add.* est *OR* (*inter. O*); *scr. et del.* a *L3*/NZ: MZ *O*/*post* equalis *scr. et del.* angulo *S*/NS: NX *R* 53 LND *corr. ex* LMD *L3* 54 erit: EF *L3*/erit . . . rectos (57) *mg. O*/angulo *om. R*/OMD: QMD *R*/SN: XN *R*/OM: QM *R*/MD *corr. ex* OMD *P1* 55 OD: QD *R*/est *om. R*/SD: XD *R* 56 SD: XD *R*/LND *om. P1* 57 *post* LMD *inter.* est *O*/EMD: EMT *C1*/EMD . . . igitur[1] (58) *mg. a. m. L3* 58 angulus *om. L3*/ZD *corr. ex* SD *L3*/*post* maior *scr. et del.* est proportio *S*/SD: XD *R* 59 OD: CD *FP1*; QD *R*

[2.437] Igitur O refertur ad Z a duobus punctis T, L, et O et Z sunt inequalis longitudinis a centro, et in diversis dyametris.

[2.438] Et quod non sunt in eodem dyametro palam ex hoc quoniam angulus SDN equalis est angulo ODM. Addito ergo communi angulo SDM, erit angulus NDM equalis angulo SDO. Sed angulus NDM minor duobus rectis, quare angulus ZDO minor duobus rectis. Quare O, Z non sunt in eodem dyametro, sed in diversis.

[2.439] **[PROPOSITIO 46]** Amplius, sumptis duobus punctis, que sint O, K [FIGURE 5.2.46, p. 604], et inequaliter distantibus a centro, reflectetur quidem unum ad aliud a duobus punctis arcus respicientis semidyametros in quibus sunt, sed non ab alio puncto illius arcus quam ab illis duobus.

[2.440] Verbi gratia, D sit centrum; K remotior a D quam O a D; GD, OD dyametri; T punctus unus reflexionis. Palam ex superioribus quod duo anguli reflexionis non erunt minores angulo ODA nec equales. Alter ergo erit maior. Sit angulus reflexionis qui est supra T maior, et ducantur linee OT, DT, KT.

[2.441] Et ex angulo illo reflexionis secetur equalis angulo ODA, qui sit OTF, et dividatur angulus FTK per equalia per lineam TE. Et a puncto K ducatur equidistans TF, que quidem concurret cum TE. Concurrat in puncto Z, et ducatur linea OK, et dividatur angulus ODK per equalia per lineam DU secantem lineam OK in puncto C, et sit KD maior OD. Cum igitur sit proportio KD ad DO sicut KC ad CO, erit KC maior CO. Item, linea DT secet lineam OK in puncto N. Dico quoniam C cadit inter N et K non inter N et O, quod sic patebit.

[2.442] Angulus KCD valet duos angulos CDO, COD, et angulus OCD valet duos angulos CKD, CDK. Sed angulus CDO equalis angulo

60 O[1,2]: Q *R*/refertur: reflectitur *R*/Z: AN *FP1*; N *S*; *corr. ex* N *L3*/T L: C Z *FP1S* 61 inequalis *corr. ex* equalis *E* 62 sunt: sint *SR*/eodem: eadem *R*/ex: E *S*/ex hoc *om. ER* 63 SDN: XDN *R*/ODM *corr. ex* OND *F*; ODN *P1*; QDM *R* 64 SDM: secundum *S*; XDM *R*/equalis . . . NDM[2] *om. E*/SDO: XDQ *R*/sed: et *R*/angulus NDM[2] *om. R* 65 *post* quare *scr. et del.* OZ non sunt in eodem diametro *C1*; *add.* magis *R*/ZDO: ZDQ *R*/O: Q *R* 66 *ante* Z *add.* et *R*/eodem: eadem *R* 68 sint: sunt *FP1L3E* 69 reflectetur: reflectitur *C1*/aliud: alium *FP1* 70 sunt: fuerint *C1* 72 remotior: remotius *R*/a D: ADG *FP1SL3C1*; ATG *O*/*ante* GD *add.* circulus *O*/GD *om. C1* 73 OD dyametri: BD semidiametri *R*/T *inter. L3*/punctus unus: punctum unum *R* 74 duo anguli *transp. C1*; uterque angulus *R*/*ante* reflexionis *add.* constans ex angulo incidentie et *R*/erunt minores: erit minor *R*/equales: equalis *R* 75 sit *corr. ex* si *O*/*post* angulus *add.* constante ex angulo incidentie et *R*/supra: super *R*/*post* maior *add.* angulo ODA *ER*/et . . . ODA (77) *mg. a. m. E* 77 reflexionis *om. R*/*post* secetur *add.* angulus *ER* 79 equidistans: equidem *FP1* 80 angulus *om. P1* 81 per equalia *inter. a. m. L3*/C: P *R*/sit: est *R* 82 KC[1,2]: KP *R*/CO: PO *R*/*post* CO *add.* erit KC ad CO *P1* 83 CO: PO *R*/quoniam: quod *R* 84 C: cum *S*; P *R*/cadit: cadat *L3*/inter . . . non *om. FP1*/non: N *S* 85 KCD: KPD *R*/CDO COD: PDO POD *R*/et . . . CDK (86) *mg. O* 86 OCD: OPD *R*/CKD: OKD *FP1SL3*/CKD CDK: PKD PDK *R*/CDK: CDH *S*/CDO: PDO *R*/*post* equalis *add.* est *OR* (*inter. O*)

CDK, et angulus KOD maior angulo OKD. Igitur angulus KCD maior angulo OCD, quare angulus KCD maior recto. Et angulus KND acutus, quod sic constabit.

[2.443] Si fiat circulus per tria puncta O, T, K, transibit infra D, quoniam, cum angulus OTK sit maior angulo ODA, erunt duo anguli OTK, ODK maiores duobus rectis, et linea ND dividet arcum illius circuli, qui est OK, per equalia infra D.

[2.444] Si a puncto divisionis ducatur linea ad medium punctum linee OK, que est corda illius arcus, erit linea illa perpendicularis super OK, et cadet inter C et K, cum CK sit maior CO. Et angulus supra N a parte illius perpendicularis et ex parte C erit acutus, et angulus supra C ex parte O est acutus. Si ergo C cadat inter N et O, impossibile erit perpendicularem illam cadere inter N et C, quia secaret DC, et fieret triangulus cuius unus angulus rectus, alius obtusus.

[2.445] Cadet ergo inter N et K, et erit angulus N ex parte perpendicularis acutus; igitur ex parte C obtusus, et ita erit triangulus cuius duo anguli obtusi.

[2.446] Palam quoniam angulus KTD est medietas anguli KTO, sed KTE medietas anguli KTF. Restat ETD medietas anguli FTO, sed FTO equalis est angulo ODA. Igitur ETD medietas anguli ODA.

[2.447] Sed angulus ODA cum angulo ODF valet duos rectos, et tres anguli trianguli ETD duos rectos. Ablato EDT communi, restat angulus TED equalis medietati anguli ODA et angulo ODN. Sed angulus ODC cum medietate anguli ODA est rectus. Igitur angulus TED est acutus, quare ei contrapositus est acutus.

[2.448] Igitur, si a puncto K ducatur perpendicularis ad TZ, cadet inter E et Z. Si enim supra E ceciderit, cum angulus TEK sit obtusus, accidet triangulum habere duos angulos rectum et obtusum. Sit ergo perpendicularis KQ. Dico quoniam KT se habet ad TF sicut KD ad DO.

87 CDK: PDK *R*/et . . . KOD *inter.* (angulus *om.*) *O*/KOD: KCD *FS*; KED *P1*/OKD: CKD *O*/KCD: KPD *R* 88 OCD: OPD *R*/KCD: KPD *R* 90 tria puncta *transp. FP1*/K: H *S* 91 *ante* cum *add.* si transeat per D *R*/OTK: OTH *FP1* 92 ODK: CDK *S*; *inter. L3*/*post* rectis *add.* si transeat supra D eadem est demonstratio *R*/ND: non *S*/dividet: dividit *C1* 94 *post* si *add.* autem *R* 96 C: P *R*/CK: C et K *S*; *corr. ex* C et K *L3*; PK *R*/CO: PO *R*/supra: super *R*/a: ex *O* 97 *post* perpendicularis *add.* erit acutus *FP1SOL3*/C: P *R*/*post* erit *scr. et del.* perpendicularis *C1*/supra: super *R* 98 C^{1,2}: P *R*/cadat: cadet *FP1C1*; cadit *R* 99 C: P *R*/secaret: secari *P1*/DC: DP *R* 100 triangulus: triangulum *R* 102 C: O *R*/*post* obtusus *add.* ergo P non cadit inter N et O quia *R*/et *om. R*/erit triangulus *transp. OC1*/triangulus: triangulum *R* 104 quoniam: quod *R*/est *om. FP1*/anguli: an *FP1S*/*ante* KTO *add.* sed *P1*/*post* KTO *add.* sed KTO *F*/sed KTE (105) *om. P1* 105 *post* KTE *add.* est *OR*/ETD *corr. ex* ED *L3*/FTO^{1} . . .anguli (106) *om. FP1* 106 *post* ETD *add.* est *S*/*post* anguli *rep.* KTF (105) . . . anguli (106) *S*; *rep. et del.* FTO^{1} (105) . . . anguli (106) *L3* 108 trianguli: tri *P1*; *corr. ex* tri *O* 109 TED equalis: DED equales *FP1* 110 ODC: ODP *R* 111 contrapositus: circapositus *S* 112 puncto *corr. ex* p *L3*/TZ *corr. ex* TE *L3* 113 et *inter. a. m. C1* 114 accidet: accidit *C1*/sit *corr. ex* si *a. m. L3* 115 quoniam: quod *R*/DO: TO *P1*

[2.449] Probatio: TO aut est equidistans KD, aut concurrit cum ea. Sit equidistans [FIGURES 5.2.46a, 5.2.46b, p. 605]. Erit angulus ODA equalis angulo TOD, et ita TOD equalis angulo OTF. OD, TF aut sunt equidistantes, aut concurrent.

[2.450] Si equidistantes [FIGURE 5.2.46a, p. 605], cum cadant inter equidistantes, erunt equales. Si vero concurrunt [FIGURE 5.2.46b, p. 605], faciunt triangulum cuius latera equalia, quia respiciunt equales angulos, et FD secat illa latera equidistans basi. Erit ergo proportio unius laterum ad DO sicut alterius ad FT, et ita TF equalis DO.

[2.451] Et hoc dico si concurrant sub KD. Et si concurrant sub TO eadem erit probatio, quia fiet triangulus cuius unum latus TO et alia duo latera equalia, et erit proportio unius ad DO sicut alterius ad TF. Item, angulus TDK equalis angulo DTO, quia DT inter equidistantes. Igitur est equalis angulo DTK, quare DK, TK sunt equalia. Igitur proportio TK ad TF sicut KD ad DO.

[2.452] Si vero TO concurrit cum KD, concurrat ex parte A in puncto P [FIGURE 5.2.46c, p. 605]. Scimus quoniam proportio KT ad TF compacta est ex proportione KT ad TP et TP ad TF. Sed proportio KT ad TP sicut KD ad DP, quoniam DT dividit angulum KTO per equalia. Et proportio TP ad TF sicut DP ad DO, quoniam angulus ODP equalis angulo PTF, et angulus supra P communis. Erit partialis triangulus similis totali. Igitur proportio KT ad TF constat ex proportione KD ad DP et proportione DP ad DO. Sed proportio KD ad DO constat ex eisdem, quare proportio KT ad TF sicut KD ad DO.

[2.453] Si vero TO concurrat cum KD ex parte G [FIGURE 5.2.46d, p. 605], sit concursus L, et a puncto D ducatur equidistans linee KT, que sit DR, concurrens cum TO in puncto R. Igitur angulus KTD equalis

116 probatio *om. R*/*post* TO *add.* enim *R*/equidistans: equidem *F*/*post* KD *scr. et del.* et con *P1*/concurrit: concurrat *FP1SL3C1*; *alter. ex* concurrat *in* concurret *O*/ea: eo *FP1* 117 equidistans: equidem *F*/*post* erit *add.* ergo *R* 118 *ante* ODA *scr. et del.* cum ea/TOD^1 *corr. ex* DTOD *F*/et ita TOD *om. FP1*; *inter. L3*/OTF: OZF *FP1*; *alter. in* ODF *O*/*post* OTF *add.* et *R*/OD TF *inter. L3* 119 concurret: concurrunt *R* 121 equidistantes: equidem *F* 122 faciunt: facient *R* 123 FD *corr. ex* secundus *L3*/equidistans: equidistanter *C1ER*/ergo *inter. a. m. S* 124 *post* unius *scr. et del.* ad *E*/ad^2 *inter. C1*/TF: F *FP1*/DO^2: TO *E* 125 et hoc (125) . . . ad TF (127) *transposui a fine* 2.445 *ad initium* 2.451/si^1 *inter. L3*/*post* si^1 *add.* linee ille *R*/sub^1: cum *FP1*/sub^1 . . . concurrant *om. L3*/sub^2 *corr. ex* supra *O*/TO *corr. ex* DO *F* 126 probatio: improbatio *C1*/triangulus: triangulum *R*/*post* latus *add.* est *R*/TO *corr. ex* DO *F* 127 unius: unus *S*/*post* unius *add.* laterum *R*/ad . . . alterius *mg. O*/DO: TO *P1E*/*post* TF *add.* et ita TF equalis DO *R* 128 *post* equalis *add.* est *OR* (*inter. O*)/quia: quare *FP1*/*post* DT *inter.* est *O* 129 DK: DF *E*/*post* DK *add.* equalis est *R*/sunt equalia *om. R* 131 TO *om. P1* 132 P: L *R*/quoniam: quod *R*/KT: BT *O* 133 TP^{1,2,3}: TL *R*/proportio *om. R* 134 *ante* sicut *add.* est *R*/DP: DL *R*/quoniam DT: quare DO *FP1*; *mg. L3*/dividit *rep. P1*/et *corr. ex* ET *O* 135 TP: TL *R*/*post* TP *scr. et del.* similis (137). . . quare (139) (*ante* similis *add.* ad) *S*/DP: DL *R*/*post* angulus *inter.* est *a. m. E*/ODP: ODL *R*/*ante* equalis *add.* est *R*/*post* equalis *inter.* est *O* 136 PTF: per TH *F*; PTH *P1*; PTK *S*; LTF *R*/supra P: super L *R*/partialis . . . similis (137): partiale triangulum simile *R* 138 DP^{1,2}: DL *R*/sed . . . DO^2 *mg. a. m. L3* 139 KD: BD *P1* 140 TO: DO *FP1*/KD *corr. ex* AD *L3* 141 L: A *P1*; S *R*/D *om. FP1*/KT *inter. a. m. C1* 142 DR: DZ *FP1*/TO: DO *FP1L3*/angulus *corr. ex* triangulus *L3*/equalis est (143) *transp. R*

est angulo TDR, sed idem est equalis angulo DTO, quare DR est equalis TR. Sed quoniam triangulus LTK similis triangulo LRD, erit proportio DR ad RL sicut KT ad TL, et ita RT ad RL sicut KT ad TL. Sed RT ad RL sicut DK ad DL. Igitur KT ad TL sicut KD ad DL.

[2.454] Sed quoniam angulus FTO equalis est angulo ODA, erit angulus ODL equalis angulo FTL, et angulus supra L communis. Erit triangulus ODL similis triangulo FTL. Igitur TL ad TF sicut DL ad DO, et ita KT ad TL sicut KD ad DL. Et TL ad TF sicut DL ad DO, quare KT ad TF sicut KD ad DO, quod est propositum.

[2.455] Sed quoniam KZ [FIGURE 5.2.46, p. 604] equidistans TF, erit angulus KZE equalis angulo ETF, et ita triangulus KZE similis triangulo ETF, quare proportio KE ad EF sicut KZ ad TF. Sed KE ad EF sicut KT ad TF, propter angulum supra T divisum per equalia. Igitur KZ equalis KT.

[2.456] Verum quoniam KQ est perpendicularis super EZ, erunt omnes eius anguli recti. Sed angulus ETD est acutus, quoniam est medietas anguli. Igitur KQ concurret cum TD. Sit concursus H**,** et ducatur linea EH, et a puncto E ducatur equidistans KH producta usque ad DH, que sit EC.

[2.457] Et mutetur figura propter intricationem linearum [FIGURE 5.2.46e, p. 606], et fiat circulus transiens per tria puncta C, T, E. Et producatur KD usque in circulum cadens in punctum M, et ducatur MT. Erit angulus TME equalis angulo TCE, quia cadunt in eundem arcum, et angulus TCE equalis angulo CHK. Erit TME equalis angulo CHK.

[2.458] Secetur ab angulo TME angulus equalis angulo DHE, qui sit FMD, et punctus in quo FM secat TC sit I. Palam quoniam triangulus IMD similis est triangulo EDH, quare proportio HD ad DM sicut EH ad IM.

[2.459] Et similiter, triangulus TMD similis triangulo KHD, et proportio KD ad DT sicut HD ad DM, et ita KD ad DT sicut EH ad IM.

143 TDR: DDN *FP1* 144 TR: ZR *E*/quoniam . . . similis: quia triangulum STK simile est *R*/LRD: SRD *R* 145 RL^1: SR *R*/sicut^1 *mg. a. m. C1*/TL^{1,2}: RL *F*; TS *R*/RL^{2,3}: RS *R*/sed . . . RL^3 *mg. a. m. C1*/RL^3 *corr. ex* TL *L3* 146 DL^1 *corr. ex* LDL *P1*/DL^{1,2}: DS *R*/KT: DT *E*/TL: TS *R*/KD: DK *R* 148 ODL: ODS *R*/FTL: FTS *R*/et . . . FTL (149) *om. ER*/angulus *corr. ex* angulo *S* 149 TL: ST *R*/DL: DS *R*/DO: TO *P1S* 150 ita: est *R*/TL^1 *corr. ex* KL *L3*/TL^{1,2}: TS *R*/DL^{1,2}: DS *R*/DO: TO *E*/quare . . . DO (151) *om. FP1* 151 TF: TC *S*; *corr. ex* TC *O* 152 KZ: HZ *FP1*/equidistans: equidistat *C1E* 153 *post* equalis *scr. et del.* e *S*/triangulus: triangulum *R*/similis: simile *R* 154 TF: EZ *P1*/sed . . . TF (155) *mg. a. m.* (KE: FE) *L3* 155 supra: super *R* 156 equalis: EK *S*/*post* equalis *add.* est *R* 157 quoniam: quod *FP1*/super: supra *L3E* 158 eius anguli *transp. L3*/recti *mg. O*/ETD: EDT *S* 159 KC: CQ *E* 160 KH: HK *R* 161 EC: EX *R* 162 figura: figuram *C1* 163 C: X *R*/T *corr. ex* B *O*; Z *E* 164 circulum: triangulum *C1*/cadens *mg. L3*/ducatur: educatur *ER* 165 TME . . . angulo *om. FP1*/TCE: RCE *S*; TXE *R* 166 TCE: RCE *S*; TXE *R* 167 angulus *om. L3ER* 168 punctus: punctum *R*/TC: TX *R*/quoniam triangulus: quod triangulum *R* 169 IMD: MID *O*/similis: simile *R*/*post* DM *scr. et del.* similis *P1* 171 triangulus: triangulum *R*/similis: simile *R*

[2.460] Sed proportio KD ad DT nota, quoniam semper una et eadem permanet, cuicumque punctus reflexionis sit T in arcu EG, quia semper linea TD una, et KD similiter. Linea etiam EH una in quacumque reflexione permanet, et non mutatur eius quantitas, quare linea IM semper erit una, quare punctus F notus et determinatus.

[2.461] Si ergo a tribus punctis arcus BG fieri posset reflexio, esset ducere a puncto F ad circulum TCE tres lineas equales, quia esset proportio KD ad DT sicut EH ad quamlibet illarum. Et patet ex superioribus quod non nisi due equales duci possunt, quare a duobus tantum punctis fiet reflexio, quod est propositum.

[2.462] **[PROPOSITIO 47]** Amplius, datis duobus punctis K, O [FIGURE 5.2.47, p. 607] in diversis dyametris inequaliter distantibus a centro, est invenire punctum reflexionis.

[2.463] Verbi gratia, sumatur linea ZT, et dividatur in puncto E ut sit proportio ZE ad ET sicut KD ad DO. Quoniam KD maior DO, erit ZE maior ET. Dividatur ZT per equalia in puncto Q, et a puncto Q ducatur perpendicularis super ZT, et fiat angulus ETD equalis medietati anguli ODA. Erit quidem acutus. Igitur TD concurret cum perpendiculari.

[2.464] Sit concursus in puncto H, et ducatur linea DEK ut sit proportio KD ad DT sicut KD ad semidyametrum spere. Et angulo quem habemus KDT fiat in speculo angulus equalis KDT. Dico quoniam T est punctus reflexionis, et si predictam probationem replicaveris, manifeste videbis.

[2.465] **[PROPOSITIO 48]** Amplius, sumptis duobus punctis in diversis dyametris, que puncta inequalis sint longitudinis a centro, si fuerint extra circulum, et reflectantur ab aliquo puncto arcus oppositi dyametris, non reflectentur ab alio eiusdem arcus.

173 sed: si *L3*/KD: KC *FP1*; KT *SOC1*/semper: super *FP1E*/semper una *transp.* (una *mg.*) *C1*/*post* semper *add.* et *L3*/et *om. O* 174 cuicumque punctus: quodcumque punctum *R*/EG: OG *S*; *alter. in* BG *L3*; BG *R* 175 *post* TD *add.* est *R* 176 eius quantitas *transp. C1*/IM *inter. O*/semper erit (177) *om. O* 177 punctus: punctum *R*/notus: notum *R*/determinatus: determinatum *R* 178 si ergo: sibi igitur *FP1*/EG: OG *FP1OC1*; *corr. ex* AG *S*; *alter. in* HG *L3*; *corr. ex* KG *a. m. E*; BG *R*/posset: possit *C1* 179 TCE: XTE *R*/*post* equales *add.* quarum cuiuslibet pars interiacens diametrum TX et circumferentiam circuli esset equalis linee IM *R*/quia: quare *FP1*/esset: semper erit *R* 180 DT: TD *FP1* 181 duci *om. ER* 183 *post* punctis *inter.* sit *L3* 186 ZT: et T *S*/et *corr. ex* aut *P1*/ut: et *S* 187 sit *om. FP1*/KD^{2} *corr. ex* QD *C1* 188 Q et *mg. a. m. C1*/Q^{2} *corr. ex* quod *S* 189 ETD *inter. L3* 190 ODA: EDA *FP1S* 193 angulo: alio *FP1SO*; *corr. ex* angulus *L3* 194 quem: quoniam *FP1SOL3*/equalis *om. L3*/*post* equalis *add.* scilicet *OC1ER* (*inter. E*)/KDT: TBDC *FP1*; CKDT *S*/quoniam: quod *R* 195 punctus: punctum *R* 198 dyametris *mg. F*/a *inter. O*/si *om. S* 199 extra: ex *O*/et *inter. a. m. E*/aliquo: alio *O*; *corr. ex* alio *L3* 200 reflectentur: reflectantur *SL3C1*; *corr. ex* reflectatur *O*/alio *corr. ex* aliquo *P1*/*post* alio *inter.* puncto *L3*

[2.466] Verbi gratia, sint A, B [FIGURE 5.2.48, p. 607] puncta in diversis dyametris extra circulum, G centrum, T punctus reflexionis, et ducantur BT, AT, GT. BT secabit arcum circuli. Sit punctus sectionis Q. AT secabit similiter arcum circuli. Sit punctus sectionis M.

[2.467] Quoniam angulus BTG equalis est angulo ATG, cadunt in arcus circuli equales, quod patebit, producto dyametro TG. Erit ergo arcus QT equalis arcui MT. Si igitur B refertur ad A ab alio puncto, sit illud H, et ducantur linee BH, AH, GH. Secet BH circulum in puncto L, AH in puncto N.

[2.468] Secundum supradictam rationem, erit HL equalis NH. Sed iam habemus quod QT equalis TM, quod est impossibile. Restat ut B non reflectatur ad A a puncto H vel ab alio puncto arcus oppositi dyametri preter quam T.

[2.469] Similiter, si fuerit alterum punctorum in circulo, alterum extra, ab uno tantum puncto arcus poterit reflecti ad aliud.

[2.470] Amplius, si linea ducta ab uno duorum punctorum ad aliud contingat circulum, aut tota sit extra, sumpto quocumque puncto in arcu opposito dyametris, altera linearum a punctis duobus ad illud punctum ductarum tota erit extra circulum. Et sic neuter punctorum ad alium reflectetur ab aliquo puncto illius arcus, et ab uno solo puncto speculi arcus oppositi reflectetur, et ita ab uno solo puncto speculi.

[2.471] **[PROPOSITIO 49]** Si vero linea ducta ab uno puncto ad alium secet circulum, fiat circulus super centrum speculi et illa duo puncta. Circulus ille aut totus erit intra circulum, aut continget ipsum, aut secabit.

[2.472] Sit totus intra, et ducantur due linee a duobus punctis ad aliquod punctum arcus oppositi. Angulus quem facient erit minor

201 puncta: puncto *FP1* 202 *post* G *scr. et del.* est *C1*/punctus: punctum *R* 203 GT: TG *L3ER*/ *ante* BT *scr. et del.* T *L3*/secabit: secat *C1*/punctus: punctum *R*/AT *corr. ex* DT *O* 204 *post* secabit *add.* arcum secabit *P1*; *scr. et del.* circulum *C1*/similiter arcum *transp. FP1*/punctus: punctum *R* 205 cadunt: cadent *R* 206 producto dyametro: producta semidiametro *R*/*post* TG *add.* in P *R* 207 QT: QP *R*/MT: MP *R*/*post* B *scr. et del.* ad *E*/refertur: reflectitur *R*/ad A *om. R*/alio *om. P1* 208 illud: illum *FP1*/BH[1]: KH *F*/BH[2] *corr. ex* LH *L3*; LH *E* 209 *post* N *add.* et producatur BG in K *R* 210 *post* secundum *add.* igitur *R*/supradictam: predictam *L3ER*/rationem: probationem *R*/ HL: LH *SOC1L3E*; LK *R*/NH: NK *R* 211 QT: QP *R*/TM: PM *R*/est *om. FP1*/*post* impossibile *add.* ut *F*; *scr. et del.* ut *P1*/B: L *FP1SOL3E*; LI *C1* 212 ad *om. S* 213 dyametri: diametris *R*/*post* quam *add.* a *L3C1ER* 214 fuerit: fuerint *FP1L3* 215 tantum puncto *transp. FP1*/poterit: potest *FP1*/aliud: alium *FP1* 216 ducta *corr. ex* data *L3*/duorum punctorum *transp. C1*/ad aliud *om. R*/aliud: alium *FP1* 217 aut *corr. ex* at *F* 218 *post* duobus *add.* punctorum duorum altero *R*/ illud: illum *FP1* 219 neuter: neutrum *OR* 220 alium: aliud *C1R*/reflectetur *corr. ex* reflectatur *E*/aliquo: alio *FP1SOE*; *corr. ex* alio *L3* 221 arcus . . . speculi[2] *om. FP1R*/reflectetur: reflectentur *O* 223 alium: aliud *R*/*post* circulum *add.* et *FP1SOL3C1*/super: supra *FP1*; per *R* 224 circulum *alter. in* speculum *O*/continget *corr. ex* contingit *O*/*post* ipsum *add.* intrinsecus *R* 226 et ducantur *corr. ex* educantur *O* 227 aliquod *om. O*

angulo quem unus dyameter facit cum alio ex alia parte centri, et quilibet angulus sic factus super arcum oppositum minor erit illo angulo.

[2.473] Quoniam angulus factus in interiori circulo per lineas a punctis ad arcum eius interiacentem ductas erit equalis illi angulo, quoniam cum angulo dyametrorum supra centrum valet duos rectos. Sed angulus arcus minoris circuli maior angulo arcus speculi.

[2.474] Igitur in arcu circuli non fiet reflexio nisi ab uno puncto, cum iam dictum sit quod non est possibile reflexionem duobus punctis fieri ut sit uterque angulus minor angulo dyametrorum ex alia parte centri.

[2.475] Si vero circulus ille contingat circulum speculi, angulus factus a lineis ab illis punctis ad punctum contactus ductis erit equalis angulo dyametrorum ex alia parte centri, quare ab illo punto contactus non fiet reflexio. Et angulus factus super quodcumque punctum aliud arcus maioris circuli erit minor illo, quare a duobus punctis arcus non fiet reflexio, secundum predicta.

[2.476] Si vero circulus interior secet circulum speculi, duo puncta aut erunt extra circulum; aut intra; aut unus intra, alius extra; aut unus in circulo, alius extra vel intra.

[2.477] Si fuerint extra, vel unus in circulo alius extra, circulus secans non secabit arcum circuli speculi interiacentem dyametris, et ita quilibet angulus factus super arcum illum erit maior angulo dyametrorum ex alia parte centri. Et iam probatum est in precedenti figura quod hec puncta ab uno solo puncto arcus interiacentis poterunt reflecti.

[2.478] Si vero duo puncta fuerint intra, secabit circulus interior arcum interiacentem in duobus punctis, et restabunt ex eo duo arcus ex diversis partibus.

228 unus: una *R*/alio *corr. ex* alia *L3*; alia *R*/alia *om. R* 229 sic: sit *FSOL3C1* 231 interiori: interiore *R* 232 ductas *corr. ex* duas *L3* 233 supra: super *R*/*post* duos *add.* angulos *L3ER*/rectos *om. FP1S*; *inter. L3*; *inter.* r *O* 234 arcus[1] . . . circuli *corr. ex* minoris circuli arcus *C1*/*post* maior *add.* est *R* 235 *post* arcu *add.* speculi vel *E*/circuli: speculi *R*/*post* circuli *add.* speculi *FSO*; *add.* vel speculi *L3C1* (*deinde del.* vel *C1*)/non: ? *inter. P1*/puncto *corr. ex* punctis *P1* 236 *post* reflexionem *add.* a *OL3R* (*inter. OL3*) 237 *post* angulus *add.* constans ex angulo incidentie et reflexionis *R*/minor *corr. ex* minori *L3* 239 circulus: arcus *E*/*post* contingat *add.* intrinsecus *R*/*post* circulum *add.* illum *C1*/speculi: spere *FP1* 240 contactus: contactis *S*; *corr. ex* contactis *O* 242 quodcumque: quocumque *S*/aliud: alium *FP1* 243 arcus[1] *om. R* 246 unus[1,2]: unum *R*/alius: aliud *SR* 247 circulo: circumferentia *R*/alius: aliud *R* 248 fuerint: fuerit *FP1SO*/vel . . . extra[2] *om. R* 249 dyametris: diametros *R*/et . . . centri (251) *om. R* 251 est *om. L3*/precedenti: precedente *R* 252 *post* interiacentis *add.* diametros *R*/poterunt: potuerit *FP1*; poterint *C1*/*post* reflecti *add.* si vero unum fuerit in circumferentia aliud extra circulus secans secabit arcum circuli speculi diametros interiacentem in unico puncto et quilibet angulus factus super arcum illum erit maior angulo diametrorum ex alia parte centri et sic ab uno puncto vel a duobus potest fieri reflexio *R*

[2.479] Si unus punctorum fuerit intra circulum, alius in circulo vel extra, secabit circulus arcum interiacentem in unico puncto, et restabit unus arcus tantum.

[2.480] Si secet in duobus punctis, omnes anguli facti super arcum interiacentem duo puncta sectionis erit maior angulo dyametrorum ex alia parte centri, et ab hoc arcu poterit fieri reflexio forsitan ab uno puncto tantum, forsitan a duobus.

[2.481] Et a duobus arcubus qui restant ex arcu totali, et ex diversis partibus, omnes anguli erunt minores angulo dyametrorum, et tantum ab uno eorum puncto fiet reflexio.

[2.482] Et in hoc situ poterit fieri reflexio a duobus punctis arcus interiacentis dyametros, aut a tribus.

[2.483] Et palam quod ab uno tantum puncto arcus oppositi fiet reflexio, et ita in hoc situ aliquando a tribus, aliquando a quatuor.

[2.484] Si vero secetur arcus interiacens dyametros in uno tantum puncto a maiori circulo, omnes anguli facti in parte illius arcus inclusa minori circulo erunt maiores angulo dyametrorum, et poterit fieri reflexio a duobus punctis illius partis, vel ab uno.

[2.485] Omnes anguli alterius partis arcus interiacentis erunt minores angulo dyametrorum, et ab uno puncto tantum illius partis fiet reflexio, et ita, cum ab uno puncto arcus oppositi semper fiat reflexio in hoc situ, aliquando a tribus, aliquando a quatuor, non a pluribus poterit esse reflexio.

[2.486] Palam ergo quod puncta inequalis longitudinis a centro, aliquando ab uno puncto tantum, aliquando a duobus, aliquando a tribus, aliquando a quatuor, numquam a pluribus, reflectuntur. Cum autem puncta eiusdem longitudinis fuerint, poterit fieri reflexio aut ab uno tantum puncto, aut a duobus, aut a quatuor, numquam a tribus.

[2.487] Ubi ab uno puncto fit reflexio, una apparet ymago; ubi due, due; ubi tres, tres; ubi quatuor, quatuor. Si vero punctus visus et cen-

256 si . . . punctis (259) *om. R* 257 *post* interiacentem *add.* et *FP1* 259 duobus punctis *transp. P1*/*ante* omnes *add.* et *R* 260 erit maior: erunt maiores *R* 261 alia: aliqua *S*/alia parte *transp.* (alia *inter.*) *L3*/poterit: possit *L3E*; posset *R* 262 puncto *om. FP1*/*post* tantum *rep.* forsitan (261) . . . tantum (262) *S*/forsitan: forsan *L3* 263 *post* et[1] *add.* si *R*/*post* arcubus *add.* fiat reflexio *R*/qui restant *transp. C1*/*post* ex[1] *scr. et del.* cu *F*/et[2] *om. O* 264 *post* partibus *add.* et *FP1*/minores *inter. L3* 268 et *om. ER*/*post* palam *add.* etiam *R* 269 tribus *rep. F*/*post* quatuor *add.* punctis fiet reflexio *R* 270 secetur . . . circulo (271): unum punctorum fuerit intra circulum aliud in circumferentia vel extra secabit circulus arcum interiacentem in unico puncto et restabit unus arcus tantum et *R*/interiacens *corr. ex* interiacentis *P1L3*/in: ab *O* 271 a *om. FP1* 272 minori: minore *C1*; a secante *R*/erunt *corr. ex* essent *P1* 273 reflexio . . . punctis: a duobus punctis reflexio *FP1* 274 *post* omnes *add.* vero *R*/arcus *om. L3ER* 275 puncto tantum *transp. R*/partis *om. S* 276 et *om. FP1*/fiat: fiet *L3* 277 situ *om. FP1*/*post* quatuor *scr. et del.* a *P1*; *add.* et *R*/a[3] *om. S* 281 pluribus: qualibet *FP1*/*post* autem *scr. et del.* a *F* 282 *post* reflexio *scr. et del.* ab *F* 283 a[2] *om. O*/*post* numquam *add.* vero *R* 284 ubi[1]: nisi *FP1S*/reflexio *inter. O*/ubi[2]: ibi *R*/due: a duobus *R* 285 ubi[1]: nisi *S*/tres[1]: a tribus *R*/ubi[2]: ibi a *R*/quatuor[2] *om. FP1*/punctus visus: punctum visum *R*

trum visus fuerint in eodem dyametro, fiet reflexio a circulo toto, et locus ymaginis erit centrum visus. Verum, si centrum visus fuerit in centro speculi, nichil videt. Si vero punctus visus fuerit in centro speculi, non videbitur, quoniam forma eius accedet ad speculum super perpendicularem, nec reflecti poterit nisi super perpendicularem.

[2.488] Cum autem centrum visus et punctus visus fuerint in diversis lineis extra centrum, linee ille ad centrum producte secabunt in diversis partibus ex circulo spere duos arcus. Ab uno puncto unius tantum fiet reflexio, ab alio forsitan a tribus. Quod si centrum spere fuerit ex una parte, centrum visus et punctus visus ex una, arcus quem secant dyametri propter oppositionem capitis abscondetur, unde tunc a tribus tantum punctis fiet reflexio. Et si dirigatur in hoc situ visus ad arcum unius reflexionis tantum, abscondetur alius trium, et unica apparebit ymago.

[2.489] Item, si integrum fuerit speculum, non erit ibi perceptio. Oportet igitur ut in eo sit abscisio, et accidet non numquam arcum interiacentem dyametros abscisum esse, et tunc nichil in eo videri, quare raro eveniet quatuor ymagines in hoc speculo comprehendi. Unde si quis hanc pluralitatem ymaginum voluerit videre, disponat visum intra speculum circa ipsum ut modicam partem eius abscondat mole capitis, et totam speculi superficiem visu discurrat.

[2.490] Cum autem aliquid in hoc speculo percipietur duplici visu, si linea reflexionis fuerit equidistans perpendiculari, erit locus ymaginis punctus reflexionis, et cum distent a se puncta reflexionis respectu duorum visuum, apparebunt duobus visibus due ymagines eiusdem puncti. Si vero linea reflexionis non sit equidistans perpendiculari, et punctus visus tantum distet ab uno visu quantum ab alio, vel modica sit differentia, erit locus ymaginis respectu utriusque visus idem, aut diversus, sed modicum distans. Unde aut una apparebit ymago, aut fere una, sicut probatum est in speculis spericis exterioribus.

286 eodem: eadem *R*/toto: tota *FP1* 287 visus^2 *om. FP1SC1E*; *mg. O*; *inter. L3* 288 centro^1: centrum *P1*/punctus visus: punctum visum *R*/*post* punctus *scr. et del.* fuerit *P1*/in: a *O*/*post* centro^2 *scr. et del.* N in eodem diametro fiat reflexio *C1* 289 *post* non *scr. et del.* fi *P1*/videbitur *alter. in* videbit *a. m. E*/accedet: accidet *L3* 290 nec: non *S*/nisi super *om. S* 291 et . . . fuerint *corr. ex* visus fuerint et punctus *S*/punctus visus: punctum visum *R*/visus^2 *om. E*/fuerint: fuerit *O* 292 lineis: locis *E*; *corr. ex* locis *L3* 294 quatuor: tribus *C1* 295 punctus: punctum *L3ER*/visus: visum *R*/*post* una *add.* parte *C1*/quem: que *FP1* 296 oppositionem: oppositum *P1*/unde: unum *S* 297 tantum punctis *transp. L3* 298 unius *inter. O*/reflexionis *corr. ex* rationis *L3*; *om. E*/*post* trium *add.* reflexionum *R* 300 item *om. R*/fuerit *om. S*/ibi *om. C1* 1 ut: quod *P1* 2 dyametros: dyametro *FP1S* 5 speculum *corr. ex* circulum *C1*/ipsum *inter. a. m. E* 7 aliquid *om. O*/speculo *om. E*/*post* speculo *inter.* aliquid *O* 8 *post* perpendiculari *scr. et del.* fuerit *F* 9 punctus: punctum *R*/distent: distant *R* 10 duorum: diversorum *C1*/due *inter. L3* 11 *post* puncti *add.* et locus cuiusque imaginis est in puncto sue reflexionis *R*/equidistans: equidem *S*/et *om. FP1* 12 punctus visus: punctum visum *R*/*post* uno *scr. et del.* p *F*/visu: viso *FP1* 14 unde: unum *S*/apparebit: apparet *C1*/aut *corr. ex* ut *F* 15 speculis . . . exterioribus *mg. O*/*post* speculis *scr. et del.* singulis *E*/spericis exterioribus *transp. SO*

[2.491] In speculis columpnaribus concavis, aliquando linea communis est linea recta. Cum superficies reflexionis transit per axem, aliquando linea communis est circulus—cum superficies illa est equidistans basibus—aliquando linea communis est sectio columpnaris. Quando fuerit linea recta, erit locus ymaginis et modus reflexionis sicut in speculis planis. Quando fuerit circulus, erit idem modus qui in concavis spericis. Cum vero fuerit columpnaris sectio, aut erit locus ymaginis ultra speculum, aut citra visum, aut in centro visus, aut inter speculum et visum, aut in ipso speculo, quod sic patebit.

[2.492] **[PROPOSITIO 50]** Sit ABG [FIGURE 5.2.50, p. 608] sectio. Ducatur perpendicularis in hac sectione, que sit DG, quam secundum predicta patet esse dyametrum circuli, et unicam posse esse, cum ab alio puncto sectionis non possit duci perpendicularis super superficiem contingentem. Sumatur aliud punctum, et sit B, et ducatur ab eo in sectione linea perpendicularis super lineam contingentem sectionem in puncto B, que quidem linea, secundum predicta, necessario concurret cum perpendiculari. Concurrat in puncto D, et sumatur B circa punctum G ut angulus BDG sit acutus.

[2.493] Deinde a puncto G ducatur in sectione linea equidistans BD, que sit GH, que quidem cadat intra columpnarem sectionem, quia erit angulus HGD acutus, cum sit equalis GDB. Et a puncto G inter D et H ducatur linea, que necessario concurret cum BD. Concurrat in puncto N, et inter N et G sumatur punctus quicumque, qui sit O. Ultra punctum N sumatur punctum T. Item, a puncto G ducatur supra GH alia linea GZ tamen intra sectionem, que necessario concurret cum BD ex alia parte. Sit concursus E. Ducatur GQ linea ut angulus QGD sit equalis angulo ZGD, et fiat angulus LGD equalis angulo HGD, et angulus MGD equalis angulo NGD.

17 reflexionis: remotionis *E* 18 est[1]: erit *R* 19 equidistans: equidem *FS*/columpnaris: piramidalis *FP1SO* 20 linea recta *transp. FP1*/*post* ymaginis *add.* ultra speculum *C1* 21 circulus: circularis *mg. a. m. L3* 22 concavis spericis *transp. R*/vero: autem *FP1*; ergo *C1*/*post* vero *add.* linea communis *R*/columpnaris: piramidalis *FP1SOE*/erit . . . ymaginis (23): locus ymaginis erit *S* 24 quod: quo *S* 27 esse[1]: etiam *FP1*/unicam: unica *FP1OL3E* 30 lineam: superficiem *FP1* 31 secundum *inter. L3* 32 cum: in *S*/*post* perpendiculari *add.* GD *R*/et *om. L3*/sumatur: sumptum sit *R* 33 ut: et *S*/BDG *corr. ex* BGD *S* 34 in *om. FP1*/equidistans: equidem *S* 35 GH: DH *P1*/cadat: cadit *C1*; cadet *R*/intra: inter *FP1S*/columpnarem: piramidalem *FP1SO*; *corr. ex* piramidem *E*/erit . . . HGD (36): angulus HGD erit *R* 36 *post* angulus *scr. et del.* GG *F*/sit *om. FP1*/GDB: GDH *SE*/*post* H *add.* et *L3* 38 et inter N *om. E*/punctus . . . qui: punctum quodcumque quod *R*/punctum (39) *corr. ex* puncta *S* 39 N . . . T *mg.* (punctum *om.*) *O*/sumatur: supponatur *FP1*/punctum: punctus *FP1*/supra: super *L3*/alia *inter. L3* 41 GQ linea *transp. C1*/equalis *om. L3* 42 angulo[1]: circulus *L3*; *om. ER*/LGD: BGD *S*/et[2]: sed *S*; *om. O*/angulus[2] *corr. ex* angulo *F*

[2.494] Palam quod, si fuerit visus in puncto Z, reflectetur punctus Q ad ipsum a puncto G, et punctus ymaginis E. Et si fuerit visus in puncto H, reflectetur ad ipsum punctus L a puncto G, et erit locus ymaginis G. Si vero fuerit visus in puncto O, reflectetur ad ipsum punctus M, et locus ymaginis N. Si autem fuerit in N, erit locus ymaginis puncti M in centro visus, id est in N. Si autem fuerit in T, erit locus ymaginis inter visum et speculum, quia in N, et ita propositum.

[2.495] Hec quidem intelligenda sunt cum punctus visus non fuerit super perpendicularem cum ipso visu, tunc enim, cum infinite superficies possunt intelligi quarum quelibet ortogonalis super superficiem contingentem speculum, et omnes sint super illam perpendicularem, quedam illarum superficierum efficiet lineam communem lineam rectam, et non fiet reflexio nisi super eandem perpendicularem, et locus ymaginis centrum visus, et non videbitur punctus nisi qui fuerit in superficie visus.

[2.496] Quedam autem illarum superficierum efficit lineam communem circulum, et tunc puncta inter que et visum fuerit centrum circuli poterunt reflecti ad visum singula a duobus punctis circuli, cum a singulis ducantur linee facientes angulum cum superficie contingente quem per equalia dividat perpendicularis ducta ad centrum. Et hoc quidem dico de punctis que sunt in illa perpendiculari, et loca ymaginum erunt in centro circuli. Alia puncta illius perpendicularis non reflectentur ad visum preter punctum quod est in superficie visus, et illud per illam perpendicularem.

[2.497] Cum autem fuerit linea communis sectio columpnaris, non poterunt puncta perpendicularis reflecti ab aliquibus punctis sectionis, cum forma accedens super perpendicularem reflectatur super perpendicularem, et in sectione unica sit perpendicularis, quare per

44 punctus: punctum *R* 45 Q: quasi *FP1*/punctus: punctum *L3R*/*post* ymaginis *add.* est *R*/fuerit visus *transp. R* 46 punctus *om. R*/et . . . G (47) *inter. a. m. L3*/erit locus *corr. ex* locus erit *C1* 47 O *corr. ex* H *S* 48 punctus: punctum *R*/*post* ymaginis *add.* erit *R*/in N: NM *FP1*/*post* N *add.* erit locus ymaginis N si autem fuerit NM *F* 49 id est: scilicet *O* 50 *post* ymaginis *add.* tunc *R*/quia: qua *S*/in N: NM *SP1*/*post* ita *add.* patet *R* 51 *post* quidem *add.* iam dicta *R*/punctus visus: punctum visum *R* 52 visu *corr. ex* visui *F*/enim: et equali *FP1*; et ? *S*/infinite: finite *S* 53 possunt: possint *R*/quarum: quoniam *FP1S*/*post* ortogonalis *add.* sit *R* 54 omnes: omnis *F*; omni *P1*/sint: sit *F*; secent se *ER*; sunt *inter. L3* 55 quedam . . . perpendicularem (56) *mg.* (non: tunc) *O* 56 rectam *inter. L3*/et[1] *om. FP1*/eandem: illam *R* 57 *post* ymaginis *add.* erit *R*/et *inter. O*/punctus: punctum *R*/qui: quod *R* 58 superficie: se *S* 59 *post* autem *scr. et del.* in se visus *S*/efficit: efficiet *R* 60 que: queque *S* 61 poterunt: poterint *C1*/a[2] *inter. O* 62 *post* cum *add.* super *F* 63 quem: que *FP1O*/quem per *mg. a. m. C1*/dividat: dividit *R*/perpendicularis: perpendicularem *E*/hoc: hec *R* 64 quidem *inter. OL3* (*a. m. L3*)/quidem dico *transp. FP1*/dico *om. S*; *corr. ex* ducta *C1*/punctis: predictis *E* 66 reflectentur: reflectuntur *L3*/quod *om. L3* 67 *post* illud *inter.* punctum *L3*/perpendicularem *om. FP1* 68 columpnaris: piramidalis *FP1SOE* 69 ab *inter. O*/*post* aliquibus *add.* aliis *R* 71 et *inter. O*/unica *corr. ex* unicam *L3*; *corr. ex* una *a. m. E*; una *R*

hanc solam perpendicularem fiet reflexio, et solus punctus superficiei visus, et locus ymaginis centrum visus.

[2.498] Si vero fuerit centrum visus in centro circuli, reflectetur portio visus quam secant perpendiculares ducte a centro visus ad circulum a portione simili in circulo quam secant similiter eidem perpendiculares. Cum quelibet linea ducta a centro visus ad circulum sit perpendicularis, fiet reflexio per perpendicularem, et locus ymaginum centrum visus, quod est centrum circuli.

[2.499] Amplius, fiat super punctum A angulus acutus quoque modo, qui sit FAG. Palam quoniam concurret FA cum GZ. Sit concursus in puncto Z, et fiat angulus CAG equalis angulo FAG. Concurret quidem AC cum GQ. Sit concursus in puncto C. Palam quoniam C refertur ad Z a puncto G, et etiam refertur ad Z a puncto A, et non ab alio puncto sectionis, quia non poterit reflecti nisi a termino perpendicularis, et una est in sectione illa perpendicularis, scilicet GA.

[2.500] **[PROPOSITIO 51]** Amplius, sumptis duobus punctis in axe columpne, erit unum reflecti ad aliud ab uno circulo columpne toto, et loca ymaginum erit circulus quidam extra columpnam.

[2.501] Verbi gratia, sit EZ [FIGURE 5.2.51, p. 609] axis, T, H duo puncta sumpta in axe, AG, BD bases columpne. Dividatur TH per equalia in puncto Q, et fiat circulus cuius Q centrum, scilicet LM, qui erit equidistans basibus, eius dyameter LM, latera columpne BLA, DMG. Fiat etiam circulus KC cuius H centrum, CK dyameter, et ducantur linee TL, TM, HL, HM.

[2.502] Palam quoniam quatuor angulorum super Q quilibet est rectus, et TQ equalis QH, et QL equalis QM. Erunt illi trianguli similes, et anguli TLQ, QLH equales; similiter, anguli TMQ, QMH equales. Si

72 perpendicularem *om. FP1*/solus punctus: solum punctum *R*/*post* solus *inter.* videbitur *O*/*post* superficie *scr. et del.* punctus *F* 73 *post* visus *add.* videbitur *FP1R*/*post* ymaginis *add.* erit *R* 74 centrum *om. R*/*post* circuli *scr. et del.* et *L3* 75 quam: quem *C1* 76 portione: proportione *P1*/in *om. R*/quam: quem *SL3C1E*/eidem *corr. ex* eisdem *S*; *corr. ex* idem *C1*; eedem *ER*/perpendiculares: perpendiculari *FP1*/*post* perpendiculares *add.* quia *R* 77 *post* sit *scr. et del.* o *P1* 78 *post* fiet *add.* f *E*/reflexio: ratio *F*/per *om. FP1OE*; *inter. L3*; super *R*/perpendicularem: perpendicularis *P1*/ymaginum: ymaginis *L3ER*/*ante* centrum *add.* erit *R* 80 fiat . . . A: super punctum A fiat *ER*/*post* angulus *scr. et del.* a *P1*/quoque: quocumque *O*; quoquo *L3C1R* 81 quoniam: quod *R* 82 Z *corr. ex* S *L3* 83 quidem: equidem *FP1SOR*; *alter. in* equidem *L3*; equidistans *E*/quoniam: quod *R* 84 refertur[1,2]: reflectetur *R*/etiam: ita *R*/refertur[2]: reflectatur *L3E*/ad[2] . . . A: a puncto A ad Z *R* 85 a termino *om. FP1S* 86 *post* una *add.* sola *OL3*/scilicet *om. FP1* 88 columpne[1]: NE CO *E*/erit: poterit *R*/aliud: alium *FP1*/columpne[2] *corr. ex* columpnari *L3*; columpnali *E*/toto: tote *S* 89 loca ymaginum: locus ymaginis *R*/circulus quidam *transp. FP1*/columpnam *corr. ex* columplam *S* 90 duo *om. R* 91 columpne: columpnale *P1*; *om. L3ER*/dividatur *om. FP1*/equalia (92): qualia *F* 92 scilicet LM: eius diameter *R*/LM: HN *S*/qui erit *om. L3* 93 equidistans: equidem *F*/eius . . . LM *om. R* 94 fiat *corr. ex* fit *L3*/KC: KP *R*/cuius *corr. ex* minus *L3*/CK: PK *R* 96 quoniam: quod *R* 97 QL: QH *FP1*/illi . . . similes: illa triangula similia *R*/similes: similis *FP1* 98 anguli[1]: angulus *L3*/TLQ: DQ *FP1*; *corr. ex* LQ *L3*

igitur fuerit H centrum visus, reflectetur punctus T ad punctum H a puncto L, et similiter a puncto M. Si igitur moveatur triangulus TLH, immoto axe TH, describet punctus L circulum, et semper duo anguli TLQ, QLH manebunt equales, et semper in hoc motu reflectetur T ad H.

[2.503] Producatur autem linea CHK donec concurrat cum linea TL, et sit concursus F. Palam quoniam F erit locus ymaginis, et motu trianguli TLH, movebitur triangulus TFH, et hoc motu punctus F describet circulum extra columpnam. Et totus ille circulus erit locus ymaginum, et hoc est propositum. Idem erit probandi modus, sumptis quibuslibet duobus in axe punctis.

[2.504] **[PROPOSITIO 52]** Amplius, punctorum extra perpendicularem visus sumptorum, quedam unicam habent ymaginem, quedam duas, quedam tres, quedam quatuor, non plures.

[2.505] Verbi gratia, sit A [FIGURE 5.2.52, p. 610] punctus visus extra perpendicularem visus, et fiat superficies transiens per A equidistans basibus speculi. Faciet quidem circulum in columpna. Sit centrum illius circuli H, et sumatur in superficie circuli aliud punctum, quod sit B, et ducantur dyametri AH, BH.

[2.506] Palam ex eis que dicta sunt in speculis spericis concavis quod ab uno puncto arcus quem intercipiunt hii duo dyametri potest A reflecti ad B forsitan a duobus, aut tribus, sed non a pluribus; ab arcu autem opposito non nisi ab uno puncto. Sit igitur quod A refertur ad B a tribus punctis arcus intercisi, et sint puncta illa G, D, E, et ducantur linee AG, HG, BG, AD, HD, BD, AE, HE, BE.

[2.507] Et a puncto A ducantur in eadem superficie tres linee equidistantes tribus dyametris HG, HD, HE, que sunt AK, AF, AN. Cum igitur AK sit equidistans HG, concurret BG cum AK. Concurrat in punc-

99 fuerit *inter. L3*/H[1,2]: T *R*/punctus T: quidem H *R* 100 triangulus: triangulum *R* 101 immoto: immote *S*/punctus: punctum *L3R*/*post* L *scr. et del.* centrum *E*/semper: super *L3* 102 QLH *mg. F*/T: H *R* 103 H: T *R* 104 CHK: PHK *R* 105 palam . . . F (106) *mg. a. m. C1*/quoniam: quod *R*/et[2] *corr. ex* in *L3* 106 movebitur: movevebitur *F*/triangulus: triangulum *R*/punctus: punctum *R* 107 columpnam: columpna *S* 108 ymaginum: imaginis *R*/et *om. C1*/hoc: quod *C1*/est *inter. L3* 109 quibuslibet *corr. ex* quibusduobuslibet *F*; quibus *P1*/*ante* duobus *add.* dyameter *P1*/duobus *om. F*/*post* duobus *add.* licet *P1*/in axe punctis: punctis in axe *R* 111 *post* visus *scr. et del.* et fiat superficies transiens per A *C1* 112 *post* quatuor *add.* et *R* 113 punctus visus: punctum visum *R* 114 et *om. O*/equidistans: equidem *F*; *corr. ex* equidem *S* 116 *post* illius *scr. et del.* cum *C1*/et . . . B (117) *inter. a. m. L3*/*post* superficie *scr. et del.* speculi *P1*/aliud: alium *F* 118 eis: hiis *FP1*/quod: quoniam *L3E* 119 puncto *om. FP1SO*; *inter. L3*/arcus: arcu *O*/quem: quam *P1*/hii duo: he due *R* 120 forsitan: forsan *SOC1*/*post* duobus *add.* punctis *R*/*post* aut *add.* a *SOC1*/sed *om. FP1*/autem: aut *P1*; *om. O* 121 non *corr. ex* nec *L3*/ab: a *L3E*/uno puncto *transp. L3E*/A refertur *corr. ex* refertur A *S*/refertur: reflectatur *R* 122 arcus intercisi *transp. R*/et[2] *om. O* 123 AD *om. E*/AD HD BD: HD BD AD *L3R* (AD *inter. L3*)/BD: DB *C1E* 124 A *inter. a. m. L3*/superficie tres *corr. ex* superficies est *L3*/equidistantes (125): equidem *FL3* 125 sunt: sint *OR*/AK *corr. ex* AE *a. m. E* 126 AK[2] *corr. ex* AE *a. m. E*/concurrat *corr. ex* concurret *a. m. C1*; concurret *R*

to K. Similiter BD concurret cum AF. Sit concursus in puncto F. Similiter BE cum AN. Sit concursus in puncto N.

[2.508] Deinde a puncto H erigatur axis, que sit HU, et a puncto B perpendicularis super superficiem circuli. Erit quidem equidistans axi, que sit BT. Et sumatur in ea punctum quodcumque, quod sit T, et ducantur tres linee TK, TF, TN, et a tribus punctis G, D, E erigantur tres perpendiculares super superficiem circuli GM, DL, EQ. Erunt quidem equidistantes TB. EQ igitur erit in superficie trianguli TBN. Igitur EQ secabit TN. Secet in puncto Q. DL secet TF in puncto L; GM secet TK in puncto M. Et erunt hee tres perpendiculares linee longitudinis columpne.

[2.509] A puncto Q ducatur equidistans linee NA, que quidem concurret cum axe UH, quoniam erit equidistans EH. Sit concursus in puncto U, et ducatur linea TA, quam secabit QU, quoniam QU ducitur a latere trianguli equidistanter basi. Sit punctus sectionis I, et ducatur linea QA.

[2.510] Palam quoniam angulus BEH equalis est angulo ENA, et angulus HEA equalis angulo EAN, et angulus BEH equalis angulo HEA. Erit angulus EAN equalis angulo ENA, quare EN equalis EA.

[2.511] Et EQ perpendicularis. Erit triangulus QEA equalis triangulo QEN; erit QN equalis QA, et erit angulus QNA equalis angulo QAN. Sed angulus TQI equalis angulo QNA, et angulus IQA equalis angulo QAN. Erit angulus IQT equalis angulo IQA, quare A refertur ad T a puncto columpne quod est Q.

[2.512] Eodem modo probabitur quod refertur A ad T a punctis L, M, et ita a tribus punctis columpne ex eadem parte.

[2.513] Nec potest a pluribus, detur enim aliud. Ducto latere ab illo puncto, cadet in circulum quem habemus, et probabitur quod a puncto casus qui est in circulo poterit reflecti A ad T, replicata probatione, quod est impossibile.

127 K *corr. ex* E *a. m. E*/BD: KD *S*/concurret: concurrat *L3* 129 a^{1} *corr. ex* in *L3*/que: qui *R*/HU: HX *R* 130 *post* circuli *add.* que *R*/quidem *om. L3ER*/equidistans: equidem *F*/axi *corr. ex* axis *L3* 132 TN: ON *S* 134 equidistantes: equidem *F*/igitur erit *transp.* (igitur *inter.*) *O*/erit: erunt *SC1R* 135 DL: D vel O *FP1*/L: LG *FP1*/GM: TM *E*/secet^{3} *om. FP1SOC1E*; *inter. L3* 136 et *om. FP1* 138 ducatur: ducantur *FP1*; *corr. ex* ducantur *L3*/equidistans: equidem *F* 139 UH: XH *R*/EH: EB *C1*/sit: sint *FP1* 140 TA *corr. ex* TH *a. m. E*/quam: quoniam *S*/ducitur: ducatur *P1* 141 *post* trianguli *add.* et linea EQ *R*/equidistanter: equidistans *E*; equidistante *R*/punctus: punctum *R* 143 quoniam: quod *R*/est *om. L3*/ENA *corr. ex* ENAF *L3* 144 HEA^{1}: BEA *FP1* 145 EA *corr. ex* EN *C1* 146 triangulus: triangulum *R*/equalis: equale *R* 147 QEN *corr. ex* QAN *a. m. E*/*post* QEN *add.* et *OR* (*inter. O*)/*post* equalis^{1} *scr. et del.* ? *O*/QA *corr. ex* QM *L3*/et *inter. O*/angulus *om. R*/QAN . . . angulo (149) *om. S*/*post* QAN *inter.* et *L3* 148 sed: erit *L3*/TQI *corr. ex* IQT *L3*/QNA . . . angulo (149) *mg. a. m. L3* 149 refertur: reflectetur *R* 151 refertur: reflectetur *R*/A *om. O* 152 punctis *om. S*; *inter. O* 153 aliud: alium *F* 154 in *om. FP1*; *inter. L3*/probabitur: probatur *SOC1* 155 qui: que *S*/A ad T: AD DT *FP1*/replicata: repetita *R* 156 est *om. SO*

[2.514] Ex arcu opposito circuli poterit reflecti A ad B ab uno puncto. Sit illud Z, et ducatur dyameter HZ, et ei equidistans AC. Et ducatur BZ, que concurrat cum AC in puncto C. Et erigatur perpendicularis OZ, que erit latus et equidistans TB, et ducatur TC, que secabitur a linea OZ. Sit sectio in puncto O. Probabitur modo predicto quod A refertur ad T a puncto O. Et si sumatur ex illa parte alius punctus columpne a quo possit reflecti, per replicationem probationis probabitur quod ab alio puncto circuli quam Z potest reflecti ex parte illa, quod est impossibile.

[2.515] Si ergo A ab uno puncto circuli refertur ad B ex aliqua parte, refertur ab uno columpne ex eadem; si a duobus, a duobus; si a tribus, a tribus; nec potest amplius; ab opposita parte ab uno circuli tantum, et ab uno columpne tantum.

[2.516] Item, TB equidistans UH, nec potest sumi superficies equalis in qua sit T cum UH preter superficiem TBUH. Similiter, non potest sumi superficies in qua sit A cum UH preter superficiem AUH, que est perpendicularis. T igitur non est in eadem superficie perpendiculari cum A, nec in eodem circulo, nec est in axe, quia est in linea ei equidistante. Superficies igitur in qua A refertur ad T est sectio columpnaris.

[2.517] Verum, producta TA ultra T et A ex utraque parte, et sit RP. Cum quatuor sint superficies reflexionis, quia a quatuor punctis, et in qualibet sint duo puncta T, A, erit RP communis quatuor superficiebus reflexionis. Et quelibet harum superficierum secat superficiem contingentem speculum in puncto super suam lineam communem, non super eandem. Linea RP perpendicularis est super unam linearum

157 *post* arcu *add.* vero *R*/opposito *alter. in* oppositio *L3*/opposito circuli *transp. R*/T: H *F*; *alter. ex* Q *in* B *P1*; *alter. ex* C *in* B *L3*; B *C1ER* 158 illud *om. L3*/Z *corr. ex* S *L3*/dyameter *om. P1*/HZ: H *FP1*; *corr. ex* HS *L3*/equidistans: equidem *F*; *corr. ex* equidem *S*/AC: AS *R* 159 BZ: Z *FP1*; *corr. ex* BS *L3*/AC: AO *E*; AS *R*/C: O *O*; S *R*/*post* perpendicularis *add.* que sit *R* 160 OZ *mg. O*; *corr. ex* OI *L3*/equidistans: equidem *F*/quidistans *C1*/TB *corr. ex* TH *E*/et[2] *om. L3*/TC: TS *R* 161 OZ *corr. ex* OS *L3*/sit *om. FP1*/quod: quia *FP1*/A *inter. OL3* 162 refertur: reflectetur *R*/si *inter. L3*; *om. E*/alius *corr. ex* illius *L3*/alius punctus *transp. E*; punctum aliud *R* 163 *post* probabitur *add.* sic *P1* 164 *post* quod[1] *scr. et del.* ab *E*/Z *corr. ex* S *L3*/reflecti *mg. O* 166 puncto *om. SE*; *inter. OL3*/refertur: reflectitur *R*/B: T *SOC1E*; *corr. ex* T *L3*/aliqua: alia *FP1S* 167 refertur: reflectitur *S*; reflectetur *R*/*post* eadem *add.* ad T *R* 168 a tribus *rep. S*/nec: non *L3*/*post* amplius *add.* ab illa parte *R*/*post* opposita *add.* vero *R*/*post* parte *add.* nisi *L3E* (*inter. L3*); *add.* non nisi *R*/*post* uno *add.* nisi puncto (*inter.* puncto) *O*; *add.* puncto *R*/tantum *om. C1* 170 equidistans: equidistat *C1ER*/equalis *om. SO* 171 T: A *S*/*post* UH *scr. et del.* AI *L3*/TBUH . . . superficiem *om. S*; *mg. O* 172 sumi superficies *L3ER*/A: E *FP1* 173 T *corr. ex* circuli *a. m. L3*/T igitur *transp. R* 174 equidistante (175) *corr. ex* quidem *L3* 175 refertur: reflectitur *R*/columpnaris *om. SOC1E*; *inter. L3* 176 *post* producta *add.* sit *R*/ex *corr. ex* et *L3* 177 reflexionis *inter. L3*; *om. E*/*post* quia *scr. et del.* a *L3*/a *rep. C1E*/*post* punctis *add.* fit reflexio *R* 178 *post* qualibet *add.* horum *R*/sint: sunt *E*/puncta TA *transp. L3*/T A *inter. OL3*/erit *mg. L3* 179 et *om. O*/contingentem (180): continentem *F* 180 *post* puncto *add.* sue reflexionis *R* 181 *post* eandem *add.* sed diversas *E*/linea *corr. ex* lineam *C1*/*post* linea *add.* ergo *R*/est *inter. O*

quatuor communium, non super duas, esset enim perpendicularis super superficiem contingentem, et ita perveniret ad axem. Sunt igitur diverse perpendiculares a puncto T ad has quatuor lineas communes, nec est nisi una tantum que transeat per A.

[2.518] Et perpendicularis aut erit equidistans linee reflexionis, aut concurret cum ea ultra speculum, vel intra. Si fuerit equidistans, erit locus ymaginis punctus reflexionis, ut probatum est, et cum quatuor sint reflexionis puncta, erunt quatuor ymagines. Si concurrit, cum quatuor sint perpendiculares, quatuor erunt concursus, et quatuor ymagines.

[2.519] Amplius, datis puncto viso et puncto visus, erit invenire punctum reflexionis. Verbi gratia, sit A punctus visus. Fiat superficies secans columpnam equidistans basi transiens per A, et faciet circulum. B aut est in superficie huius circuli, aut non. Si fuerit, inveniemus punctum reflexionis in illo circulo sicut dictum est in sperico concavo. Si non fuerit, ducatur a puncto B perpendicularis super superficiem huius circuli, et replicetur supradicta probatio, et invenietur punctus reflexionis. Duplici autem visu adhibito, una ymago in veritate efficientur due, sed contigue vel admixte, unde videbuntur una.

[2.520] In speculis piramidalibus concavis, linea communis superficiei reflexionis et superficiei speculi aut erit linea longitudinis speculi, aut erit sectio piramidalis. Si fuerit linea longitudinis, erunt loca ymaginum in ipso speculo. Si fuerit sectio piramidalis, erunt loca ymaginum aliquando citra visum, aliquando in visu, aliquando inter visum et speculum, aliquando ultra speculum, sicut ostensum est in speculo columpnari concavo.

[2.521] Amplius, si in perpendiculari ducta a centro visus ad superficiem contingentem piramidem sumatur punctus corporeus inter visum et speculum, non reflectetur forma eius ad visum per perpendicu-

182 enim *corr. ex* IBN *L3*/*post* enim *add.* esset *E* 185 est *corr. ex* esse *mg. F*/nisi *inter. a. m. E*/*post* nisi *add.* perpendicularis *ER*/transeat: transit *L3ER* 186 et: sed *FP1*/erit *inter. a. m. E*; est *R* 187 concurret: concurrit *R* 188 punctus: punctum *R*/et *om. L3* 189 sint: fuerint *FP1*/ concurrit: concurrant *L3*; concurrerint *C1*; concurrunt *E*; *corr. ex* concurrat *O* 190 sint: sunt *R*/ quatuor . . . concursus: erunt concursus quatuor *R* 193 sit *corr. ex* si *O*/punctus visus: punctum visum *R*/*ante* fiat *inter.* et *O*; *add.* B centrum visus *R* 194 equidistans: equidistanter *R*/et *inter. O* 195 inveniemus; invenimus *L3*/punctum reflexionis (196) *om. FP1* 196 reflexionis: rationis *S* 197 super *om. S* 198 punctus: punctum *R*/reflexionis (199) *om. P1* 199 in . . . due (200): efficietur due in veritate *O*/efficientur; efficietur *FP1* 200 videbuntur: videbitur *R*/*post* una *inter.* ymago *a. m. C1* 201 piramidalibus *corr. ex* pluribus *O* 202 reflexionis et superficiei *mg. a. m. L3*/superficiei: superficie *P1*/aut[1] . . . speculi *transp. post* piramidalis (203) *mg. O* 203 si *mg. O* 204 in . . . ymaginum *mg. a. m. L3* 206 *ante* aliquando *add.* et *L3ER*/speculum *om. O*/columpnari (207): columpnali *E* 209 piramidem: piramidalem *FP1*/punctus corporeus: punctum corporeum *R*/corporeus *inter. a. m. E* 210 per *om. FP1S*

larem, quoniam punctus ille occultabit terminum perpendicularis, et ob hoc non reflectetur ab eo. Si autem nullus fuerit punctus in perpendiculari illa, reflectetur quidem ad visum per hanc perpendicularem punctum visus, quod iterum secat perpendicularis ex eo, et ille solus.

[2.522] Verum, visu existente in hac perpendiculari et in axe, efficietur circulus ad cuius quodlibet punctum linea ducta a visu erit perpendicularis super superficiem contingentem, unde a quolibet puncto illius circuli fieri poterit reflexio ad visum per perpendicularem. Et fiet reflexio partis visus quam secant due perpendiculares maiorem angulum in eo continentes.

[2.523] Si vero inter visum et speculum fuerit axis, non fiet ad ipsum reflexio per perpendicularem nisi puncti eius quem secat perpendicularis.

[2.524] **[PROPOSITIO 53]** Amplius, existente visu et puncto viso in axe, erit reflecti unum ad aliud.

[2.525] Verbi gratia, sit H [FIGURE 5.2.53, p. 611] centrum visus, T punctus visus. Fiat superficies secans piramidem transiens super axis longitudinem, que sit ABGH, AH axis, AB, AG latera piramidis. A puncto T ducatur perpendicularis super lineam AB, que sit TQ, et producatur usque QL. Sit equalis QT. Et a puncto H ducatur linea ad punctum L, que secabit lineam longitudinis que est AB. Secet in puncto B, et a puncto B ducatur equidistans linee TQ, que necessario perveniet ad axem. Perveniat in puncto D, et ducatur linea TB.

[2.526] Palam, cum TQ sit perpendicularis super AB, et TQ equalis QL, erit triangulus BTQ equalis triangulo BQL, et erit angulus QLB equalis angulo QTB. Sed angulus QTB equalis est angulo TBD, et angulus DBH equalis est angulo QLB. Igitur angulus TBD equalis est angulo DBH, et ita T refertur ad H a puncto B, et locus ymaginis L.

211 punctus ille: punctum illud *R*/*post* terminum *add.* illius *E*/*post* perpendicularis *add.* illius *R* 212 ob: ab *SO*/ob hoc *corr. ex* ad hoc *mg. a. m. C1*/*post* hoc *scr. et del.* re *S*/reflectetur: reflectitur *L3*/nullus: nullum *R*/punctus: punctum *R* 214 punctum *alter. in* puncti *L3*/*post* visus *add.* punctum *OL3C1* (*inter. L3*; *inter. a. m. O*)/iterum *om. SOR*/ille solus; illud solum *R* 215 existente *corr. ex* exigente *a. m. E* 216 cuius quodlibet *corr. ex* cuiuslibet *L3*/ducta *corr. ex* producta *F*/ducta a visu: a visu ducta *L3* 217 illius circuli (218) *transp. C1* 218 *post* reflexio *add.* a *C1*/per *om. FP1E*; *inter. SL3*; secundum *R*/perpendicularem: perpendiculares *R* 219 due perpendiculares *transp. L3ER* 220 continentes: contingentes *S* 221 *post* ipsum *add.* visum *L3E* (*inter. a. m. L3*) 222 per *om. FP1*; *inter. SL3*; propter *O*/quem: quod *R*/secat: secant *L3ER*/perpendicularis (223): perpendiculares *L3ER* 225 erit: poterit *R*/aliud: alium *FP1* 227 punctus visus: punctum visum *R* 228 longitudinem *corr. ex* superficiem *L3*/AB AG *transp. FP1* 230 usque: ut *O*; quousque *R* 231 *post* AB *scr. et del.* in puncto *P1* 232 et . . . B *mg. O* 233 TB: DB *FP1* 235 triangulus . . . equalis: BTQ triangulum equale *R*/BQL *corr. ex* QL *L3* 236 est *om. P1*/TBD: QBD *O* 238 DBH *corr. ex* QLB *P1*/refertur: reflectitur *R*/*post* ymaginis *add.* est *R*

[2.527] Igitur, moto triangulo TLH, describet punctus B circulum in piramide, et a quolibet puncto illius circuli reflectetur T ad H. L vero extra circulum describet circulum qui totus erit locus ymaginis puncti T.

[2.528] **[PROPOSITIO 54]** Amplius, sumptis duobus punctis extra perpendicularem visus et extra axem in hoc speculo, scilicet Z,E [FIGURE 5.2.54, p. 612], fiat superficies equidistans basi super Z. Faciet circulum in speculo. E aut erit in hoc circulo, aut in alia superficie.

[2.529] Sit in superficie illius circuli, et ducatur linea EZ. Palam quoniam Z refertur ad E a circulo illo ex una parte aut ab uno puncto, aut a duobus, aut a tribus; ex alia ab uno.

[2.530] Sumatur punctus circuli a quo refertur ad ipsum, et sit H, centrum circuli T. Et ducantur linee ZH, EH. Et dyameter TH dividet quidem angulum illum per equalia, et secabit lineam EZ. Secet in puncto Q, et sit A conus piramidis, AH linea longitudinis.

[2.531] A puncto Q ducatur linea cadens perpendiculariter super lineam AH, que sit QM, que quidem perveniat ad axem, qui est AT. Cadat in ipsum in puncto D, et ducantur linee ZM, EM. A puncto Z ducatur in superficie circuli linea equidistans linee QH, que sit ZL. Concurrat quidem EH cum illa. Sit concursus in puncto L, et a puncto H ducatur perpendicularis super LZ, que sit HC.

[2.532] Deinde in superficie trianguli EMZ ducatur linea equidistans linee QM, que sit ZO. Concurrat EM cum ea in puncto O, et ducatur linea LO. Et a puncto C ducatur equidistans LO, que sit CN, et ducatur linea NM.

[2.533] Palam quoniam angulus EHQ equalis est angulo QHZ et angulo HLZ, et angulus QHZ equalis est angulo HZL. Erit HL equalis

239 triangulo *corr. ex* angulo *C1*/describet *corr. ex* describetur *C1*/punctus: punctum *R* 240 L *om. FP1* 241 circulum[1] *scr. et del. L3*; speculum *R*/circulum[2] *om. O*/qui *rep. E*/locus . . . puncti *corr. ex* puncti ymaginis locus *E* 243 *post* punctis *add.* et *ER* 244 Z *corr. ex* S *L3* 245 equidistans *corr. ex* equidem *F*/Z *corr. ex* S *L3* 246 E *inter. O*/alia: illa *P1*/*post* superficie *add.* ipsi equidistante *R* 247 EZ *corr. ex* ES *L3* 248 quoniam: quod *R*/Z *corr. ex* S *L3*/refertur: reflectitur *E*; reflectetur *R*/E: A *SOE*; *corr. ex* A *L3*/parte *om. O* 249 *post* alia *add.* vero *R* 250 *post* sumatur *add.* igitur *R*/punctus: punctum *R*/refertur: reflectitur *R* 251 ZH *corr. ex* SH *L3*/EH: EB *S*/dyameter: dyametrum *FP1* 252 quidem: equidem *FP1*/lineam: linea *E*/EZ *corr. ex* ES *L3* 253 et *inter. O*/conus: vertex *R* 254 cadens *om. R*/perpendiculariter: perpendicularis *R* 255 perveniat *alter. in* perveniet *O*; perveniet *R*/AT: AD *R*/AT cadat (256): et cadat *inter. L3*/*post* AT *add.* et *ER*; *scr. et del.* et *C1* 256 in puncto *inter. O*/ZM *corr. ex* SM *L3*/ZM . . . linee (257) *om. FP1* 257 ZL *corr. ex* SL *L3* 258 concurrat: concurret *ER* 259 H *om. P1*/LZ *corr. ex* LS *L3*/HC *corr. ex* HZ *E*; HP *R* 260 trianguli *corr. ex* circuli *E*; *om. R*/EMZ *corr. ex* EMS *L3* 261 linee QM *corr. ex* QM linee *C1*/ZO *corr. ex* SO *L3*; *corr. ex* TZO *C1*/*post* ZO *add.* et *R*/*post* EM *scr. et del.* ducatur linea *S*/*post* et *scr. et del.* L *F*/ducatur *corr. ex* duducatur *S* 262 linea . . . ducatur[2] *om. O*/et *om. S*; *inter. L3*/C: P *R*/CN: PN *R* 264 quoniam: quod *R*/EHQ: EQH *C1*/QHZ *corr. ex* QHS *L3* 265 HLZ *corr. ex* HLS *L3*/QHZ *corr. ex* QHS *L3*/*post* angulo[2] *add.* coalterno *ER*/HZL *alter. ex* HLS *in* HLZ *L3*/erit . . . HZ (266) *om. P1*/*post* erit *add.* igitur *R*

HZ, et HC perpendicularis super LZ. Erit triangulus LCH equalis triangulo CHZ, et erit LC equalis CZ.

[2.534] Et CN equidistans OL; erit proportio LC ad CZ sicut ON ad NZ, quare ON equalis NZ. Item, cum OZ sit equidistans QM, erit superficies ZLO equidistans superficiei QMH. Et superficies EOL secat illas duas super lineas communes que quidem erunt equidistantes, scilicet MH, OL, quare HM, CN equidistantes. Et quoniam HC cadit inter LZ, HQ equidistantes, et est perpendicularis super LZ, erit perpendicularis super HQ, quare CH erit contingens circulo.

[2.535] Igitur superficies AHC est superficies contingens piramidem. In hac superficie est CN et NM, et super hanc superficiem est perpendicularis linea DM. Igitur perpendicularis est super lineam NM, quare NM perpendicularis super OZ, et ON equalis NZ. Erit MO equalis MZ, et proportio EM ad MO sicut EM ad MZ.

[2.536] Sed EM ad MO sicut EH ad HL, et EH ad HL sicut EH ad HZ, et EH ad HZ sicut EQ ad QZ. Igitur EM ad MZ sicut EQ ad QZ, quare angulus EMQ equalis angulo QMZ, quare Z refertur ad E a puncto M. Si ergo Z refertur ad E a puncto circuli H, refertur ad ipsum a puncto piramidis M. Et si a duobus circuli, a duobus piramidis; si a tribus, a tribus; si a pluribus, a pluribus. Eodem modo ex alia parte circuli fiet probatio quod ab uno piramidis sicut ab uno circuli.

[2.537] Si vero E non fuerit in circulo equidistante basi transeunte super Z, erit E [FIGURE 5.2.54a, p. 613] quidem supra aut infra. Sit

266 HZ *corr. ex* HS *L3*/HC: HP *R*/*post* perpendicularis *add.* est *R*/super: SF *E*/LZ *corr. ex* LS *L3*/triangulus . . . equalis: triangulum LPH equale *R* 267 triangulo angulo *O*/CHZ: CHT *P1*; *alter. ex* THS *in* THZ *L3*; CHE *E*; PHZ *R*/LC: LP *R*/CZ *corr. ex* CS *L3*; PZ *R* 268 CN: PN *R*/*post* equidistans *add.* est *R*/LC: LP *R*/CZ *corr. ex* CS *L3*; PZ *R* 269 NZ[1,2] *corr. ex* NS *L3*/OZ *corr. ex* OS *L3*; *corr. ex* CZ *a. m. E*/equidistans *corr. ex* E *O*/*post* QM *add.* et HQ equidistans LZ *R* 270 ZLO *corr. ex* SLO *L3*; ZOL *R*/equidistans: equidem *S* 271 super *om. FP1*/lineas *corr. ex* illas *S*/scilicet (272) *om. C1* 272 OL: LO *R*/CN: N *S*; PN *R*/*ante* equidistantes *add.* sunt *R*/et *scr. et del. L3*/quoniam: quare *L3*/HC *corr. ex* HI *L3*; HP *R* 273 LZ[1]: Z *F*; ZH *P1*/LZ[1,2] *corr. ex* LS *L3*/equidistantes: EQ *C1*/erit . . . HQ (274) *om. R* 274 CH . . . circulo: PH continget circulum *R* 275 igitur: quare *ER*/AHC: AHP *R*/superficies[2] *corr. ex* superficiei *L3*/contingens: contingentes *F* 276 CN: ON *SC1*; PN *R*/et[1] *inter. C1* 277 *post* perpendicularis[1] *scr. et del.* est super *F*; *scr. et del.* super OE et ON equalis NZ *C1*/linea *corr. ex* lineam *F*/*post* igitur *rep.* perpendicularis[1] . . . igitur *S* 278 *post* NM *add.* est *R*/super OZ et *mg. O*/OZ: OE *SL3C1E*/NZ *corr. ex* MS *L3*/erit . . . MZ[1] (279) *inter. a. m. L3* 279 proportio *om. R*/EM *alter. ex* EZN *in* ZN *F*; EZN *P1*/MZ[2] *corr. ex* MS *L3* 280 et . . . HL[2] *inter. a. m. L3*/EH[3]: HL *SC1E*; HF *O*; HE *L3* 281 HZ[1] *corr. ex* HS *L3*/et . . . HZ[2] *mg. O*/HZ[2]: ZH *L3*/QZ[1] *corr. ex* QHZ *F*; *corr. ex* QS *L3*/igitur EM *rep. P1*/MZ *corr. ex* MS *L3*/QZ[2] *corr. ex* Z *mg. F* 282 *post* EMQ *add.* est *L3*/*post* equalis *inter.* est *O*/QMZ *corr. ex* QMS *L3*/Z *corr. ex* S *L3*/refertur: reflectitur *R* 283 M: E *S*; *corr. ex* E *L3*/M . . . puncto[1] *mg. O*/refertur[1]: reflectitur *R*/refertur[2]: reflectetur *R* 284 piramidis[1]: piramidalis *S*/a[1] *inter. O*/*post* piramidis[2] *add.* et *FP1* 286 probatio *corr. ex* proportio *L3*/*post* ab *add.* u *S*/*post* uno *add.* puncto *C1R*/piramidis: piramide *FP1*/*post* circuli *add.* reflexio fiat *R* 287 E *corr. ex* est *F*; est *P1*/equidistante: equidistans *F*; equidistanti *E* 288 Z *corr. ex* S *L3*/*post* Z *scr. et del.* equi *F*/E *om. FP1SER*/aut: vel *L3ER*/aut infra *corr. ex* infra aut *S*/sit *corr. ex* si *a. m. L3*; si *E*

supra, quia utrobique eadem est probatio. Ducatur linea AE donec contingat superficiem illius circuli, et sit punctus contactus H, Q centrum circuli. Palam quoniam H potest reflecti ad Z ab aliquo puncto circuli. Sit illud T, et ducatur dyameter QT. Et linea HZ secabit hunc dyametrum in puncto quod sit N. Et ducatur EZ et linea longitudinis AT.

[2.538] Palam, cum punctus Z sit ex una parte dyametri QT, ex alia E, linea EZ secabit superficiem AQT. Secet in puncto O, et a puncto O ducatur perpendicularis super lineam AT, que sit OC, que necessario cadet super axem. Cadat in puncto D, et ducantur linee EC, ZC. Dico quoniam E refertur ad Z a puncto C.

[2.539] Probatio: ducatur a puncto Z linea equidistans QT, que sit ZF, et producatur linea HT donec concurrat cum illa. Sit concursus in puncto F. Similiter, a puncto Z ducatur equidistans linee OC, que sit ZK, et producatur linea EC donec concurrat cum illa. Sit concursus in puncto K.

[2.540] Palam, cum linea ZF sit equidistans QT, et ZK equidistans OC, erit superficies ZKF equidistans superficiei OCT que est superficies AQT. Et superficies HFK secat has duas superficies super lineas CT, KF. Igitur CT, KF sunt equidistantes.

[2.541] Ducatur a puncto T perpendicularis super lineam ZF, que sit TP. Palam, cum cadat inter duas equidistantes, erit equidistans linee NZ, et ita erit contingens circulo. Igitur superficies ATP contingit piramidem super lineam AT, et linea OC est perpendicularis super hanc superficiem. Superficies igitur ATQ erit ortogonalis super superficiem ATP, et superficies ATP secat duas superficies ATQ, ZKF, que sunt equidistantes. Igitur linee communes sectionum sunt equidistantes, una harum linearum est CT, alia sit PI. Sed iam patet quod CT est equidistans KF. Igitur PI est equidistans KF.

289 est *om. L3*/probatio *corr. ex* proportio *L3*/AE: ab E *FP1*; *corr. ex* A *L3*; a puncto E *E*; a vertice A per punctum E *R*/*ante* donec *scr. et del.* puncto *C1*/contingat (290): secet *R* 290 *ante* superficiem *scr. et del.* super *C1*/punctus: PG *S*; punctum *R*/contactus: sectionis *R* 291 quoniam: quod *R*/Z *corr. ex* S *L3*/ab *corr. ex* et *L3* 292 illud: illum *FP1*/dyameter *corr. ex* dyatmeter *F*/HZ *corr. ex* HS *L3*/hunc: hanc *R* 293 et[1] *om. S*/EZ *corr. ex* ES *L3* 294 *ante* palam *scr. et del.* et *E*/punctus: punctum *R*/Z *corr. ex* S *L3*/*post* QT *add.* et *P1ER* 295 E *om. E*/EZ *corr. ex* ES *L3*/secet *rep. S* 296 OC: OP *R*/necessario cadet (297) *transp.* (cadet *alter. ex* sit *in* cadat) *L3* 297 EC: EP *R*/ZC: SC *L3*; ZP *R* 298 quoniam: quod *C1R*/E: Z *R*/refertur: reflectetur *R*/Z *corr. ex* S *L3*; E *R*/C *corr. ex* Z *S*; P *R* 299 probatio: proportio *L3*; *om. R*/Z *corr. ex* S *L3* 300 ZF *corr. ex* SF *L3*/et *inter. a. m. C1*/concurrat *corr. ex* conconcurrat *L3* 1 Z *corr. ex* S *L3*/OC *corr. ex* AC *E*; OP *R* 2 ZK *corr. ex* ZE *E*/EC: EP *R* 3 K *corr. ex* E *a. m. E*/*post* K *add.* et ducantur linee KF KH *R* 4 linea *corr. ex* lenea *O*/ZK *corr. ex* ZE *a. m. E* 5 OC: OP *R*/*ante* erit *add.* quod *R*/ZKF *corr. ex* ZEF *a. m. E*/superficiei *om. ER*/OCT: OPT *R* 6 *ante* AQT *scr. et del.* super lineas *F*/HFK *corr. ex* HFE *E*; HKF *R* 7 CT[1,2]: PT *R*/KF[1] *corr. ex* EF *a. m. E*/KF[2]: EF *E*/equidistantes: equidem *S*/*post* equidistantes *add.* et *L3* 9 TP: TS *R*/cadat: cadit *C1*/equidistans . . . NZ (10): angulus QTS rectus *R* 10 erit . . . circulo: continget circulum *R*/ATP: ATS *R* 11 et *inter. O*/OC: OP *R* 12 ortogonalis: ortogonaliter *E* 13 ATP[1,2]: ATS *R*/et . . . ATP[2] *inter. a. m. C1*/ZKF *corr. ex* ZEF *a. m. E* 14 sunt: fiunt *P1* 15 CT[1,2]: PT *R*/PI: SI *R*/patet: patuit *R*/est[2] *om. FP1*/est equidistans (16) *transp. R* 16 equidistans[1]: equidem *S*/igitur . . . KF[2] *om. C1*/igitur . . . sed (17) *om. FP1*/PI: SI *R*/KF[2] *corr. ex* EF *a. m. E*

[2.542] Sed planum est quod angulus NTZ equalis est angulo TZF, et angulus HTN equalis angulo TFZ, et TP perpendicularis. Erit FP equalis PZ. Sed proportio FP ad PZ sicut KI ad IZ; erit KI equalis IZ.

[2.543] Ducta autem linea CI, cum superficies ATPI sit ortogonalis super superficiem ZKF, erit CI ortogonalis super ZK, et erit angulus CKZ equalis angulo KZC. Sed angulus ECO equalis est angulo CKZ, et angulus OCZ equalis est angulo CZK, quare angulus ECO equalis est angulo OCZ. Et ita E refertur ad Z a puncto C, quod est propositum.

[2.544] Si autem sumatur aliud punctum in circulo a quo H reflectatur ad Z, probabitur quod ab alio puncto piramidis quam C refertur E ad Z. Et si reflectatur H ad Z a tribus punctis circuli, reflectetur E ad Z a tribus piramidis; si a quatuor, a quatuor.

[2.545] Punctum autem reflexionis a quo E refertur ad Z facile est invenire, invento puncto circuli a quo punctus H refertur ad Z, et erit inventio modo predicto.

[2.546] Si vero dicatur quod a pluribus punctis piramidis quam quatuor possit punctus E reflecti ad Z, per replicationem predicte probationis poterit ostendi quod punctus H refertur ad Z a pluribus punctis circuli quam quatuor, et ubi accidet punctum H reflecti ad Z ab aliquot punctis circuli vel ab uno tantum, accidet punctum E reflecti ad Z a totidem punctis piramidis aut ab uno tantum, et econverso. Quod si dicatur contrarium, poterit improbari predicto modo.

[2.547] Palam ergo quod punctorum quedam habent unicam ymaginem, quedam duas, quedam tres, quedam quatuor, sed non possibile quod plures. Verum adhibito speculo duplici visu eiusdem ymaginis

17 TZF: DZF *FP1*; *corr. ex* TZP *a. m. E* 18 TFZ *corr. ex* TZF *C1*/TP: TS *R*/FP: FS *R* 19 PZ[1,2]: SZ *R*/FP: SP *SE*; FS *R*/*post* PZ[2] *rep.* sed . . . PZ[2] *P1*/KI[1,2] *corr. ex* EI *a. m. E*/*post* erit *add.* ergo *R*/IZ: ZI *O* 20 CI: PI *R*/cum: circuli *S*/superficies ATPI: super fiat PL *FP1*/ATPI *corr. ex* AQF *E*; ATF *R* 21 ZKF *corr. ex* ZEF *a. m. E*/CI *corr. ex* Q *O*; *corr. ex* QCI *L3*; PI *R*/ZK *corr. ex* ZE *a. m. E*/et *om. O* 22 CKZ[1,2]: PKZ *R*/KZC: KZP *R*/ECO: EPO *R*/est *om. R* 23 OCZ: OPZ *R*/est *om. R*/CZK: CZH *P1*; CKZ *S*; *corr. ex* CZE *E*; PZK *R*/*ante* quare *add.* et angulus CEZ equalis est angulo CZK *S*/ECO: EPO *R* 24 est[1] *om. C1*/OCZ: COZ *S*; OPZ *R*/E *inter. L3*; Z *R*/refertur: reflectitur *R*/Z: E *R*/a puncto C *om. L3*/C: P *R* 25 sumatur: assumatur *L3*/aliud: alium *F*/H: Z *R* 26 reflectatur: refertur *L3*/*post* reflectatur *add.* H *C1*/Z: H *R*/probabitur . . . Z[2] (27) *om. O*/C: P *R* 27 refertur *om. P1*; reflectitur *L3*; reflectetur *R*/E ad Z: Z ad E *R*/*post* Z[1] *scr. et del.* a tribus pyramidis *C1*/H ad Z: Z ad H *R*/reflectetur: reflectitur *L3* 28 E *om. FP1*/E ad Z: Z ad E *R*/*post* tribus *add.* punctis *R* 29 E: Z *R*/*post* E *add.* punctus *O*/refertur: reflectitur *SL3ER*/Z: E *R* 30 punctus H: punctum Z *R*/refertur: reflectitur *L3R*/Z: H *R* 31 inventio *corr. ex* mutatio *C1* 32 dicatur *inter. O*; *rep. C1*/*post* quam *add.* a *C1* 33 possit: posset *O*/punctus E: punctum Z *R*/E *om. FP1*/Z: E *R*/replicationem: reflexionem *FP1SL3C1E*; conversionem *R* 34 probationis: proportione *L3*/punctus H: punctum Z *R*/refertur: reflectitur *L3R*/Z: H *R* 35 *post* circuli *scr. et del.* reflectetur E ad Z *C1*/H: E *C1*; Z *R*/Z: H *R*/ab . . . Z (37) *mg. C1* 36 punctis: punctus *F*/E: Z *R* 37 Z: E *R*/et econverso: aut econtrario *R*/quod *inter. L3* 38 poterit *corr. ex* verum *L3* 39 habent unicam *transp. R* 40 sed *inter. a. m. E*/*post* non *add.* est *O* 41 quod *om. R*/adhibito . . . visu: duplici visu adhibito speculo *R*/ymaginis: ymagines *FP1*

diversa erunt loca, que diversitas, propter sui imperceptibilitatem, non inducit errorem.

42 *post* diversa *scr. et del.* sunt et *E*/sui: suam *R*/imperceptibilitatem *corr. ex* imperceptibilibtatem *F*; *corr. ex* imperceptibilem *a. m. C1*

FIGURES FOR TRANSLATION AND COMMENTARY

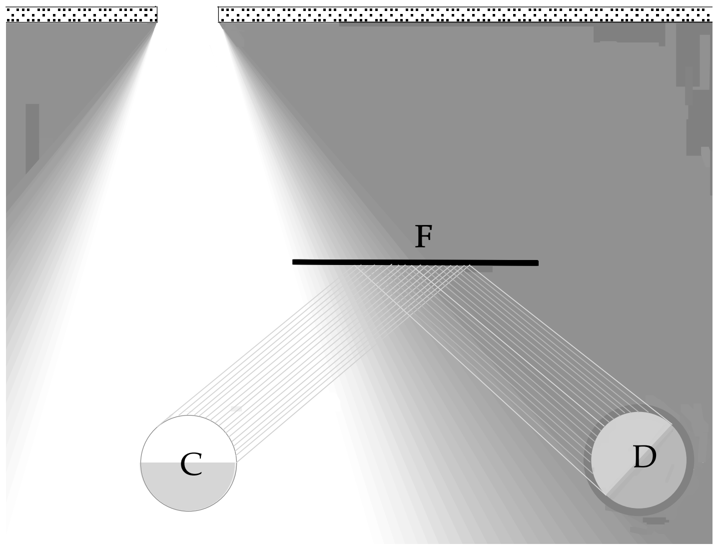


figure 4.2.1

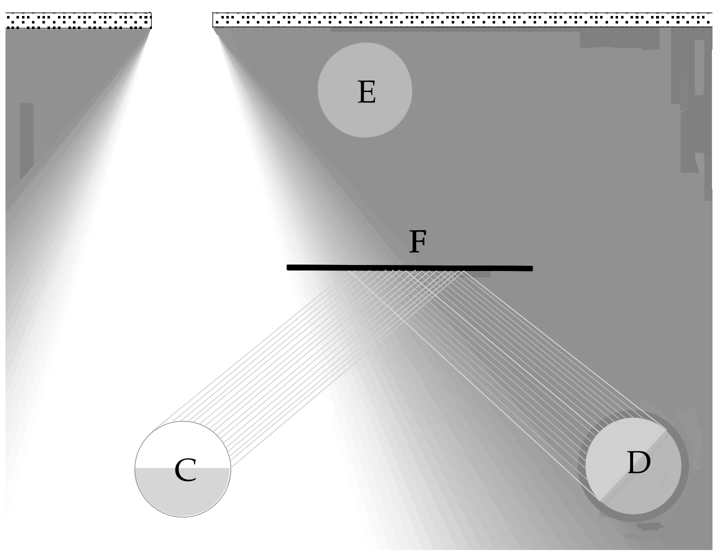


figure 4.2.2

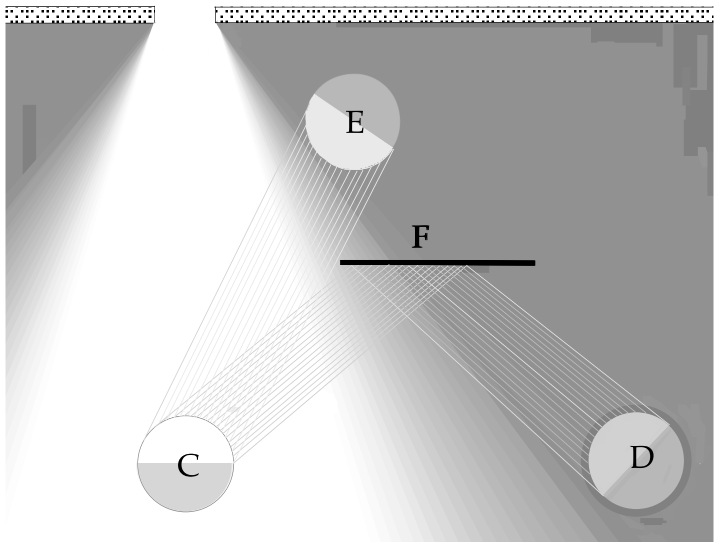


figure 4.2.3

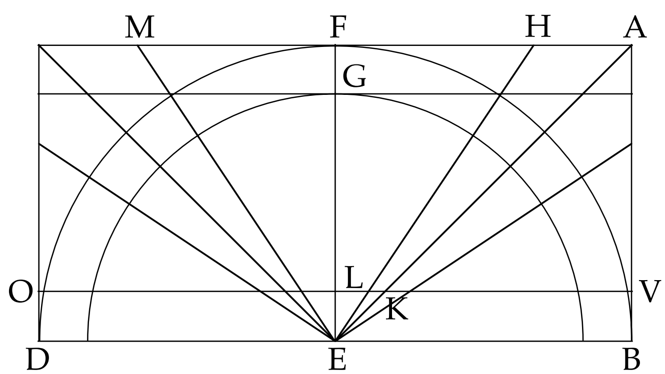


figure 4.3.1

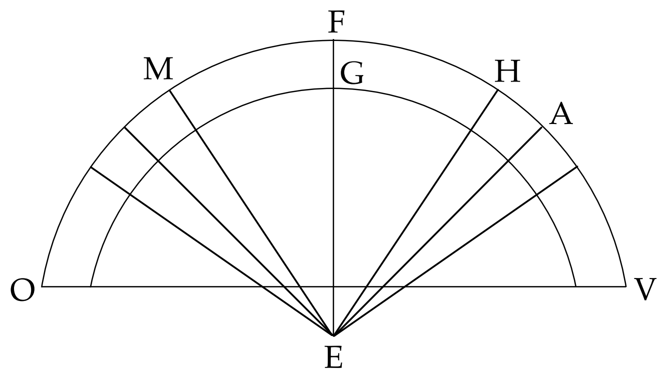


figure 4.3.2

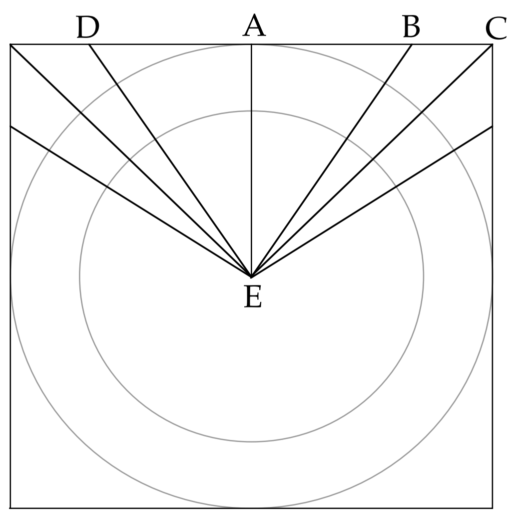


figure 4.3.3

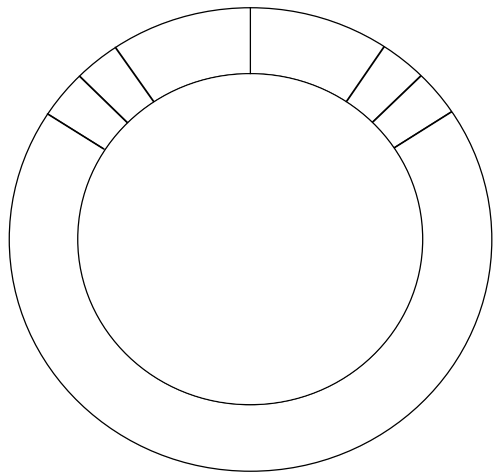


figure 4.3.4

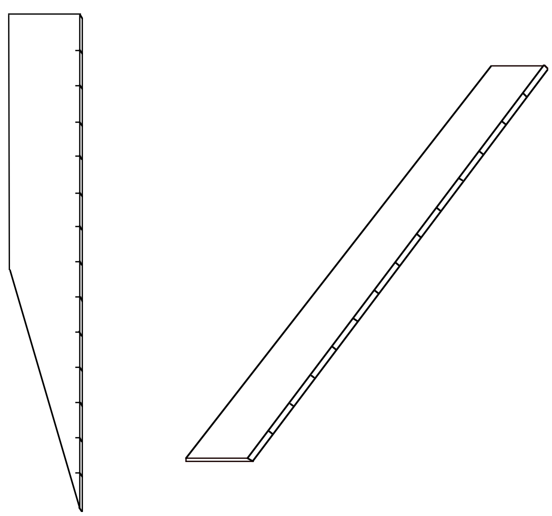


figure 4.3.5

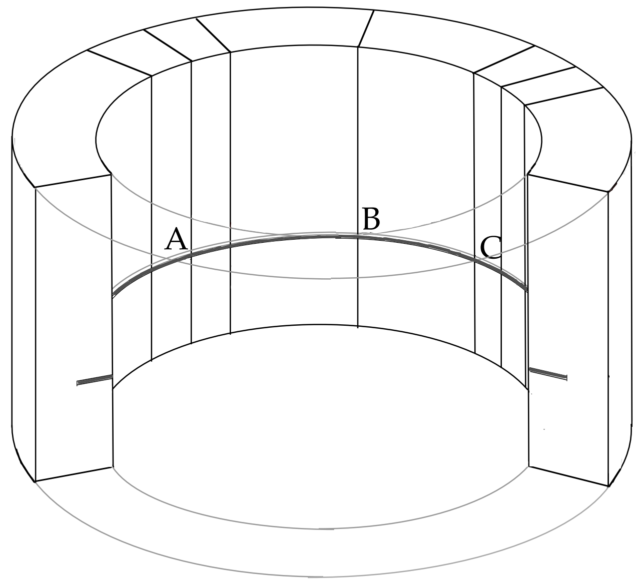


figure 4.3.6

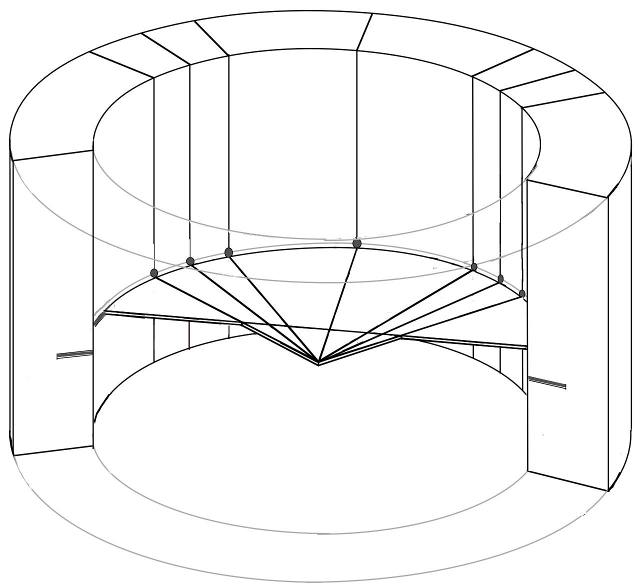


figure 4.3.7

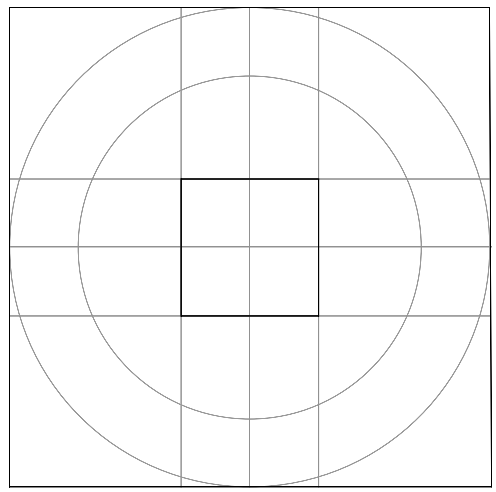


figure 4.3.8

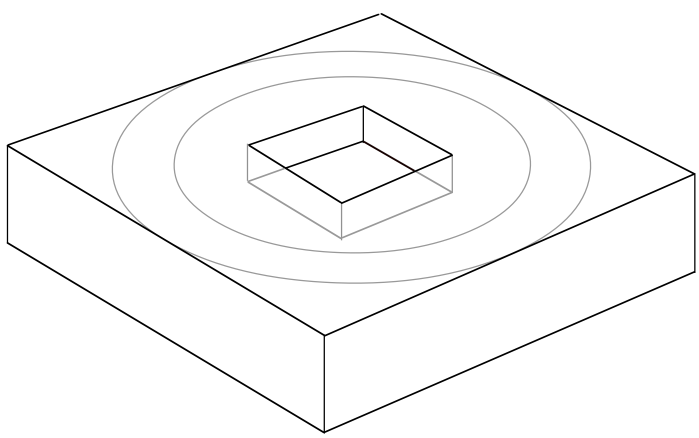


figure 4.3.9

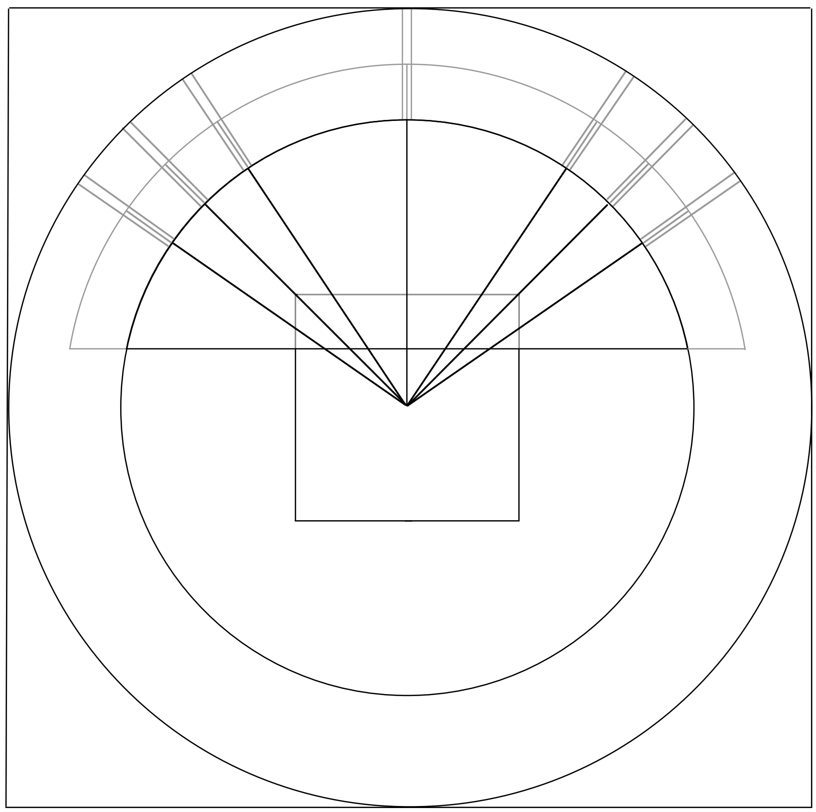


figure 4.3.10

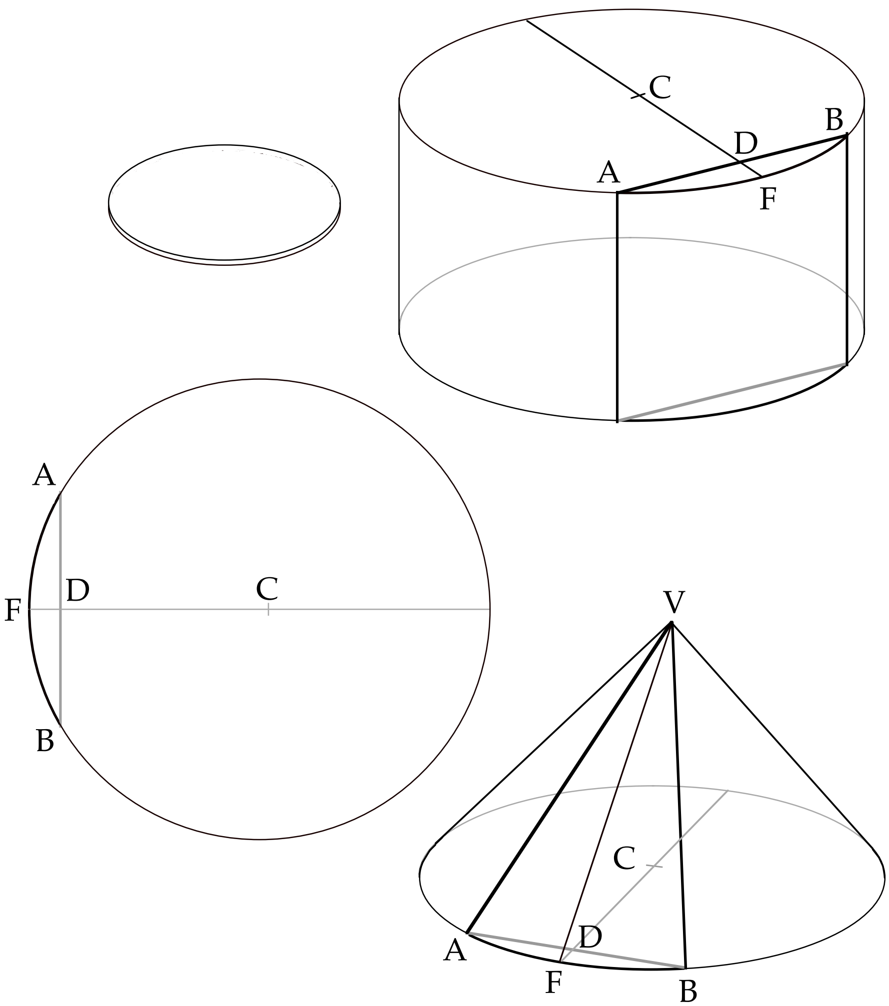


figure 4.3.11

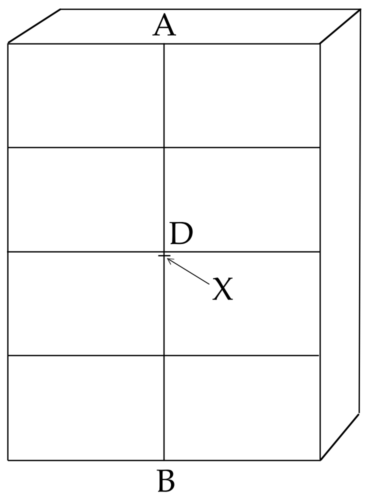
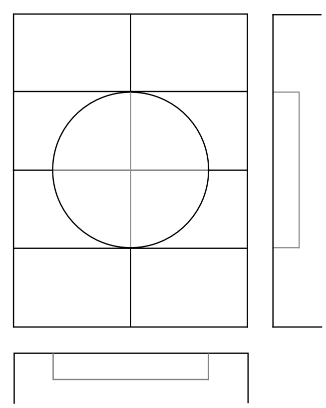


figure 4.3.12



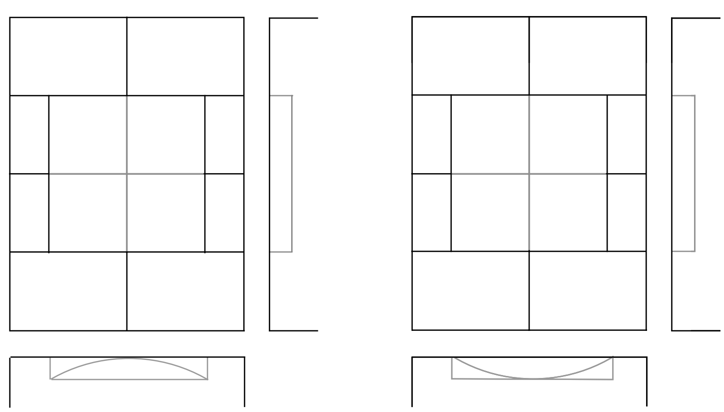


figure 4.3.13

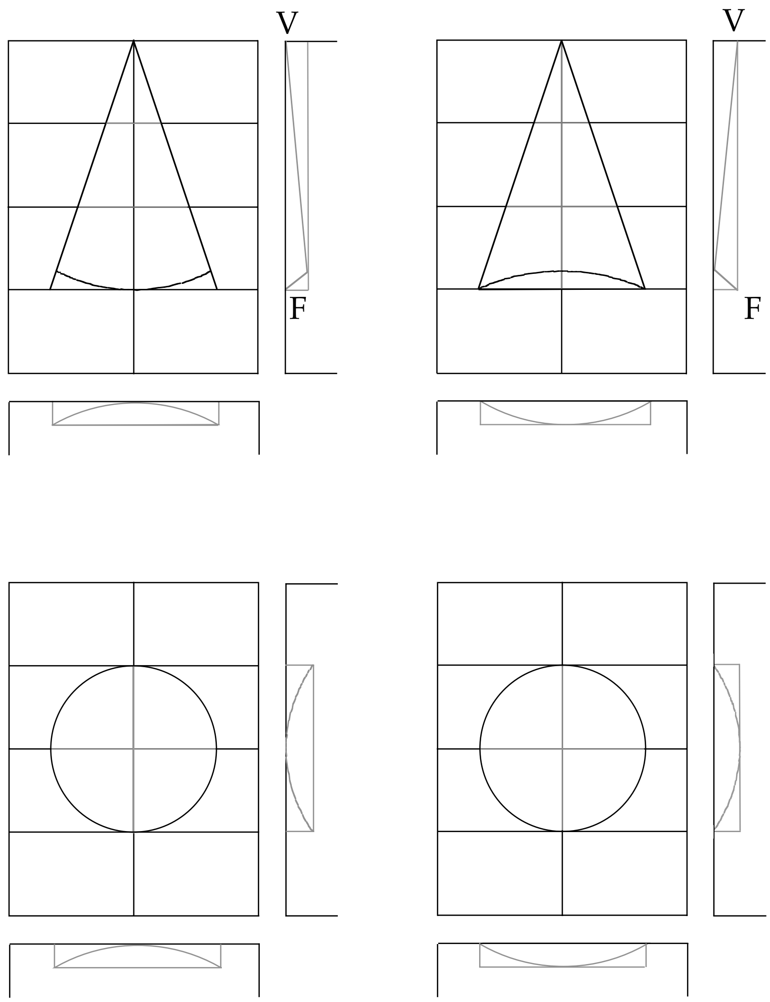


figure 4.3.14

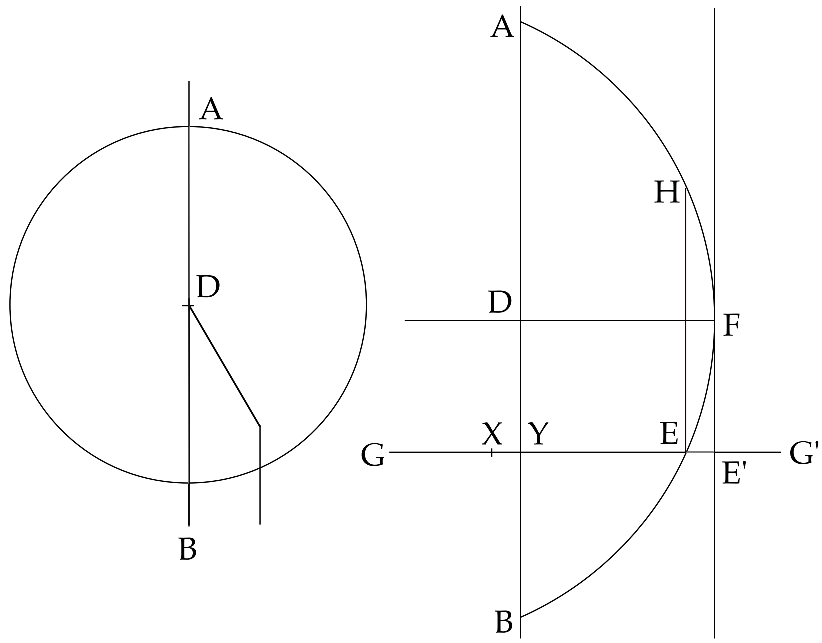


figure 4.3.15

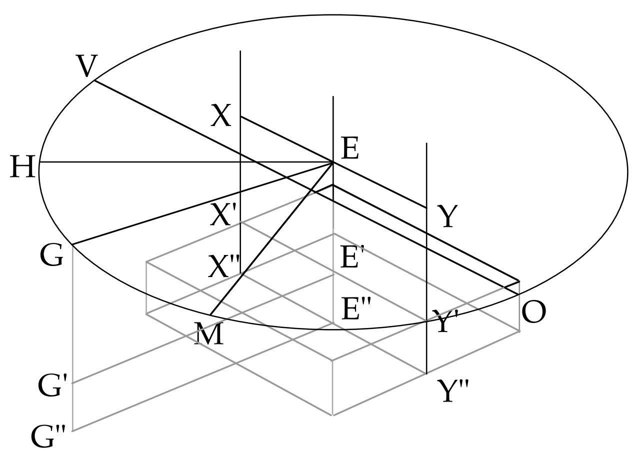


figure 4.3.16

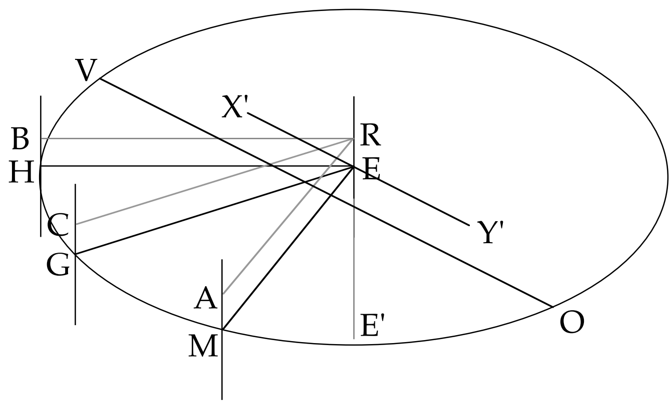


figure 4.3.17

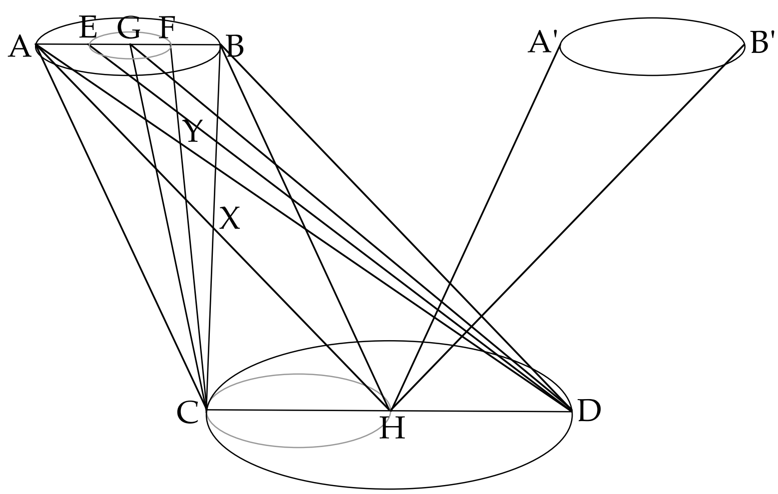


figure 4.3.18

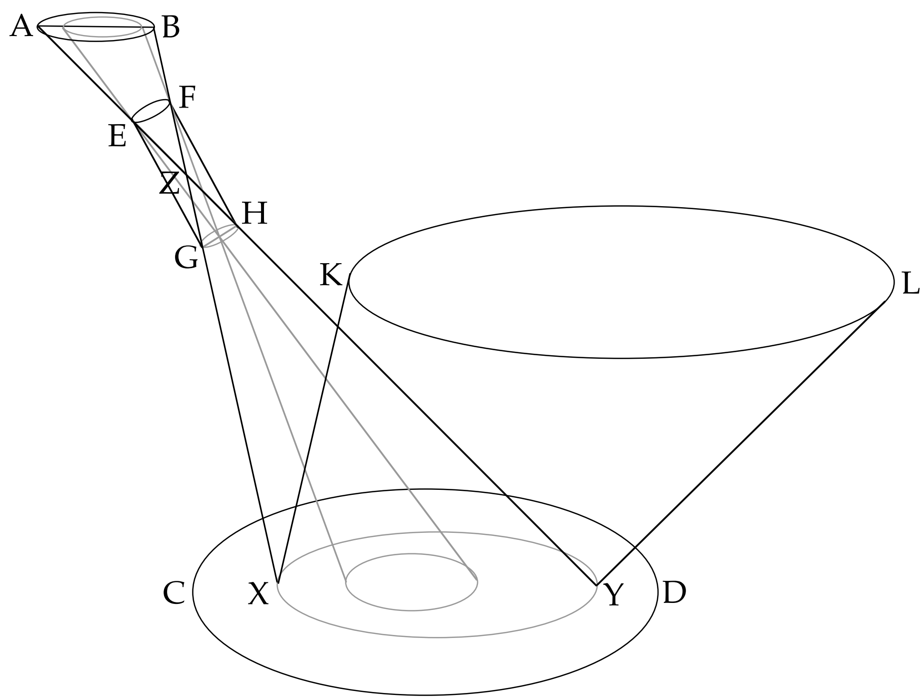


figure 4.3.19

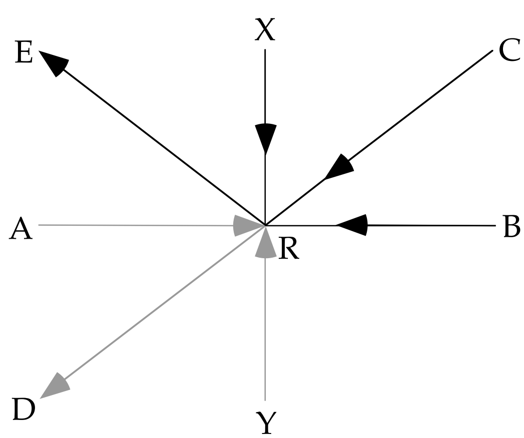


figure 4.3.20

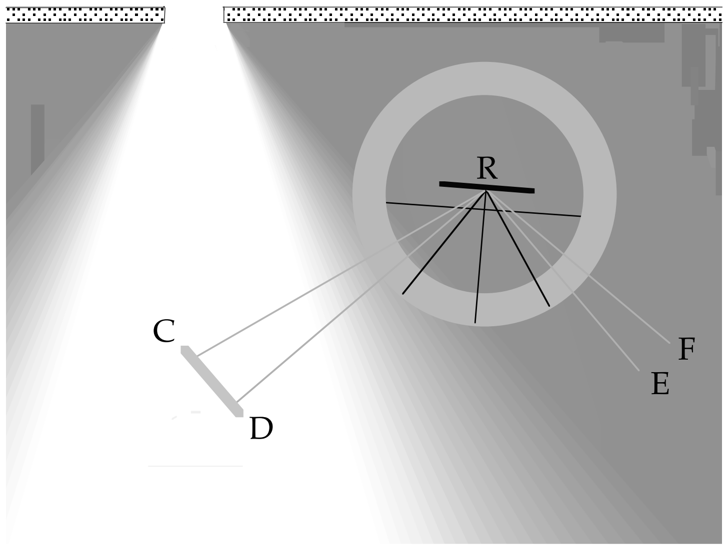


figure 4.3.21

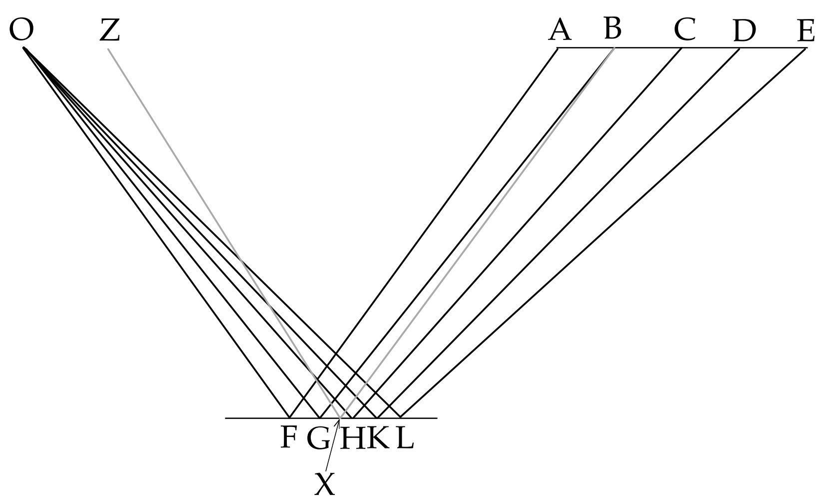


figure 4.5.1

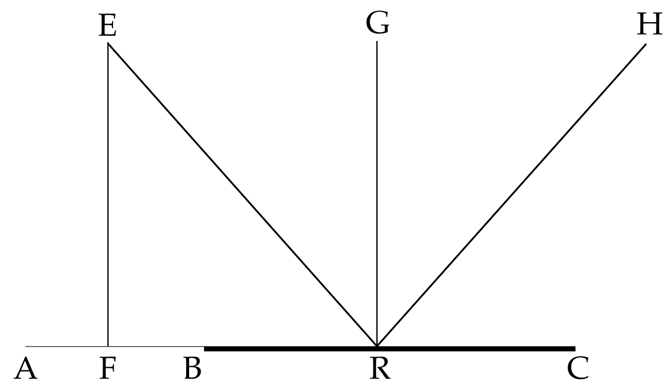


figure 4.5.2

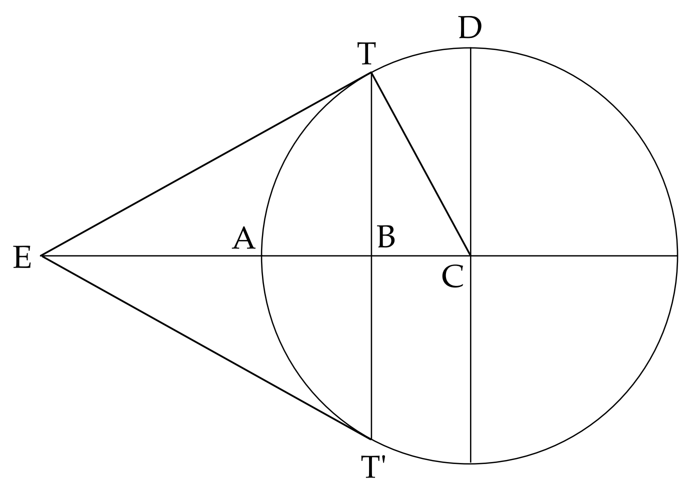


figure 4.5.3

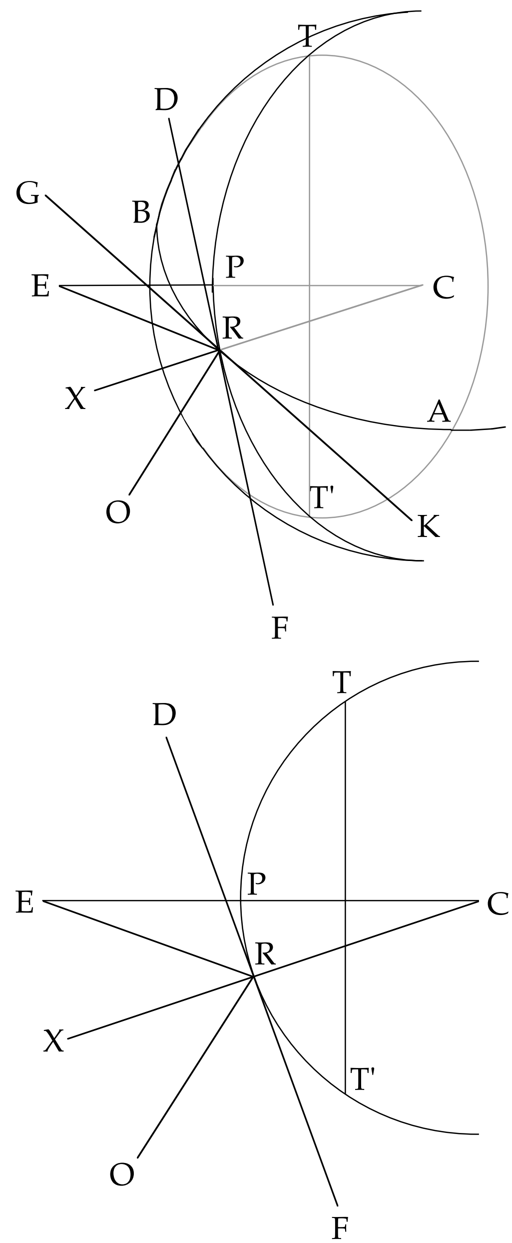


figure 4.5.4

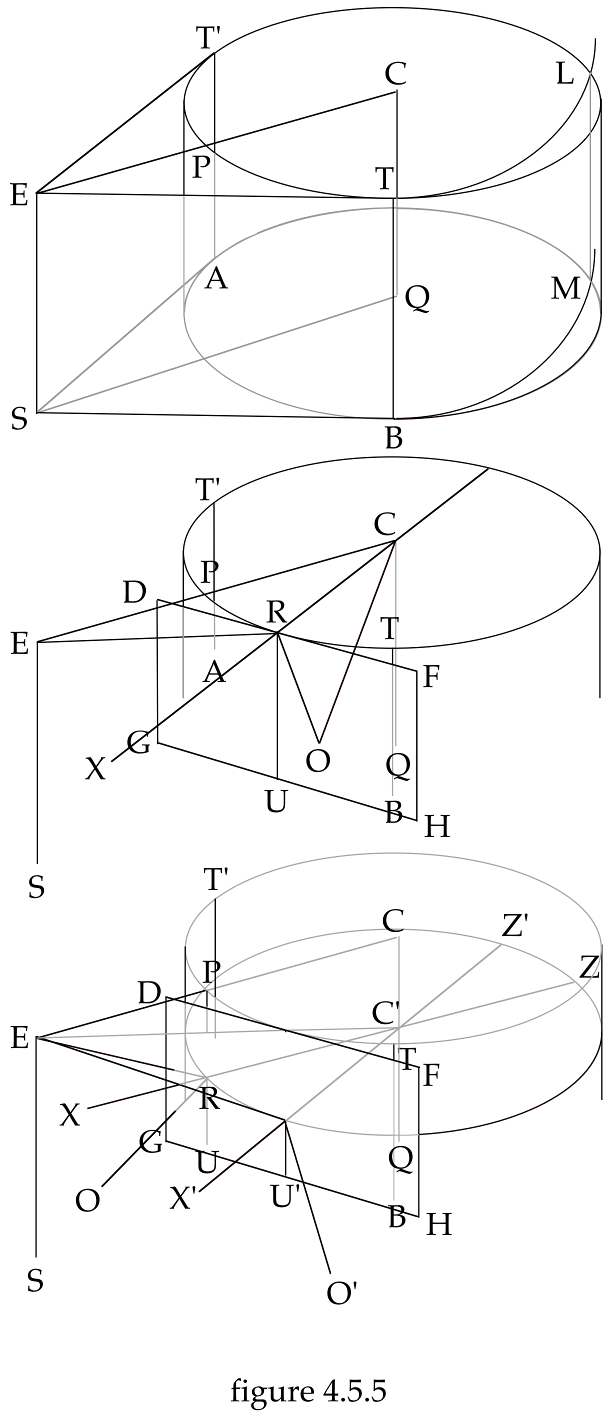

T'
C
L
P
T
E
A
Q
M
S
B
T'
C
P
D
R
T
E
A
F
G
X
O
Q
U
B
H
S
T'
C
Z'
P
Z
D
C'
E
T
F
X
R
G
U
Q
O
X'
U'
B
H
S
O'

figure 4.5.5

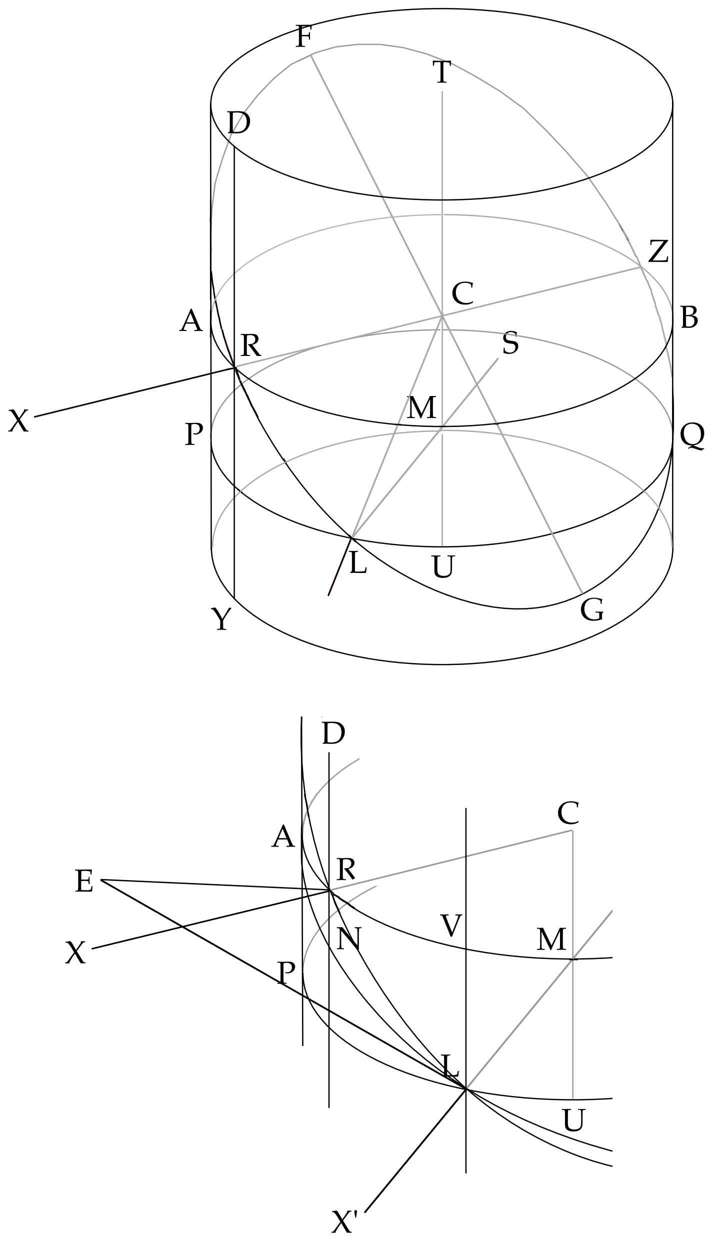


figure 4.5.6

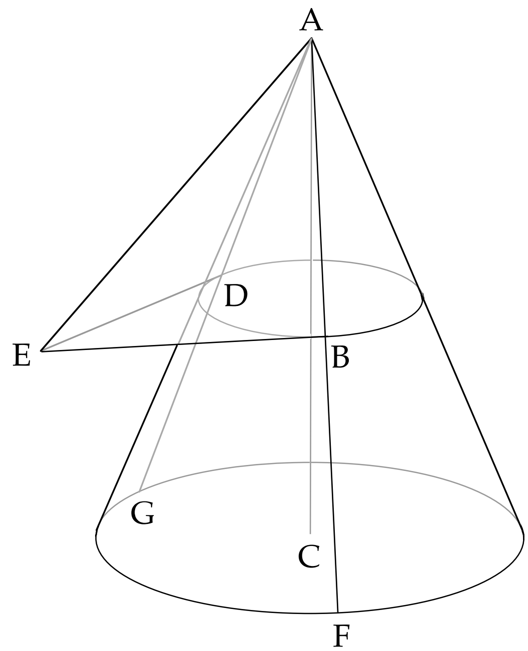


figure 4.5.7

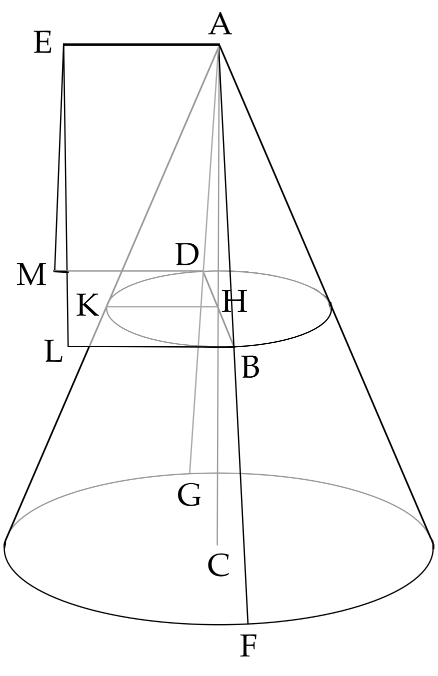


figure 4.5.8

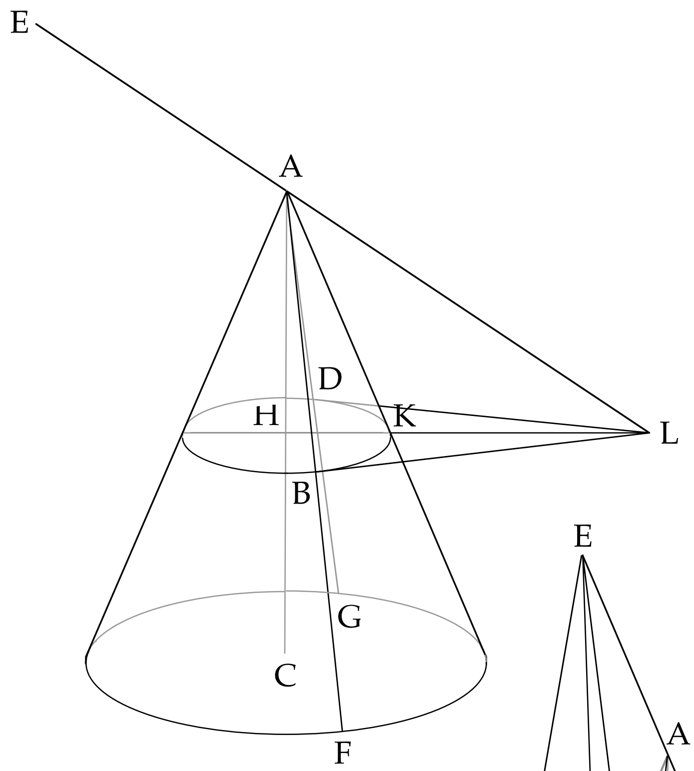


figure 4.5.9

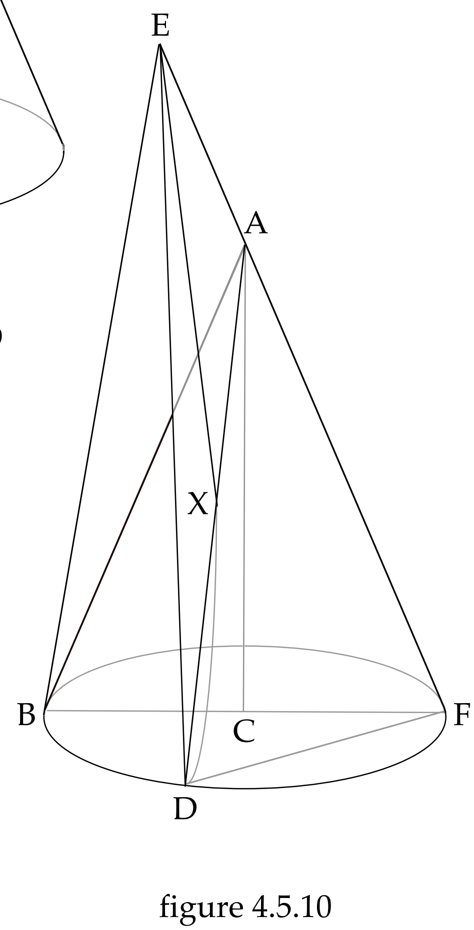


figure 4.5.10

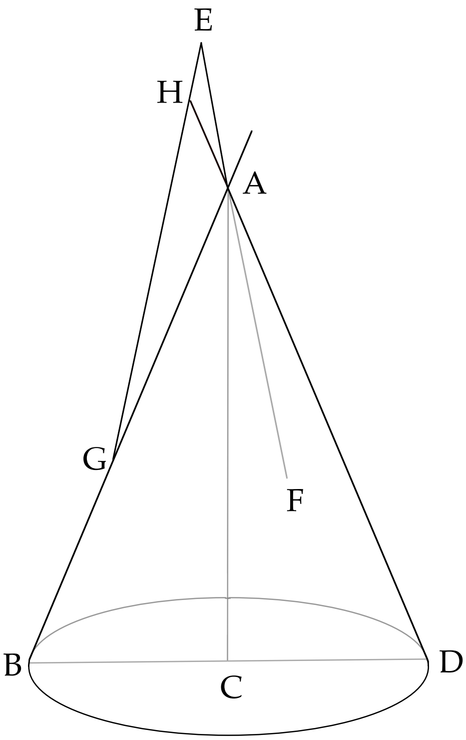


figure 4.5.11

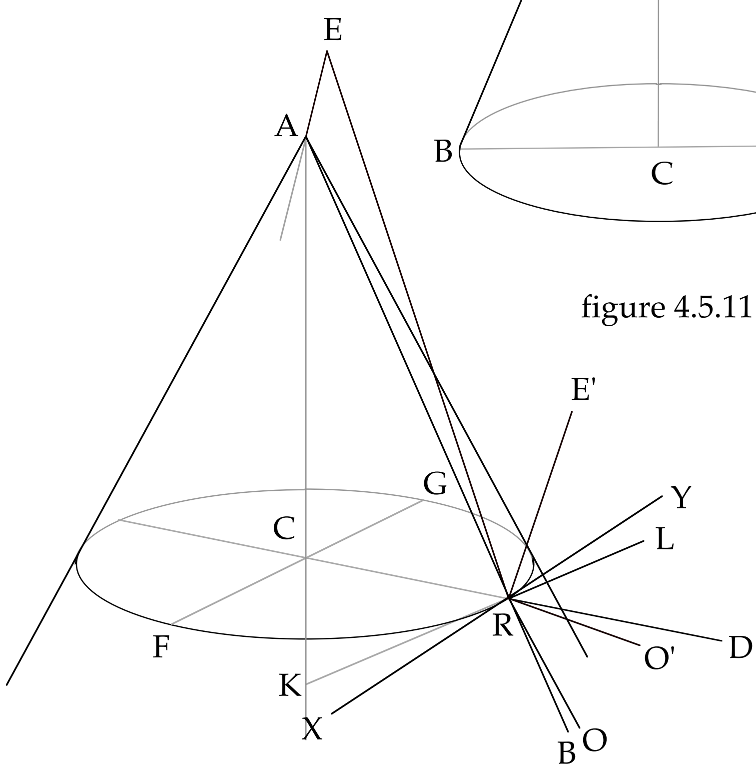


figure 4.5.12

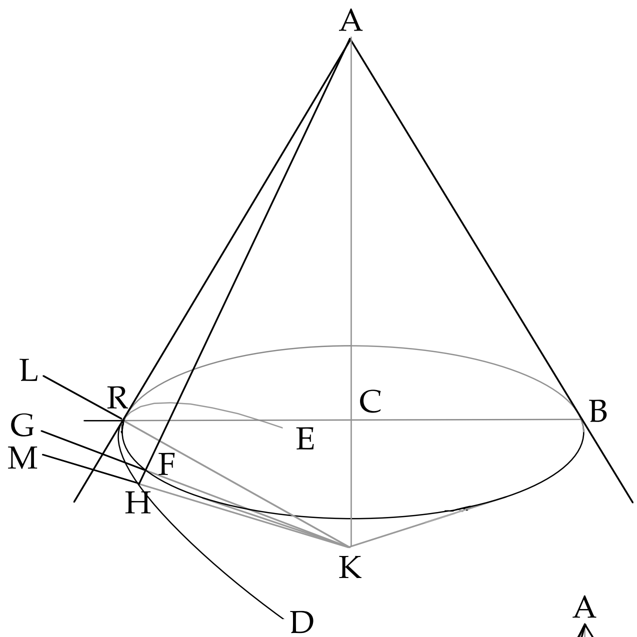


figure 4.5.13

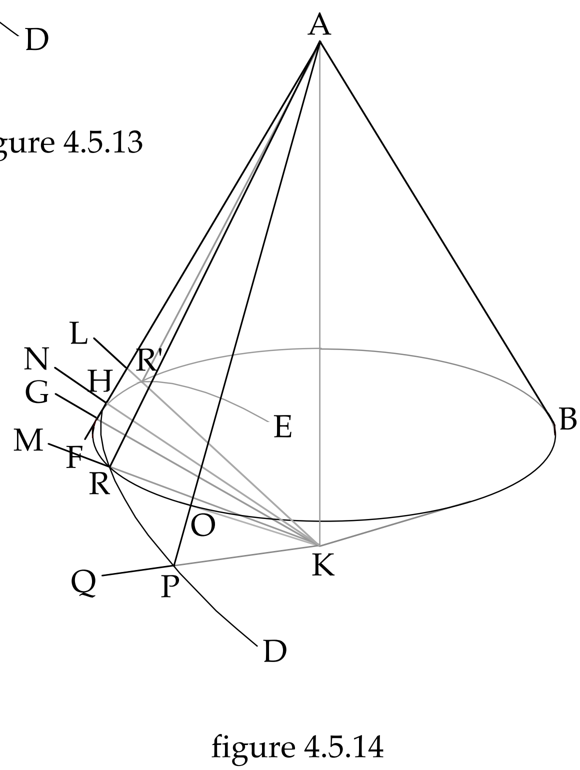


figure 4.5.14

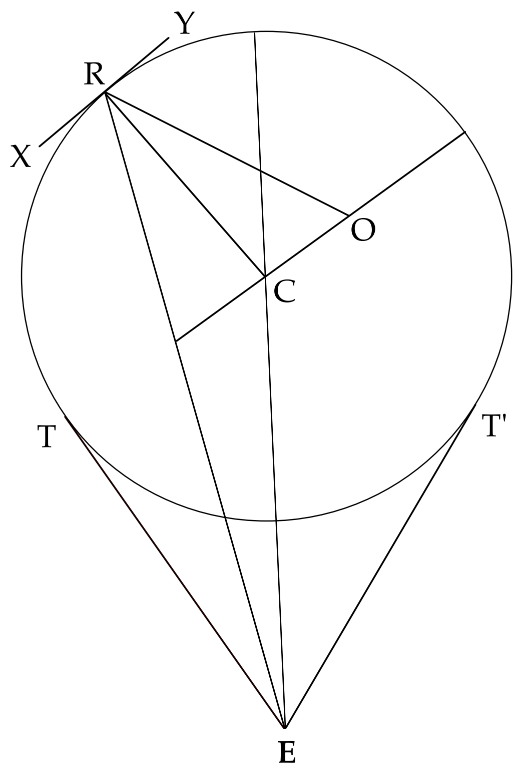


figure 4.5.15

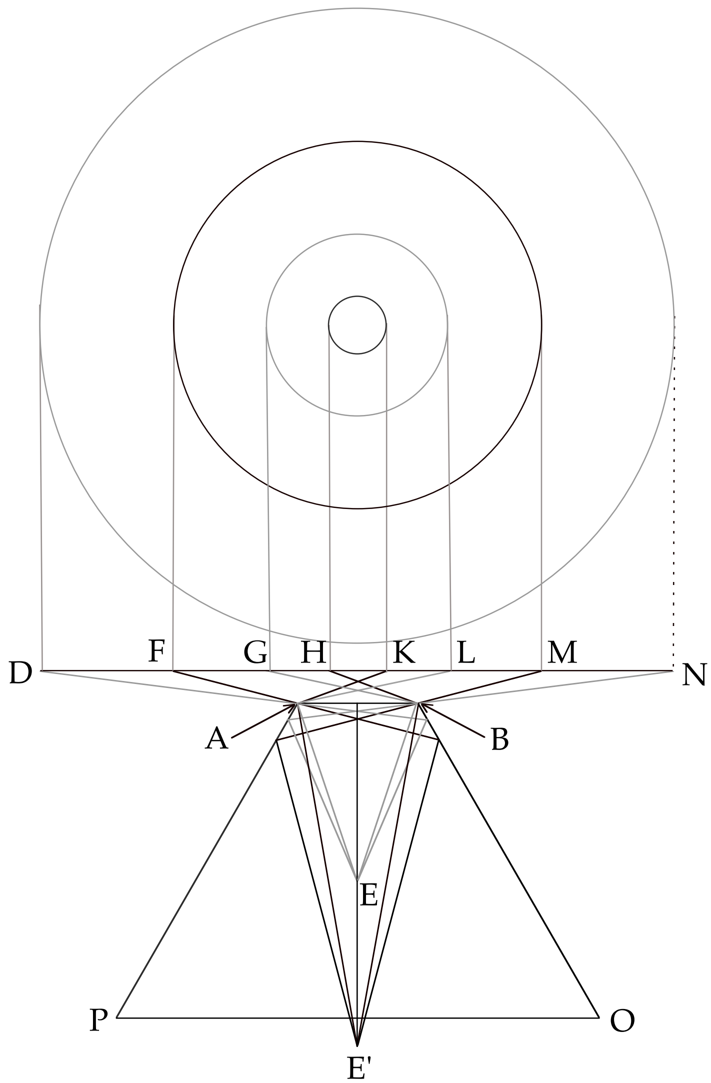


figure 4.5.16

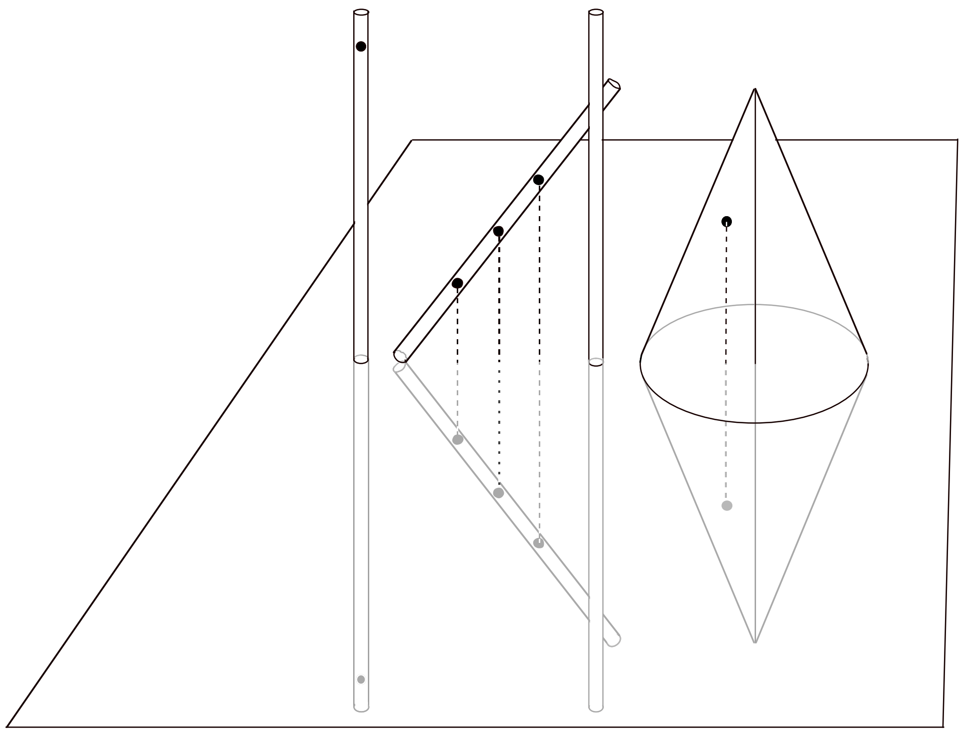


figure 5.1

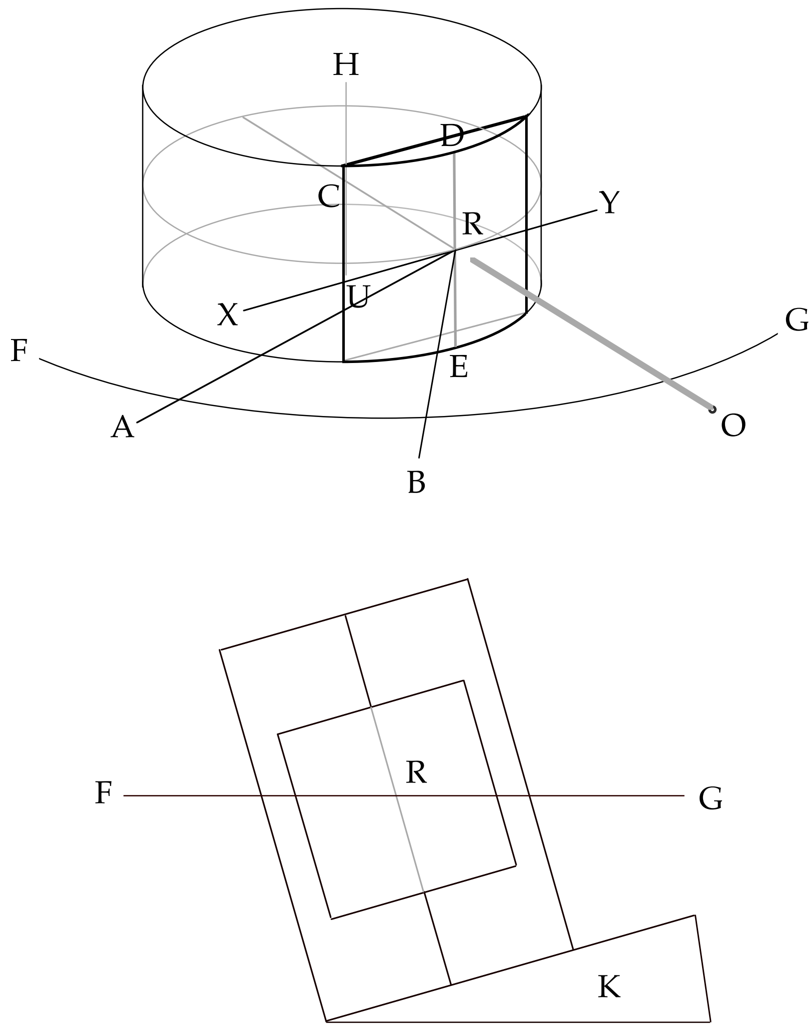


figure 5.2

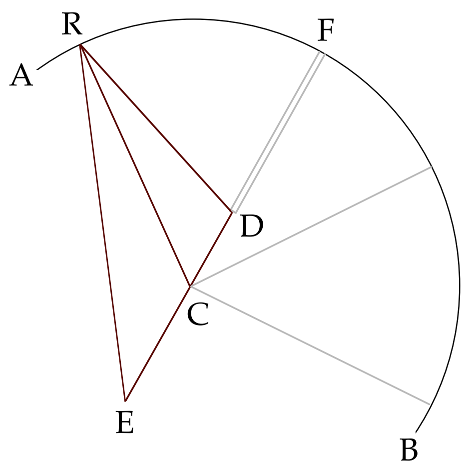


figure 5.3

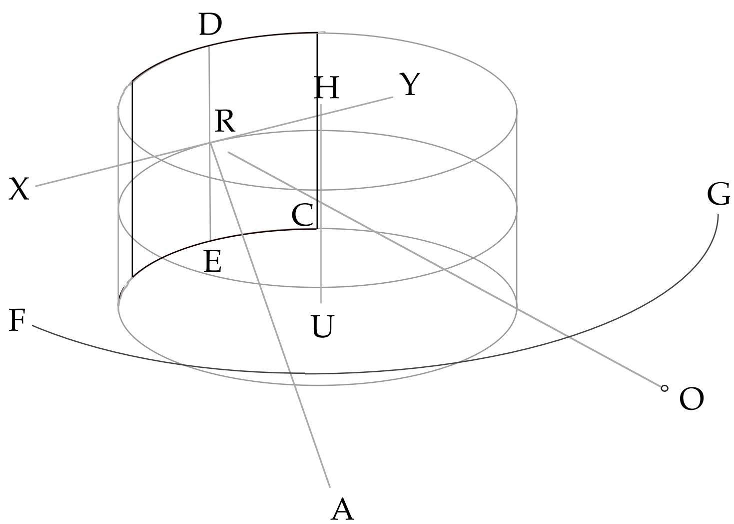


figure 5.4

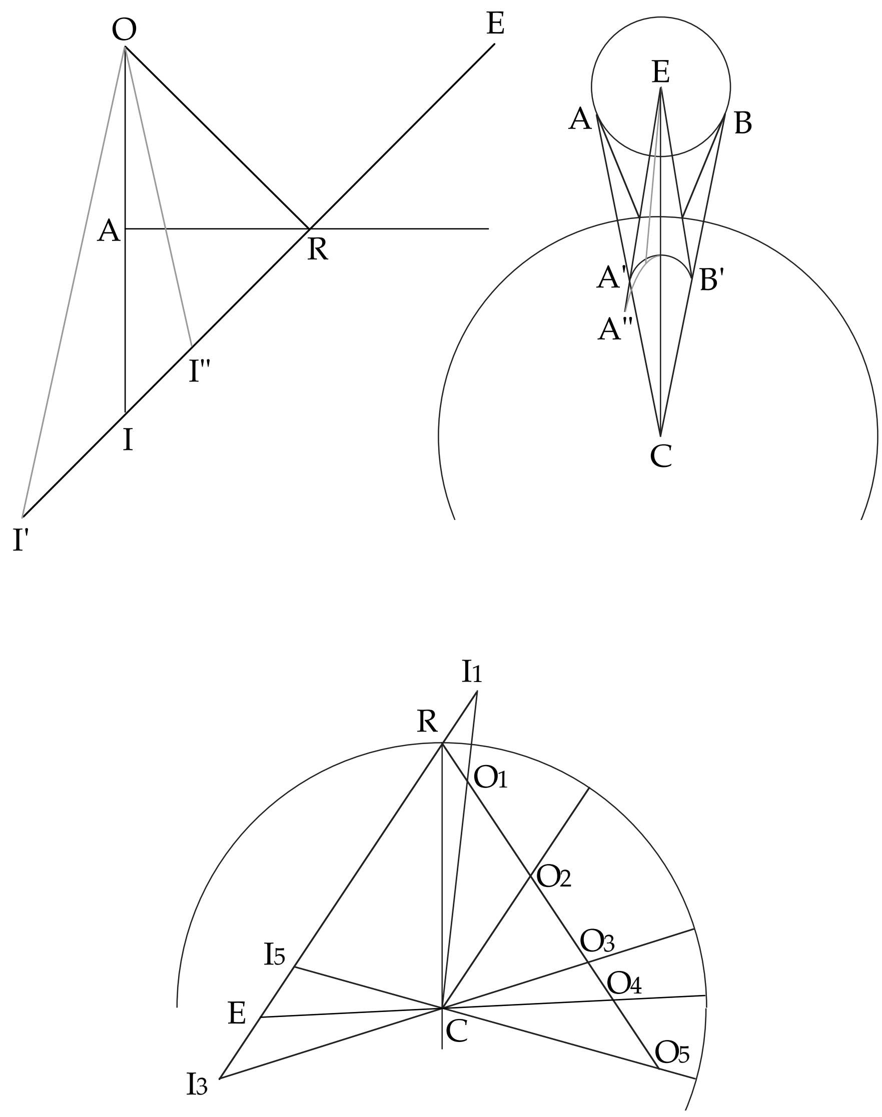


figure 5.5

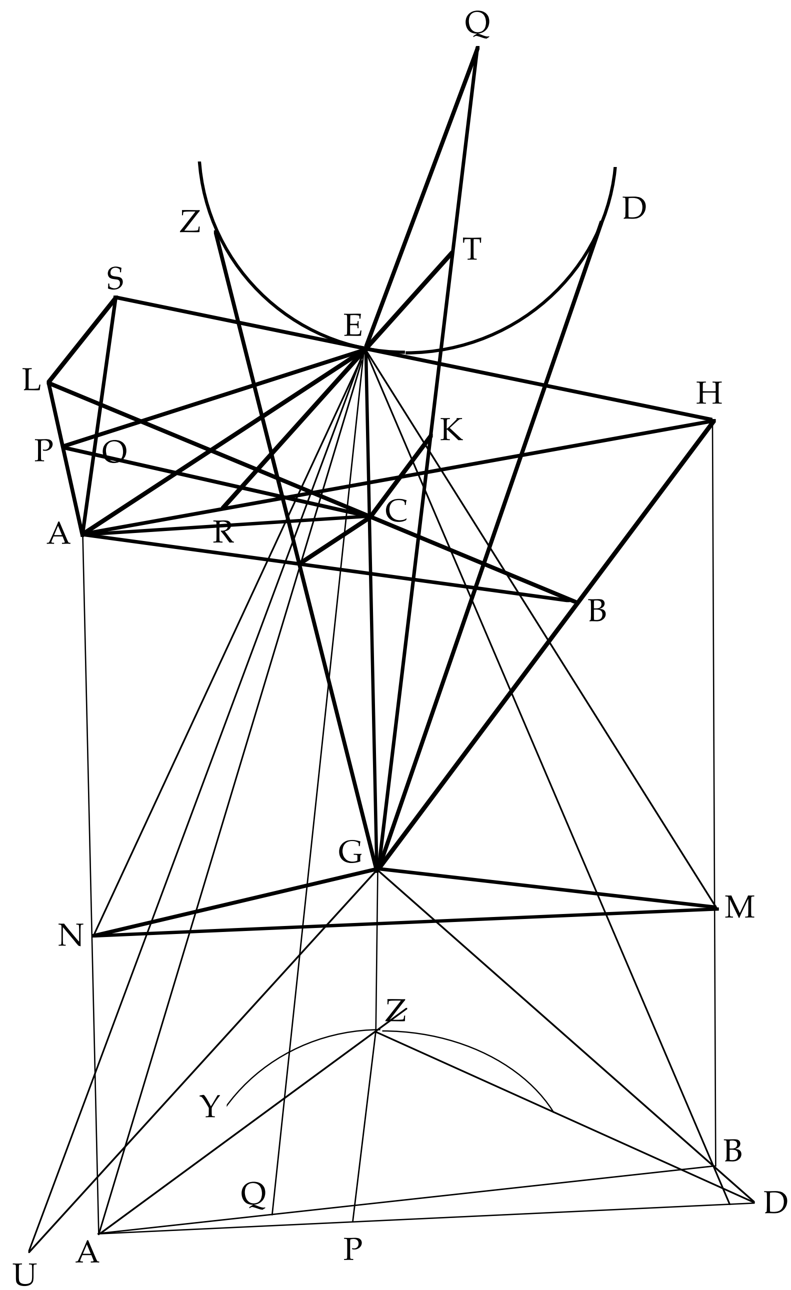


figure 5.6

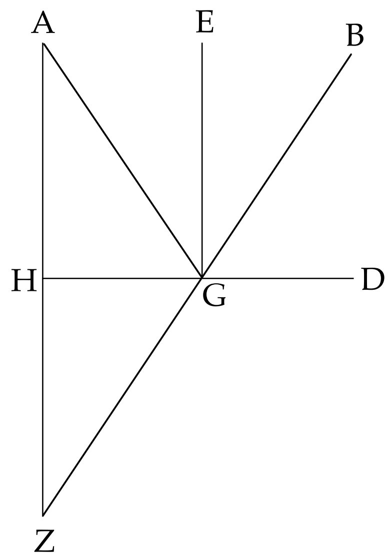


figure 5.2.1

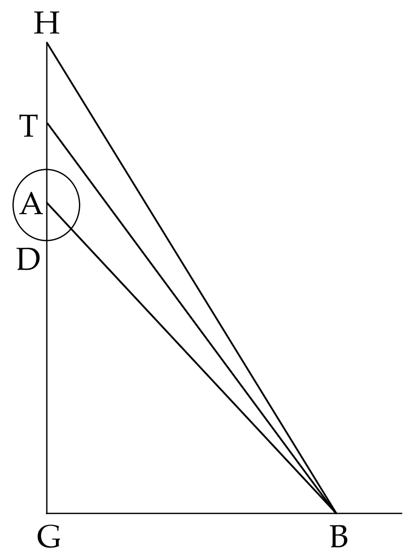


figure 5.2.2

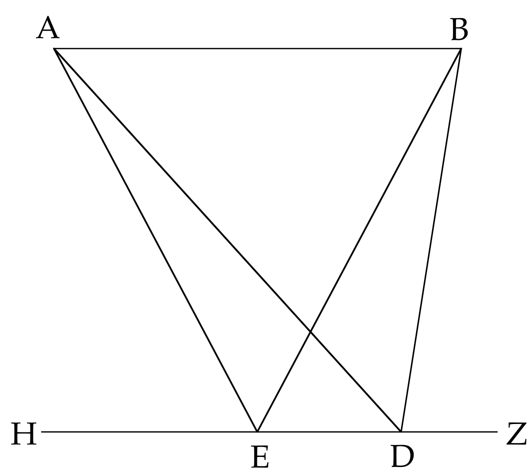


figure 5.2.3

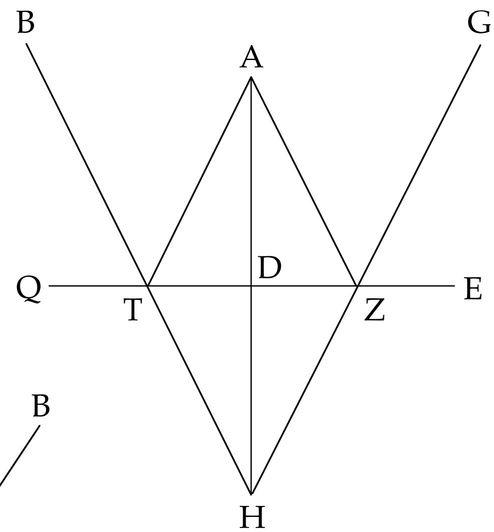


figure 5.2.4

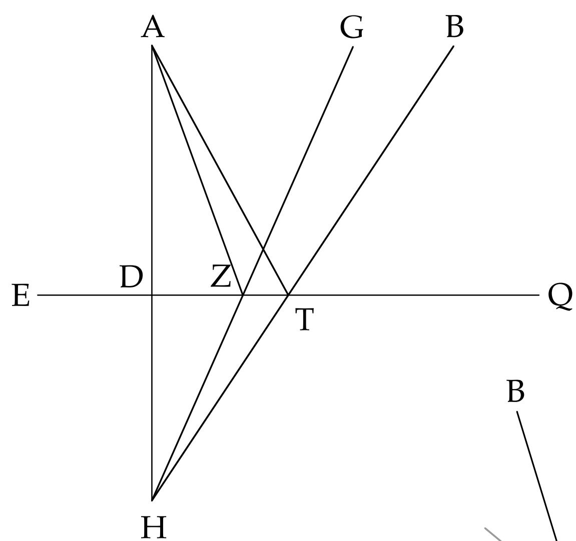


figure 5.2.4a

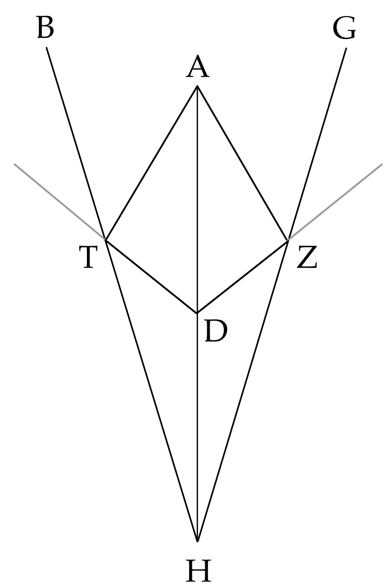


figure 5.2.4b

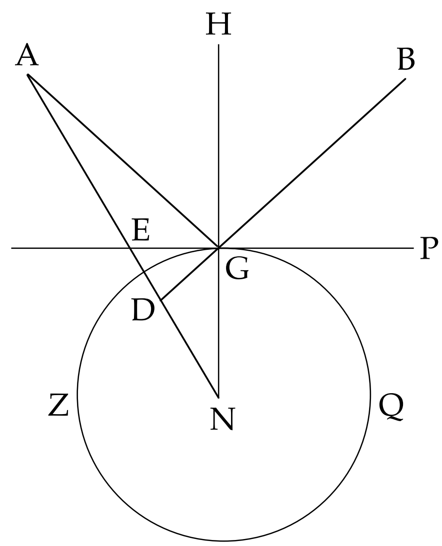


figure 5.2.5

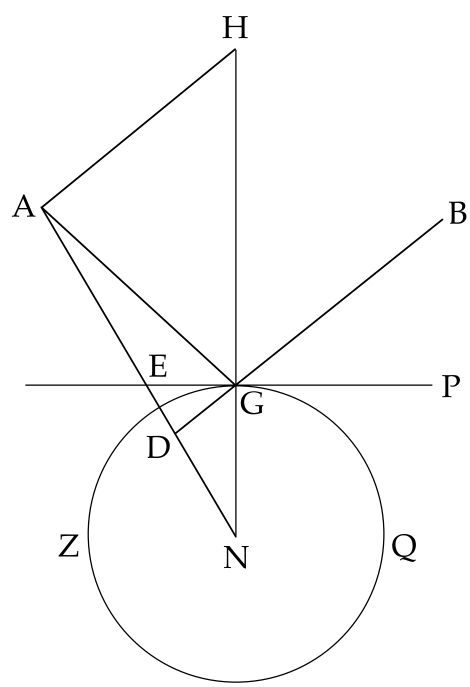


figure 5.2.6

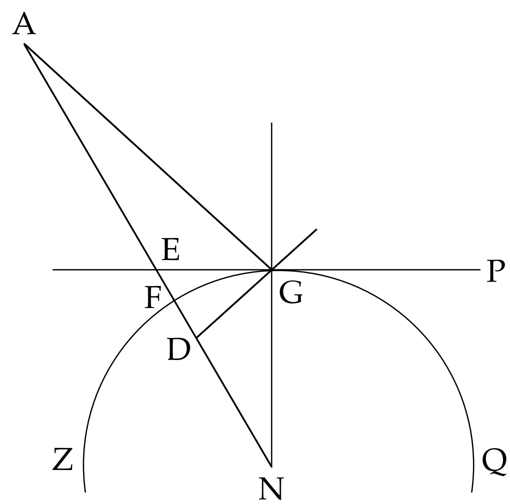


figure 5.2.7

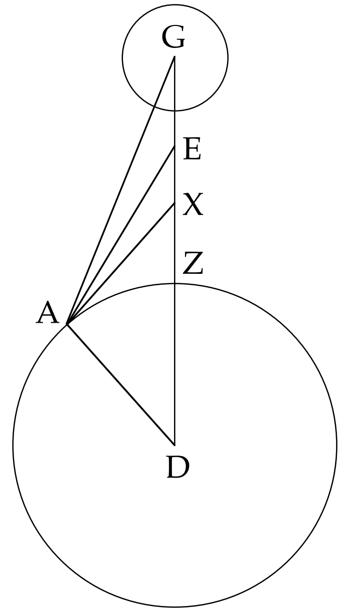


figure 5.2.8

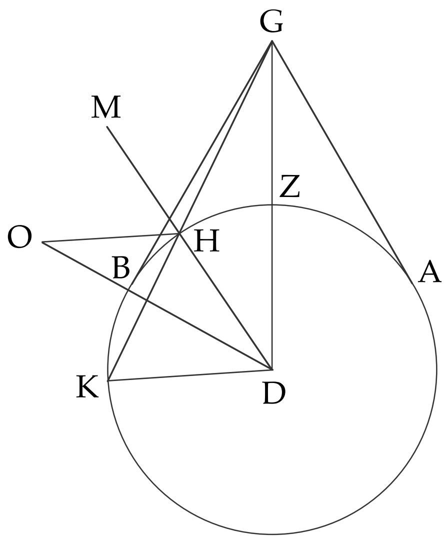


figure 5.2.9

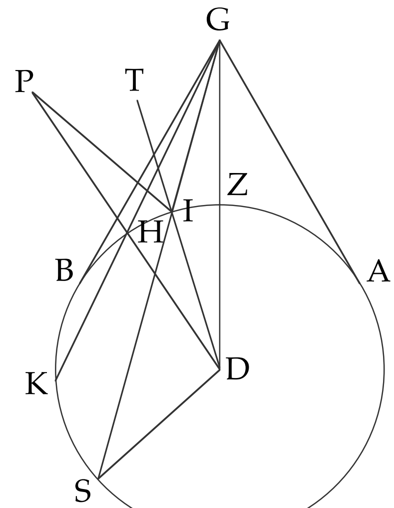


figure 5.2.9a

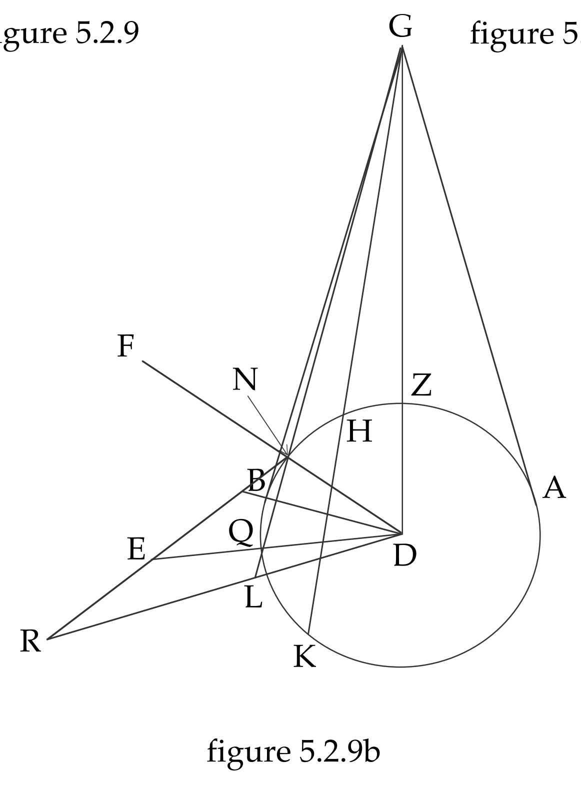


figure 5.2.9b

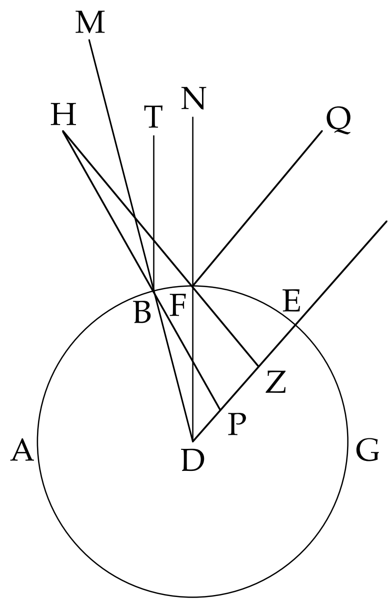


figure 5.2.10

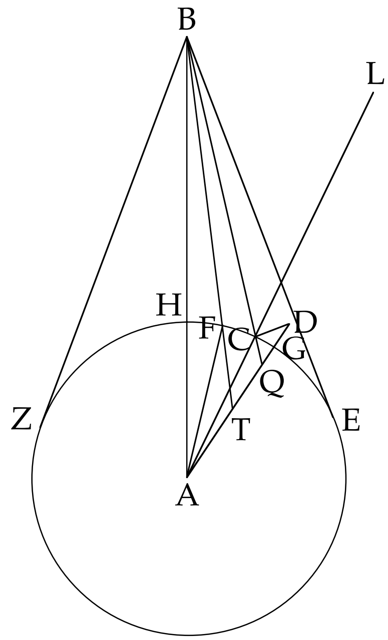


figure 5.2.11

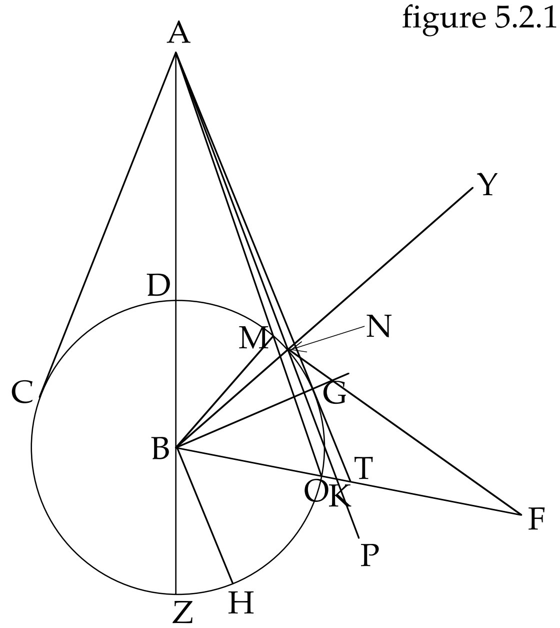


figure 5.2.12

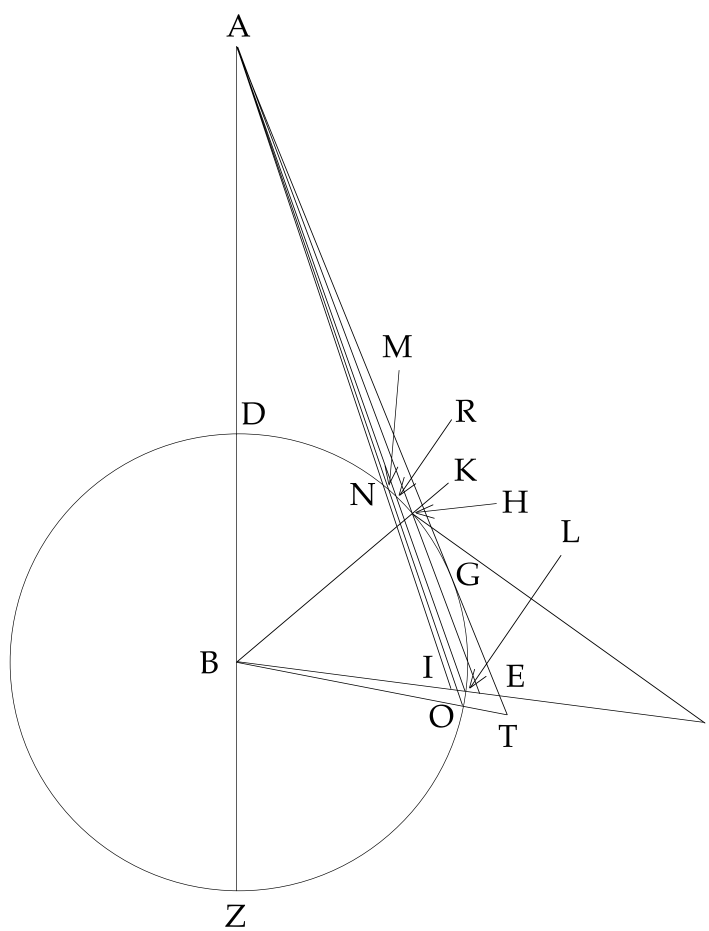


figure 5.2.13

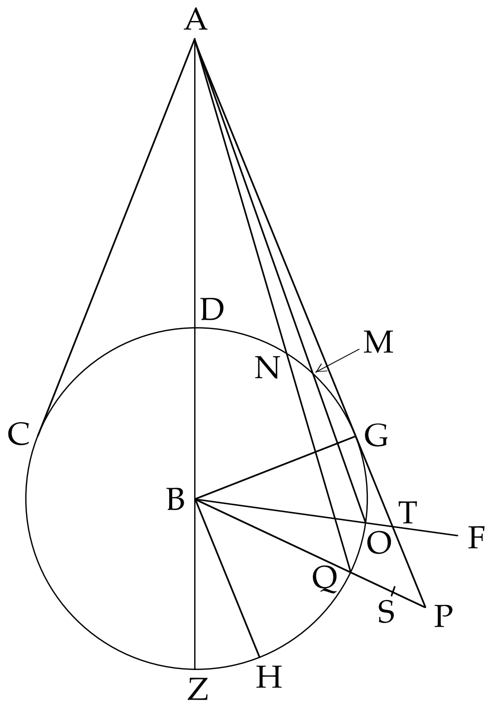


figure 5.2.14

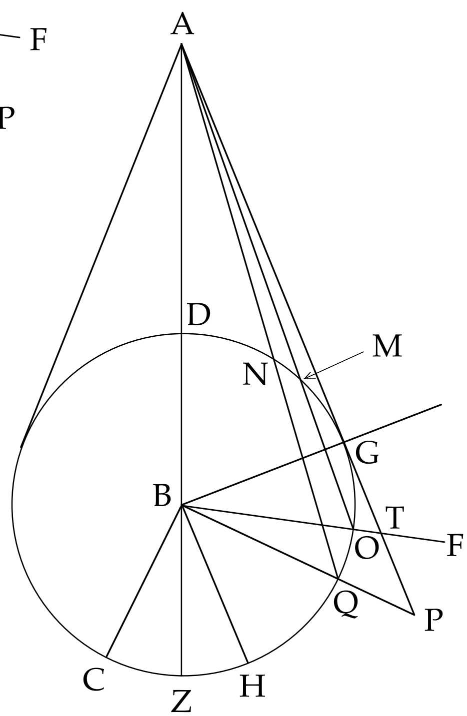


figure 5.2.15

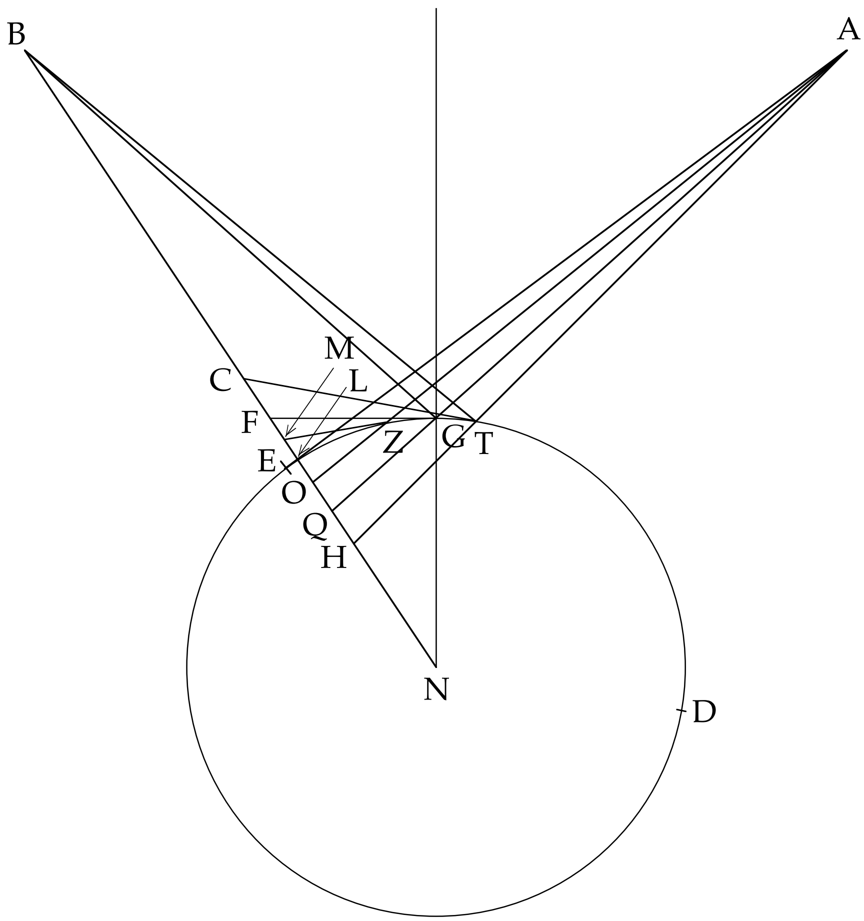


figure 5.2.16

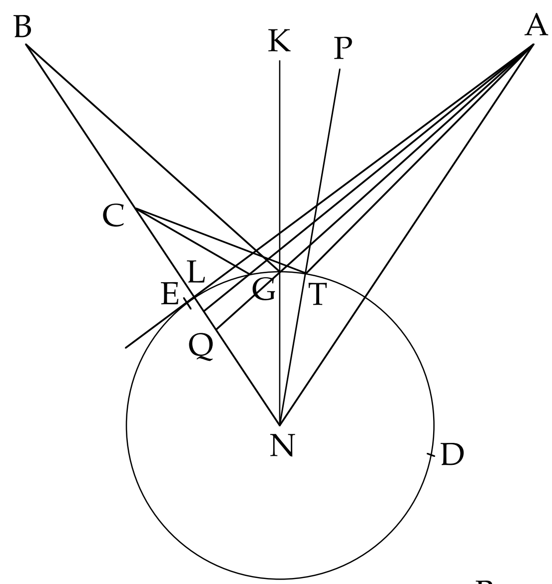


figure 5.2.17

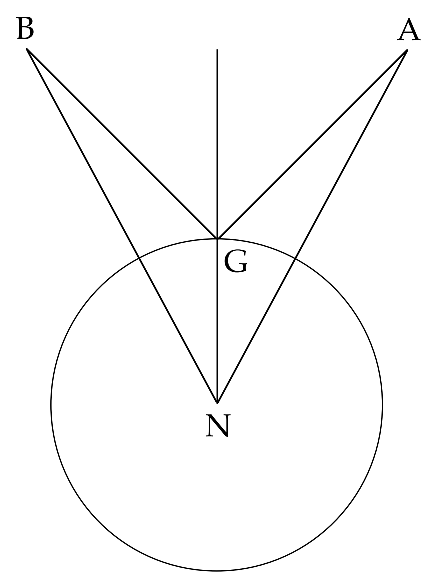


figure 5.2.18

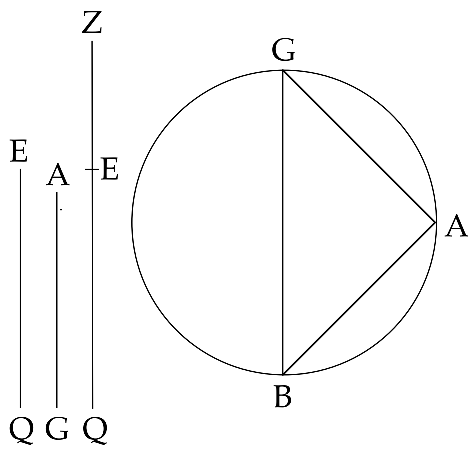


figure 5.2.19

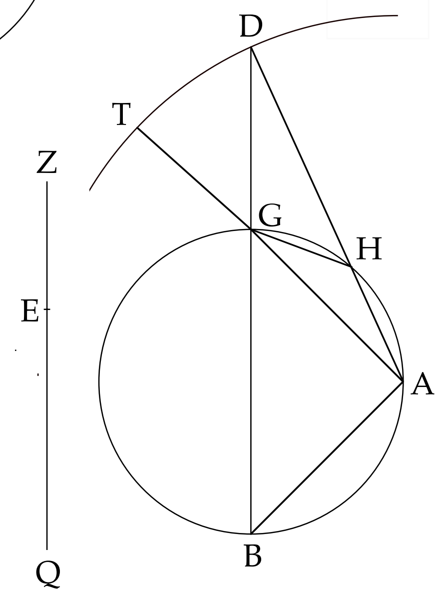


figure 5.2.19a

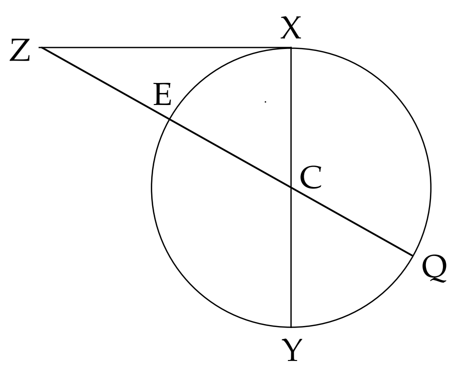


figure 5.2.19b

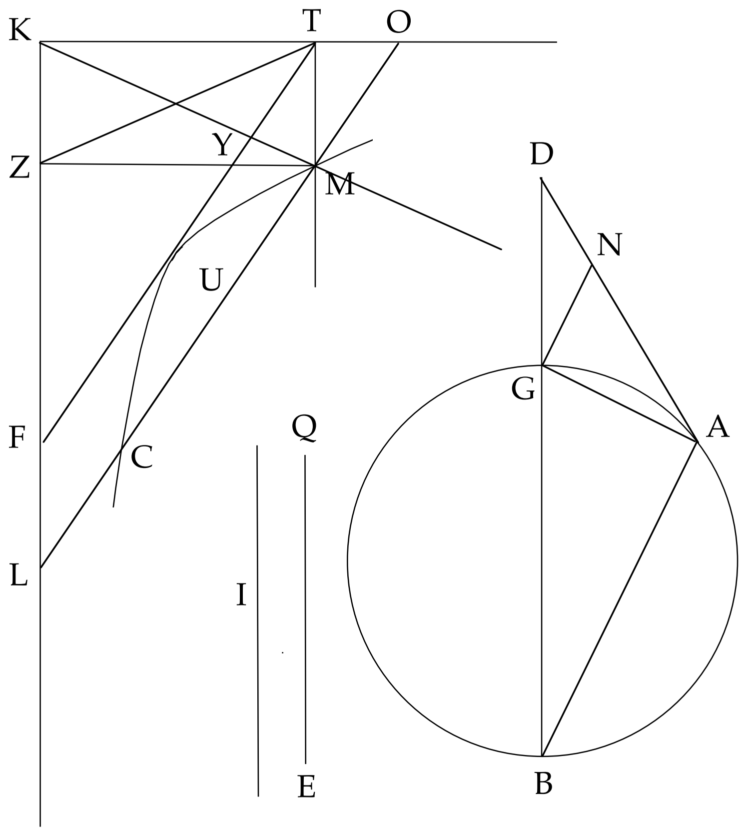


figure 5.2.19c

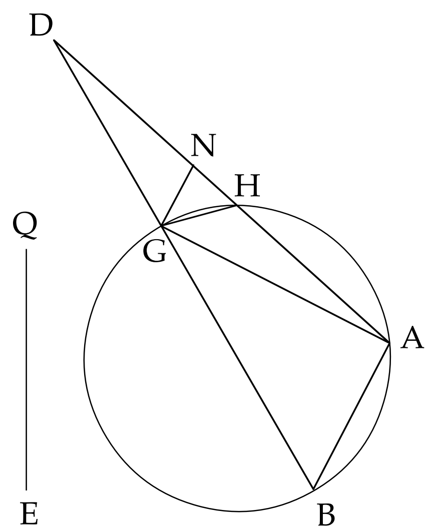


figure 5.2.19d

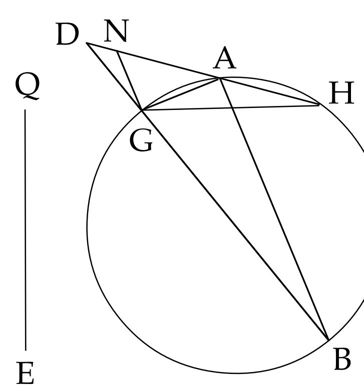


figure 5.2.19e

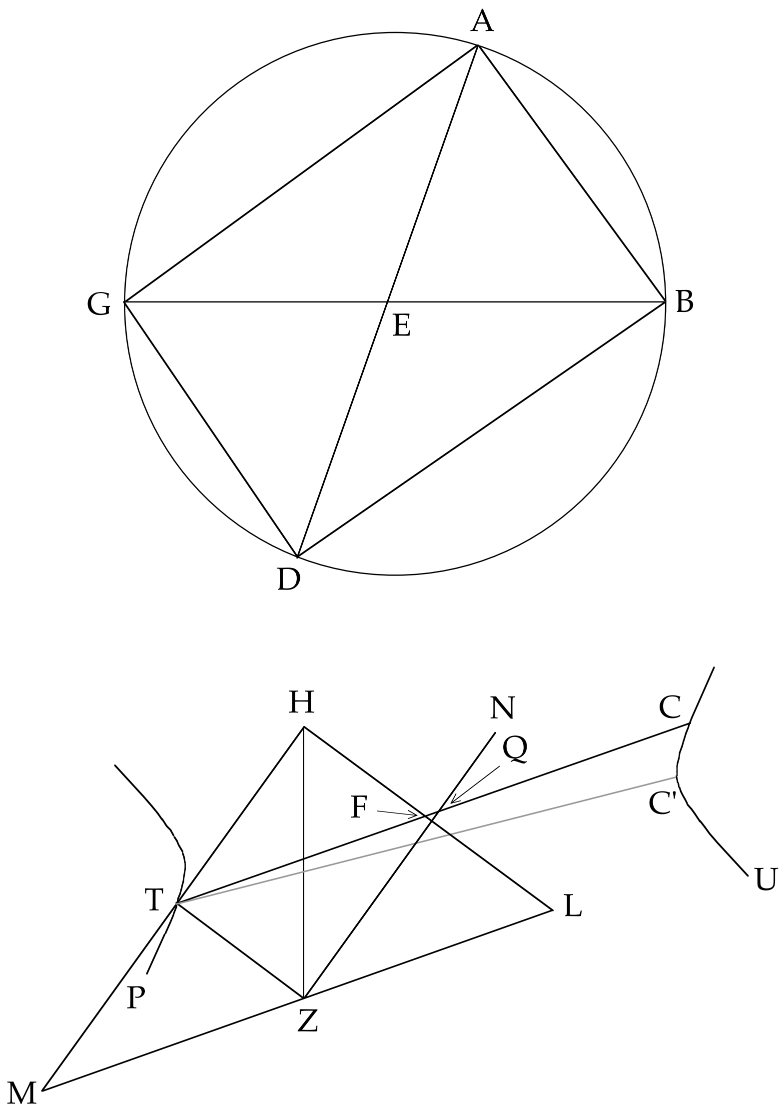


figure 5.2.20

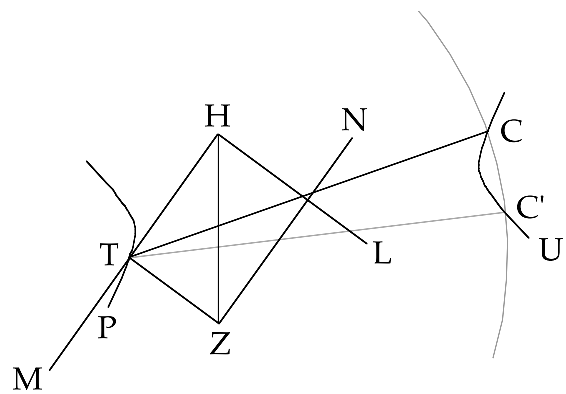


figure 5.2.20a

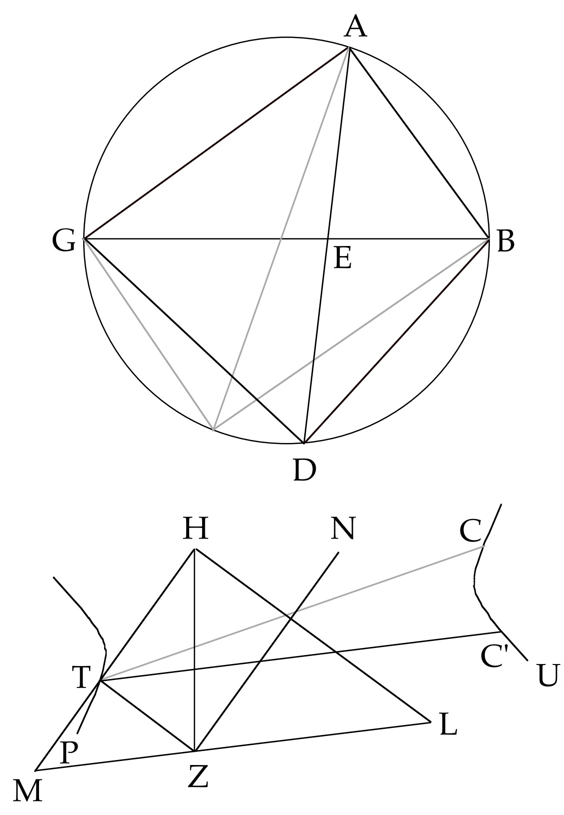


figure 5.2.20b

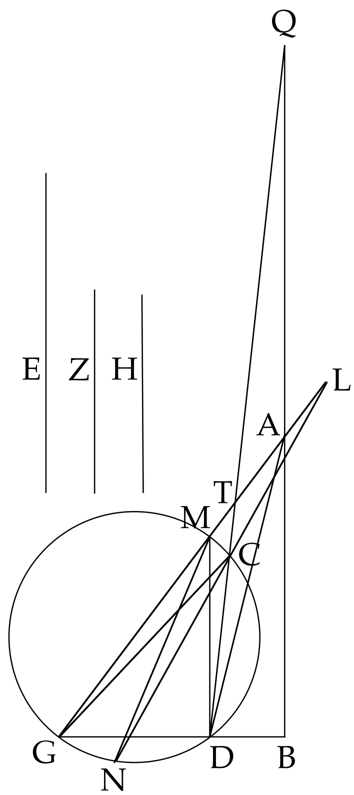


figure 5.2.21

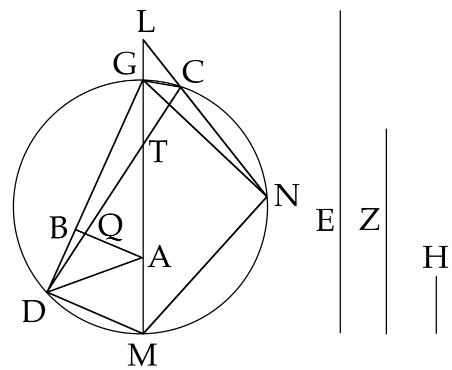


figure 5.2.21a

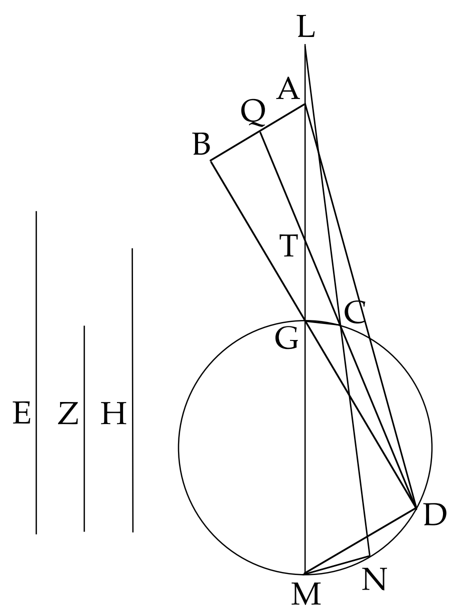


figure 5.2.21b

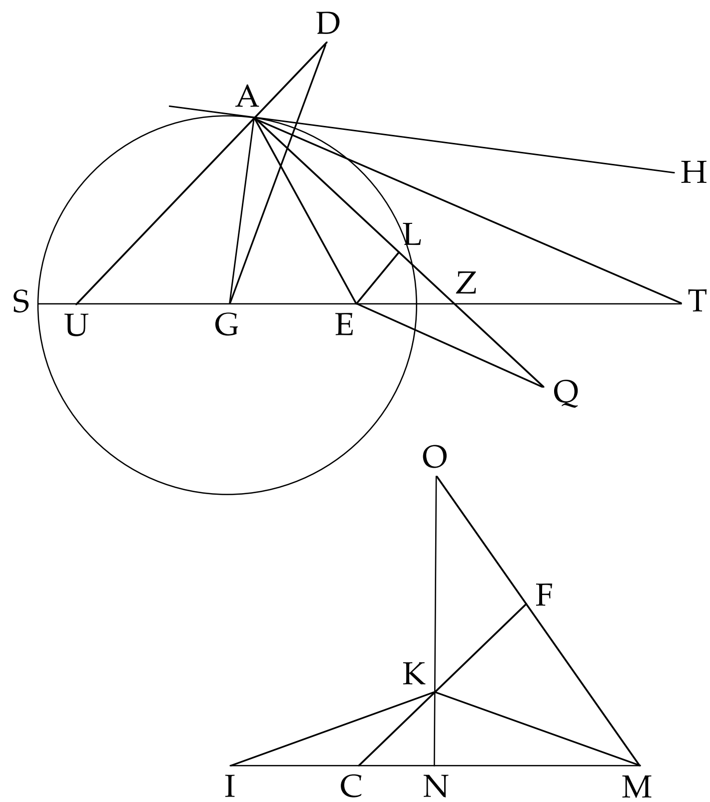


figure 5.2.22

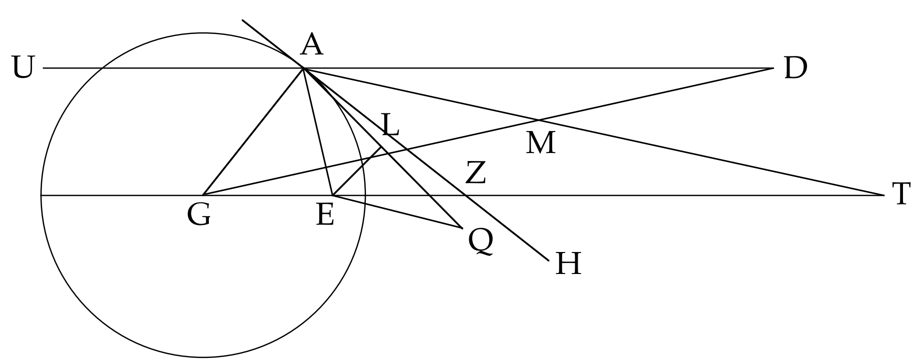
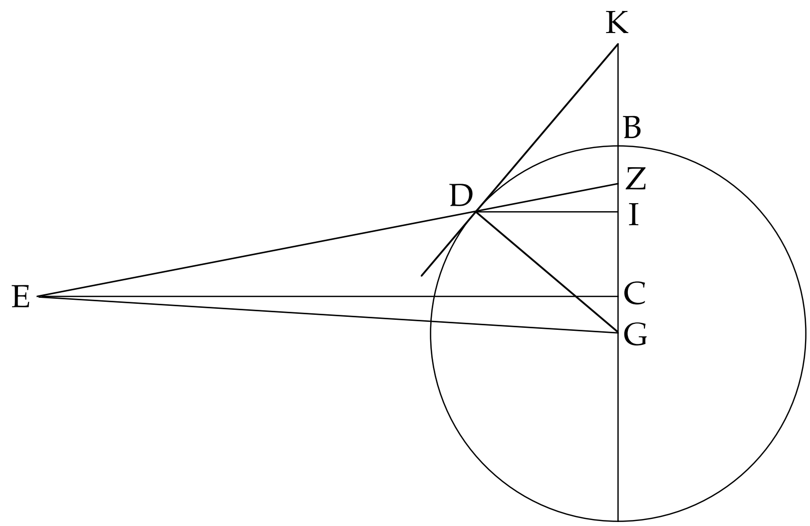


figure 5.2.22a



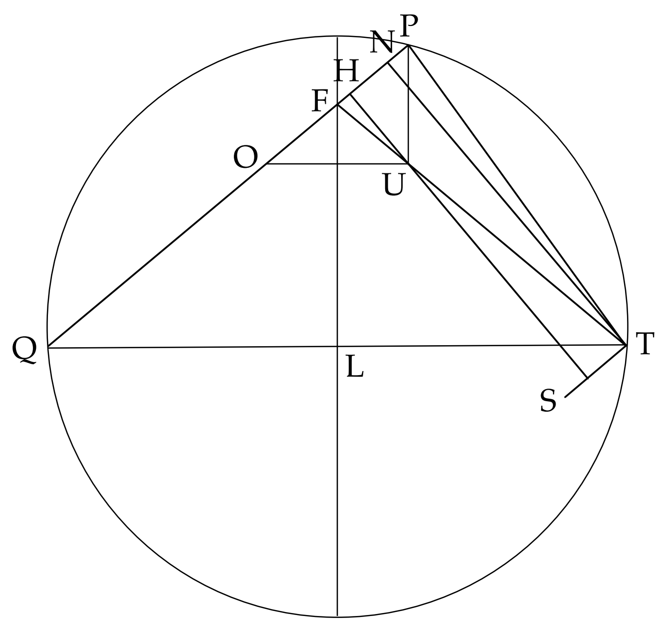


figure 5.2.23

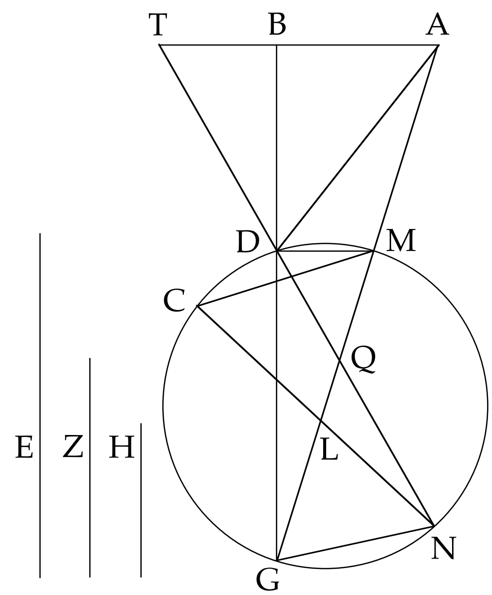


figure 5.2.24

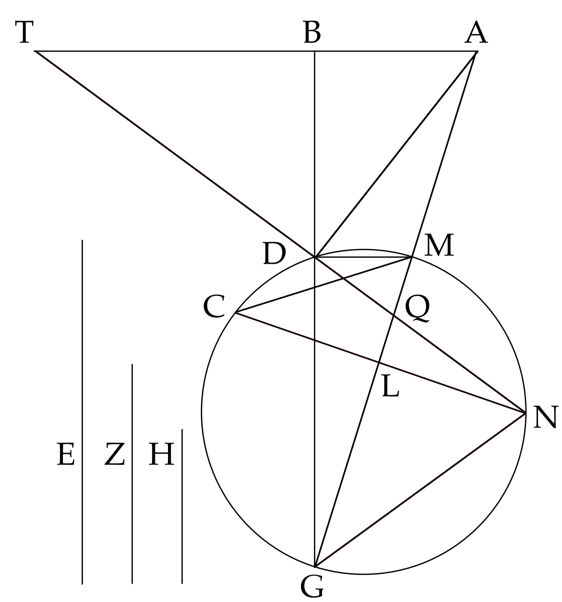


figure 5.2.24a

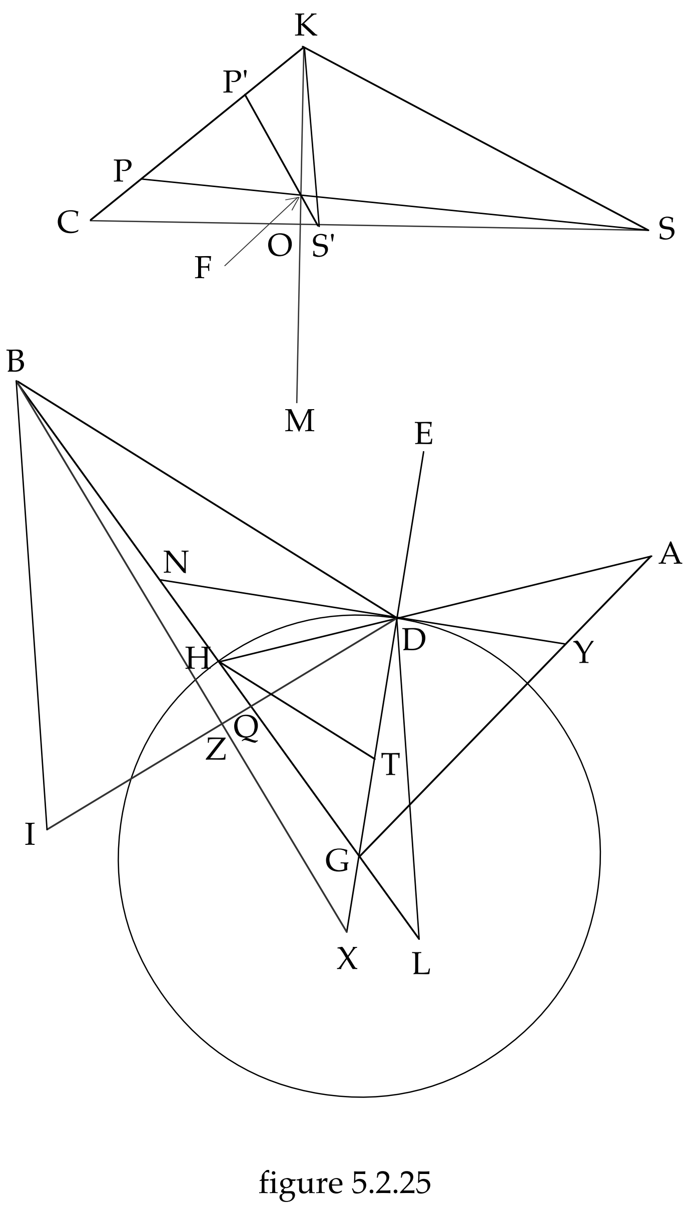


figure 5.2.25

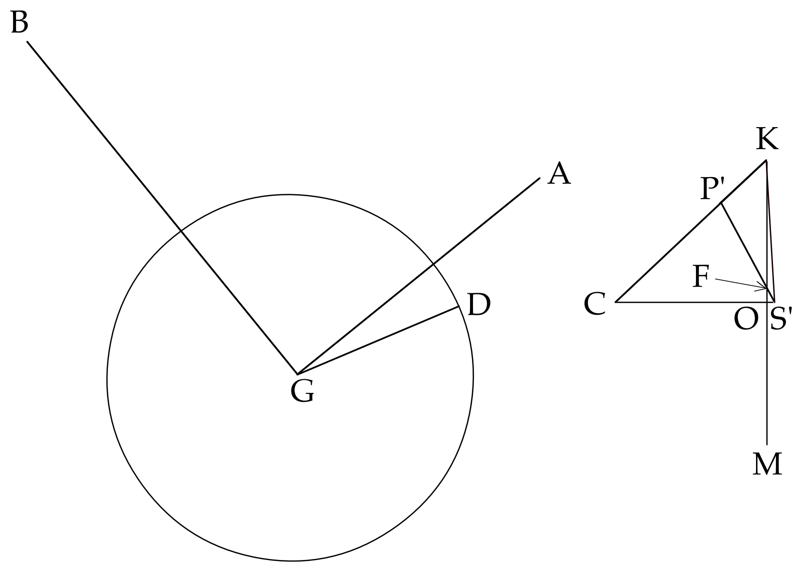


figure 5.2.25a

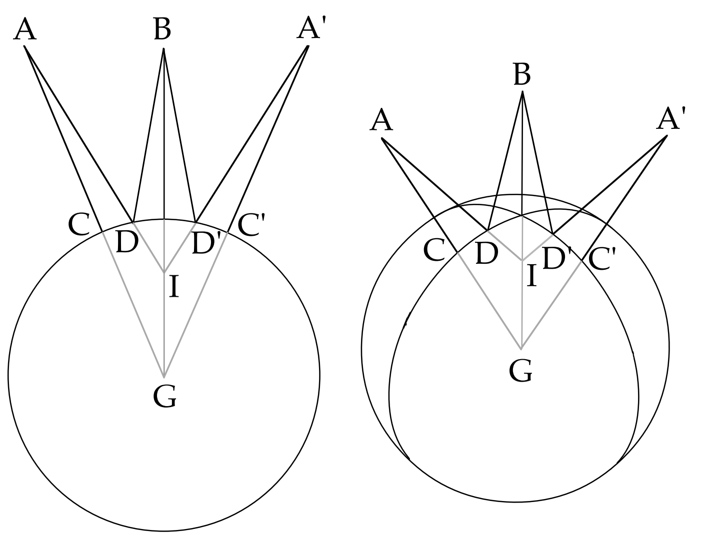


figure 5.2.25b

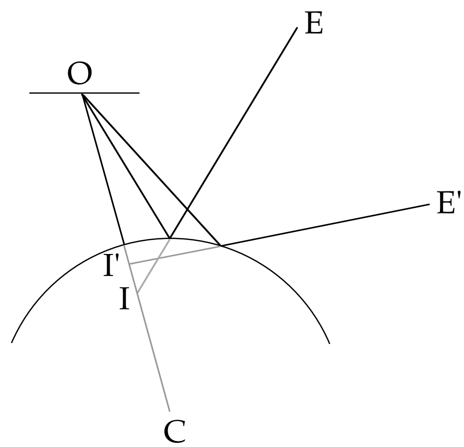


figure 5.2.25c

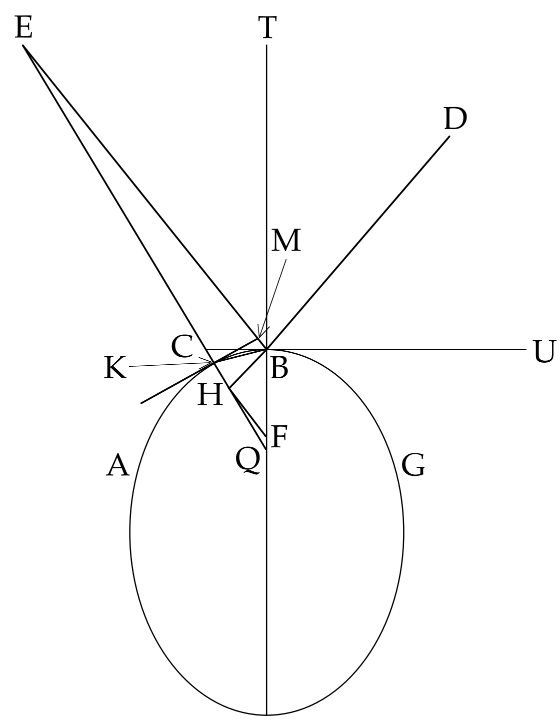


figure 5.2.26

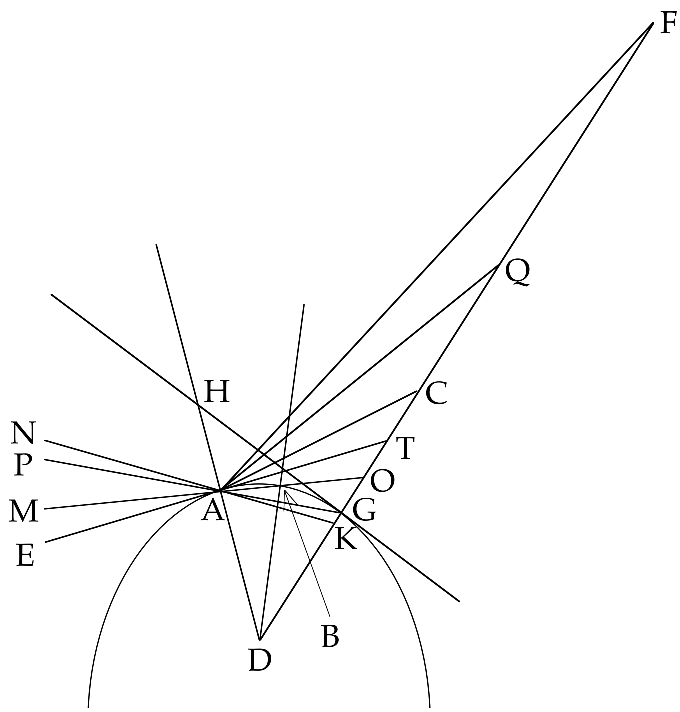


figure 5.2.27

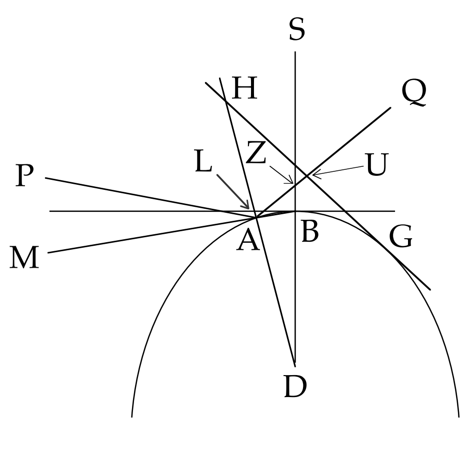


figure 5.2.27a

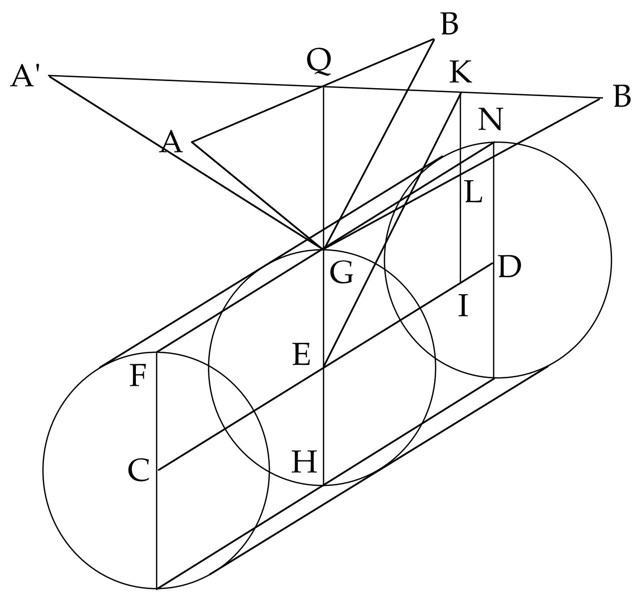


figure 5.2.28

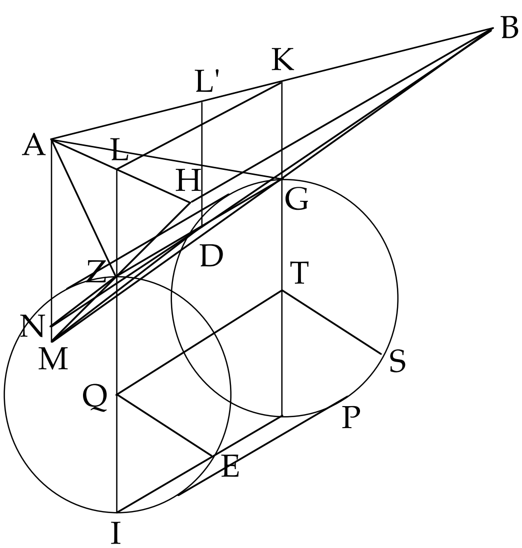


figure 5.2.28a

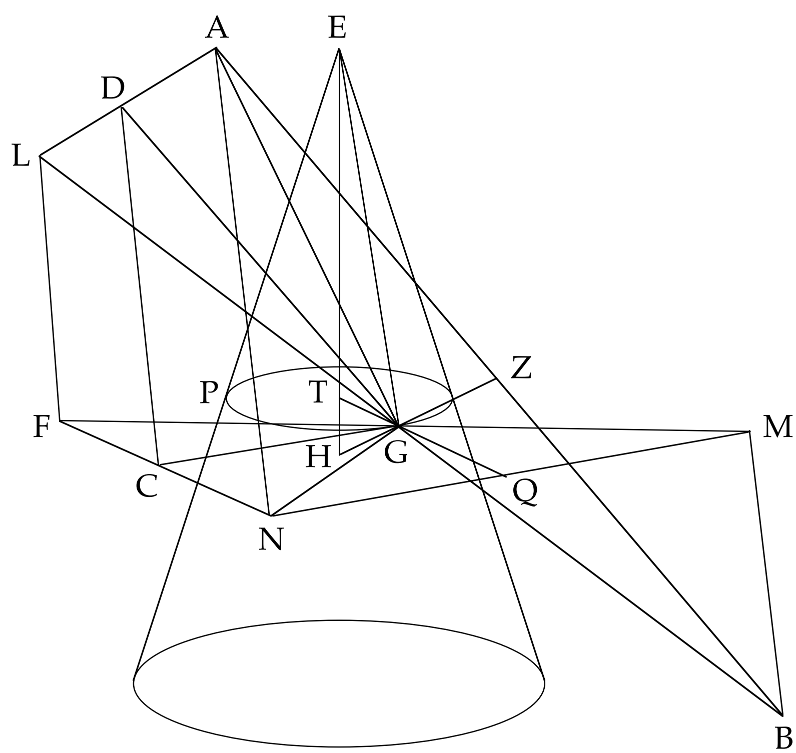


figure 5.2.30

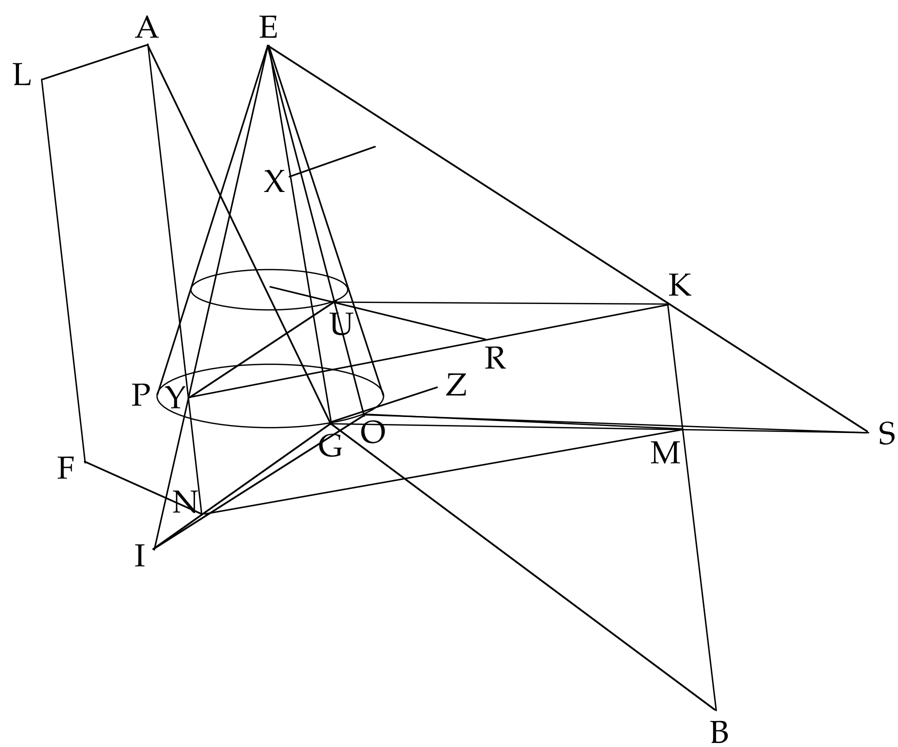


figure 5.2.30a

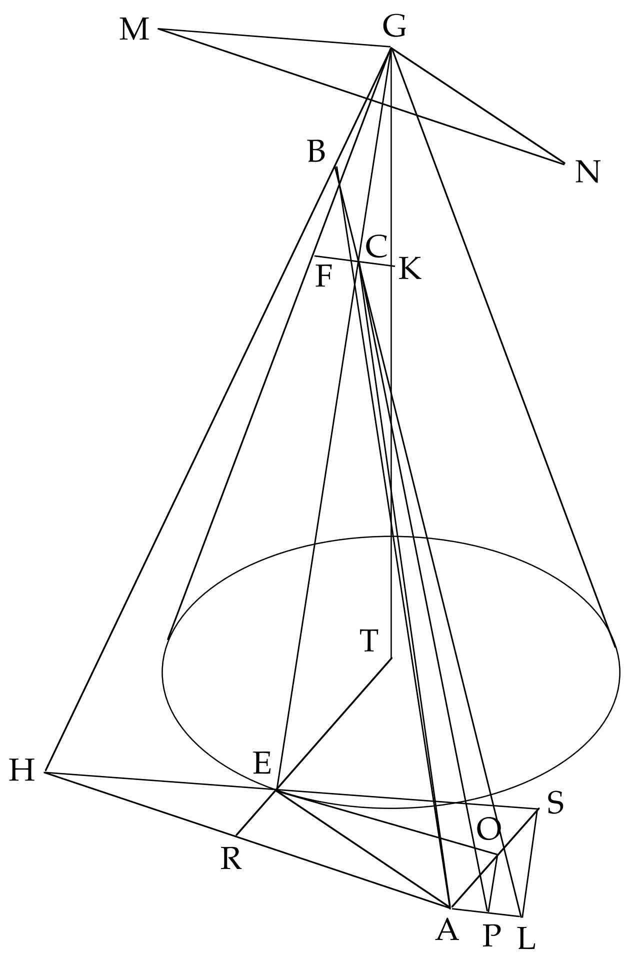


figure 5.2.31

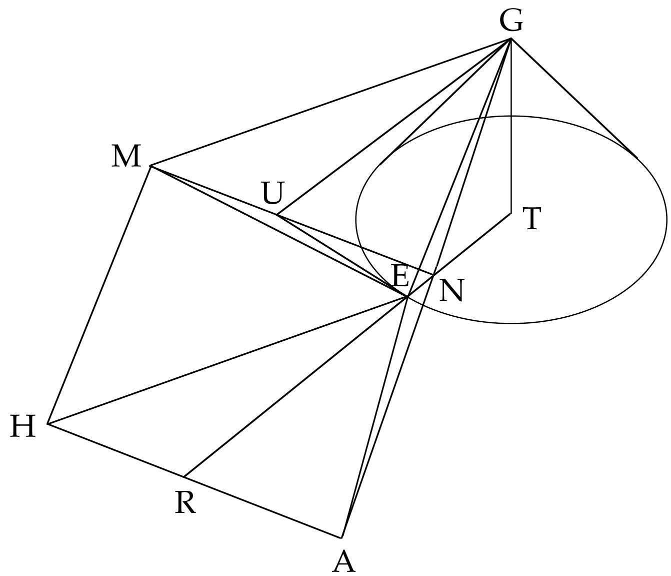


figure 5.2.31a

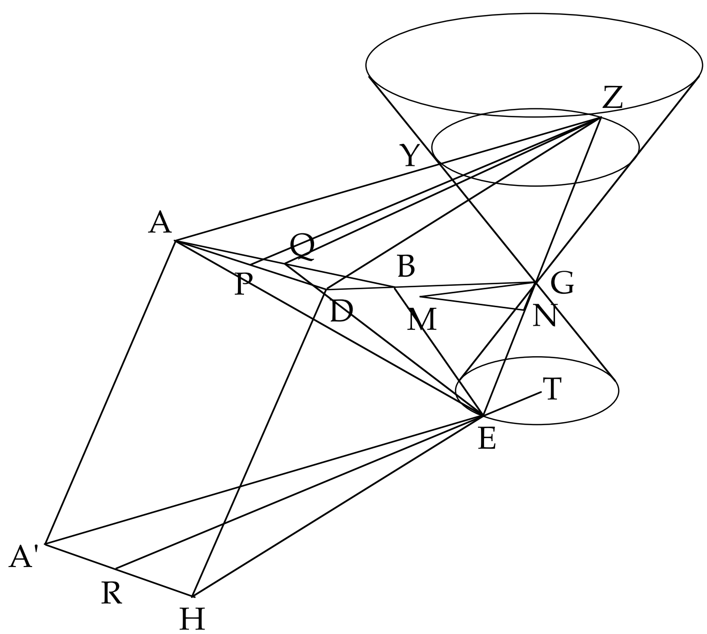


figure 5.2.31b

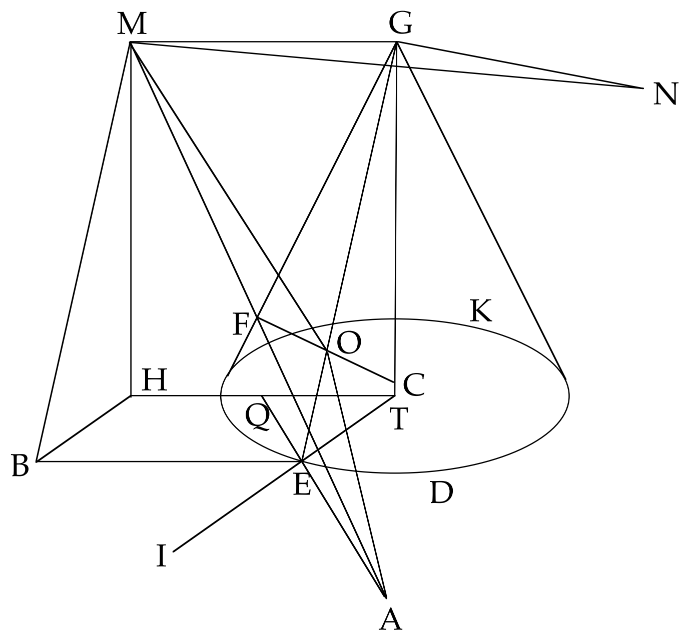


figure 5.2.31c

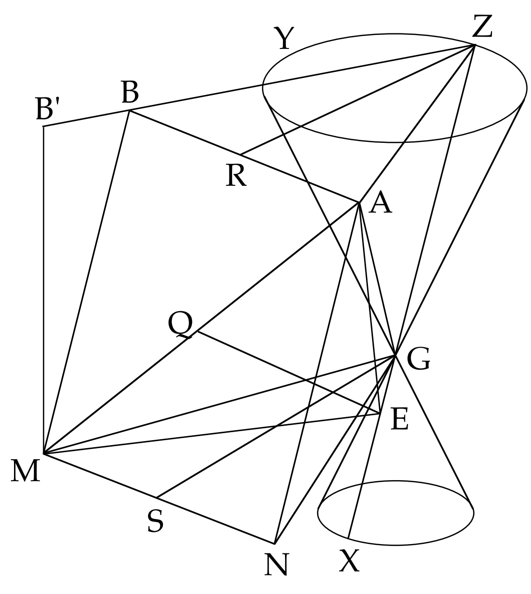


figure 5.2.31d

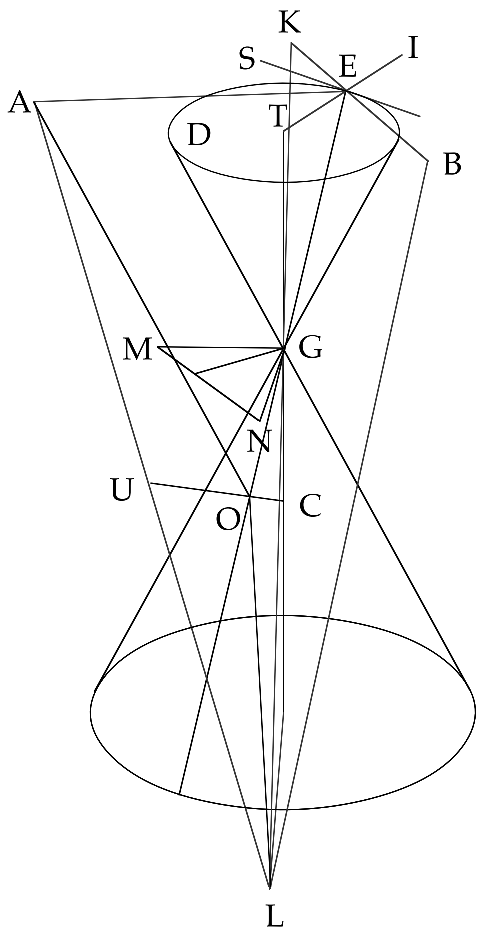


figure 5.2.31e

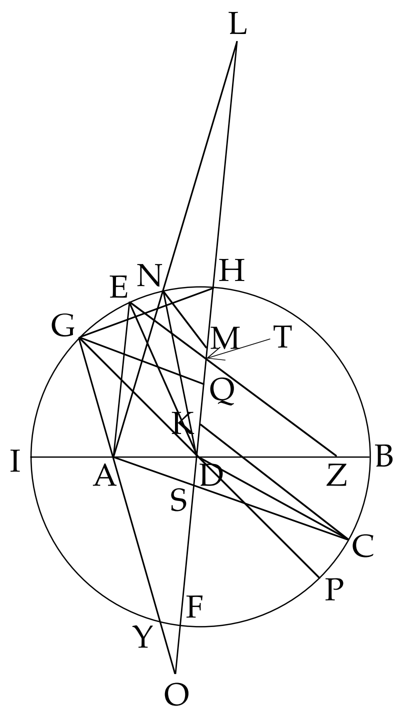


figure 5.2.32

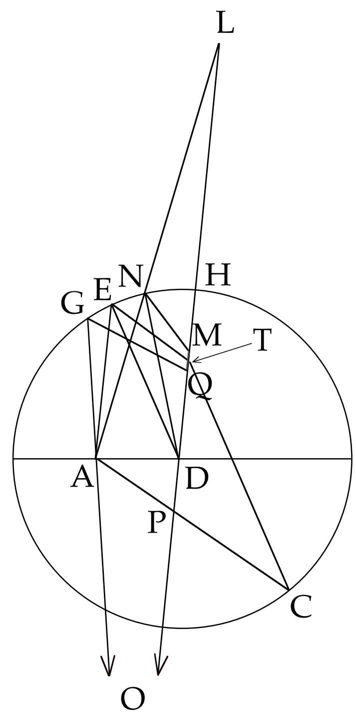


figure 5.2.32a

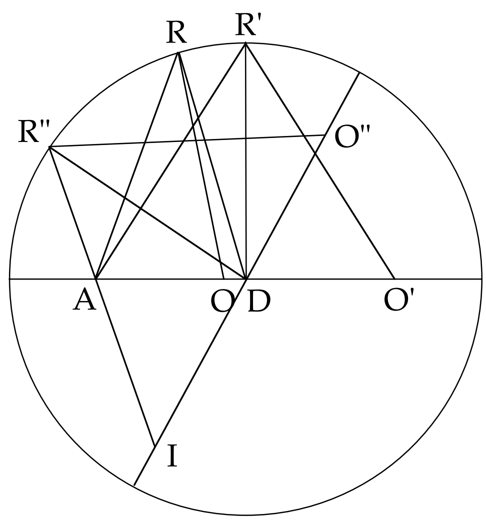


figure 5.2.32b

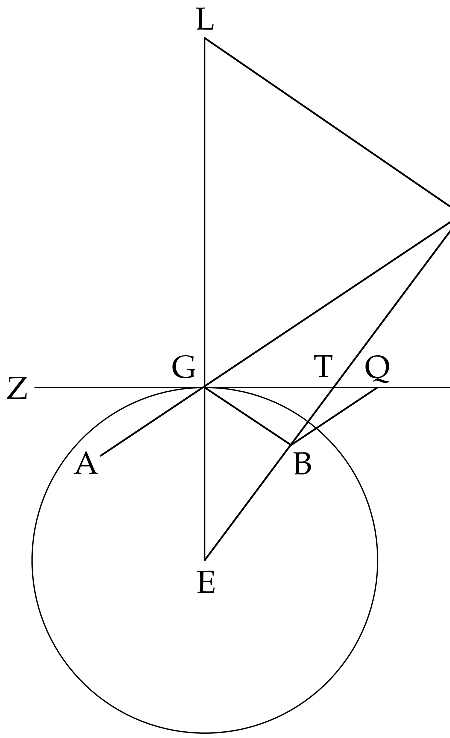


figure 5.2.33

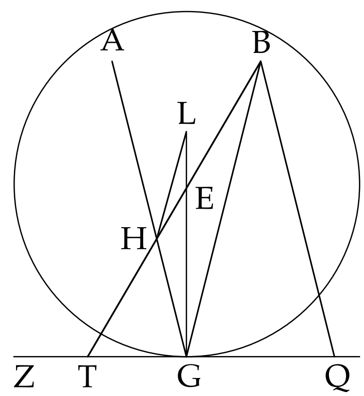


figure 5.2.33a

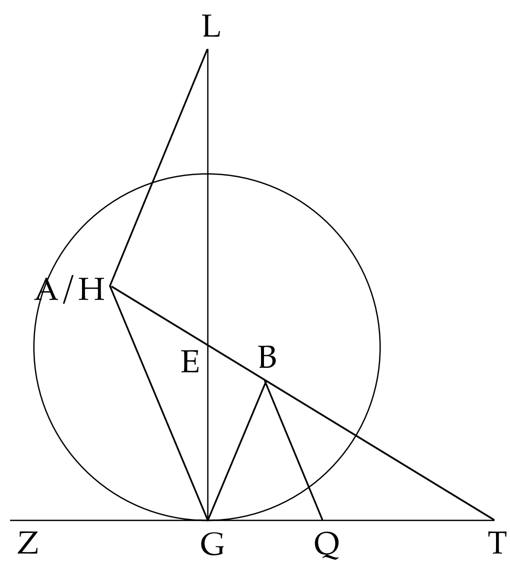


figure 5.2.33b

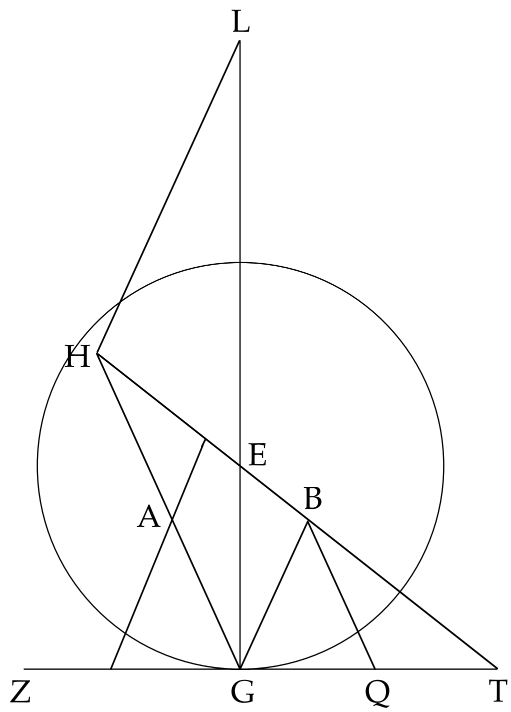


figure 5.2.33c

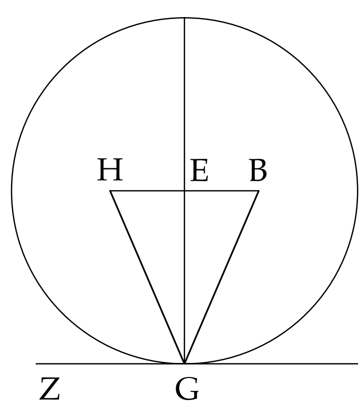


figure 5.2.33d

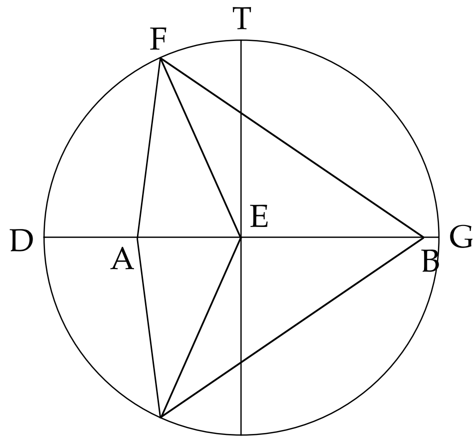


figure 5.2.34

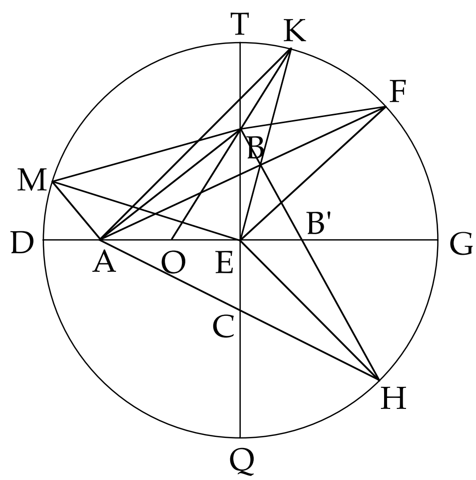


figure 5.2.34a

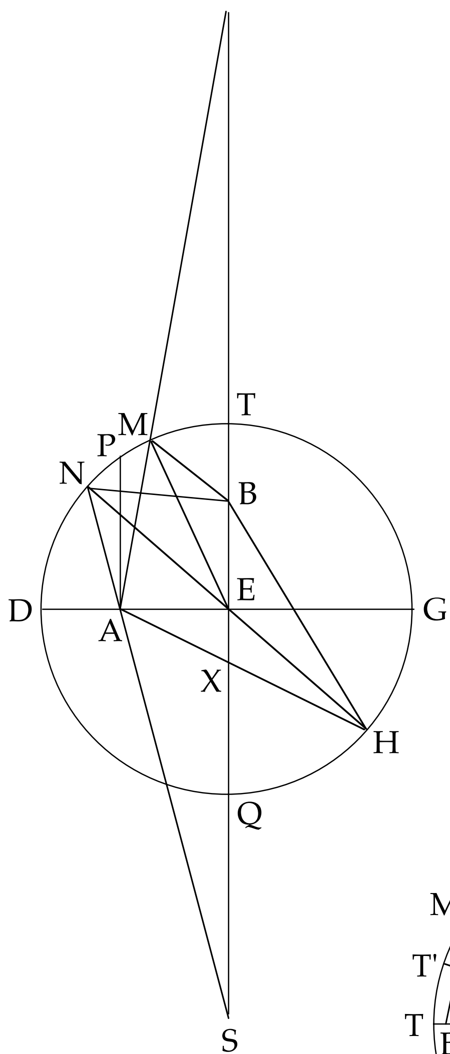


figure 5.2.34b

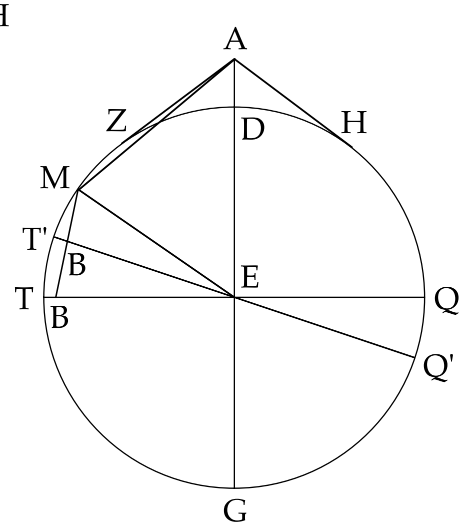


figure 5.2.34c

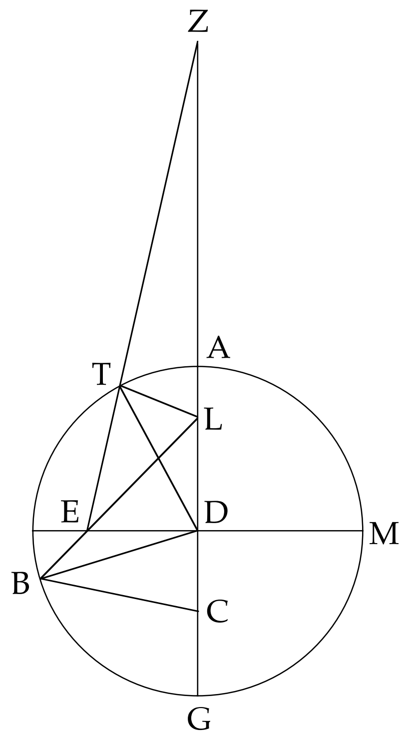


figure 5.2.35

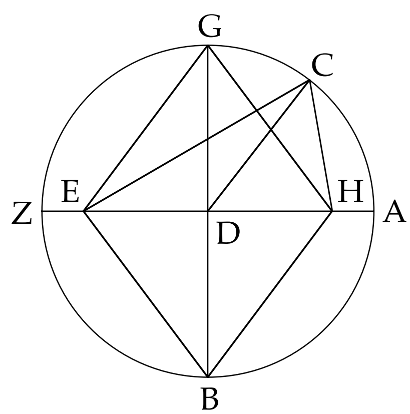


figure 5.2.36

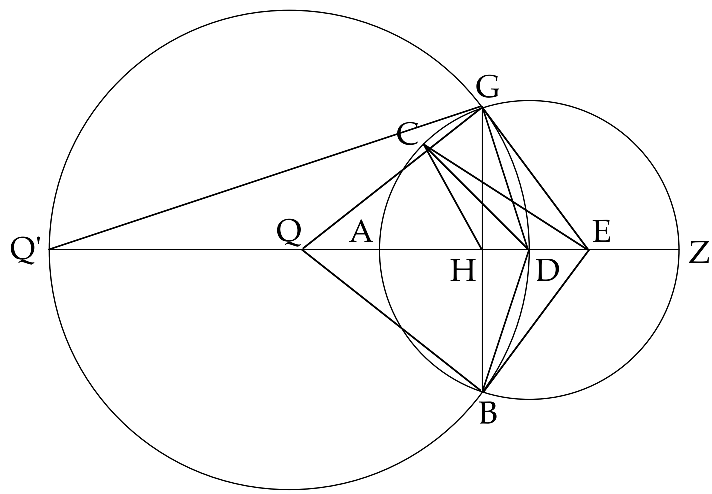


figure 5.2.36a

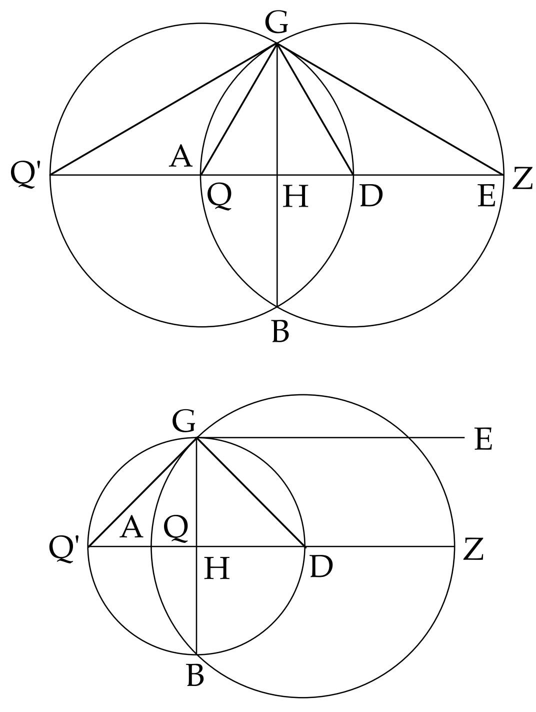


figure 5.2.36b

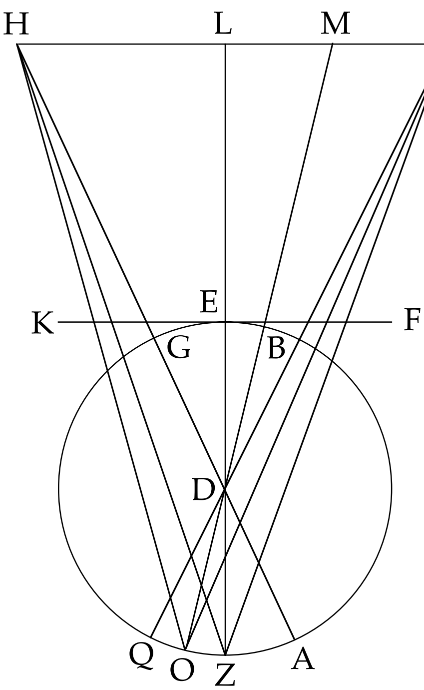


figure 5.2.37

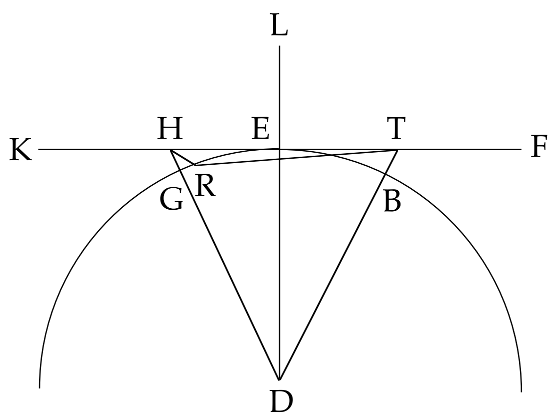


figure 5.2.37a

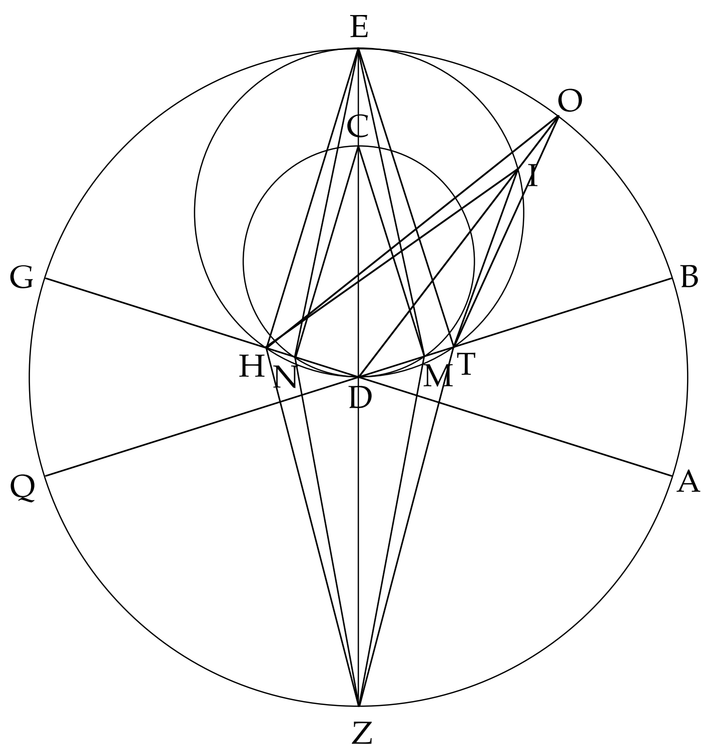


figure 5.2.37b

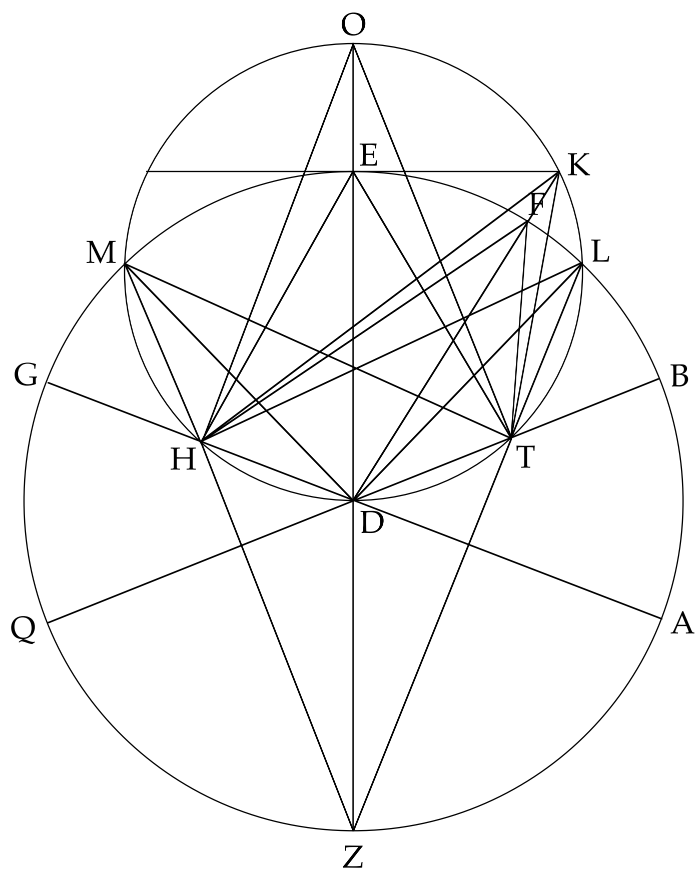


figure 5.2.37c

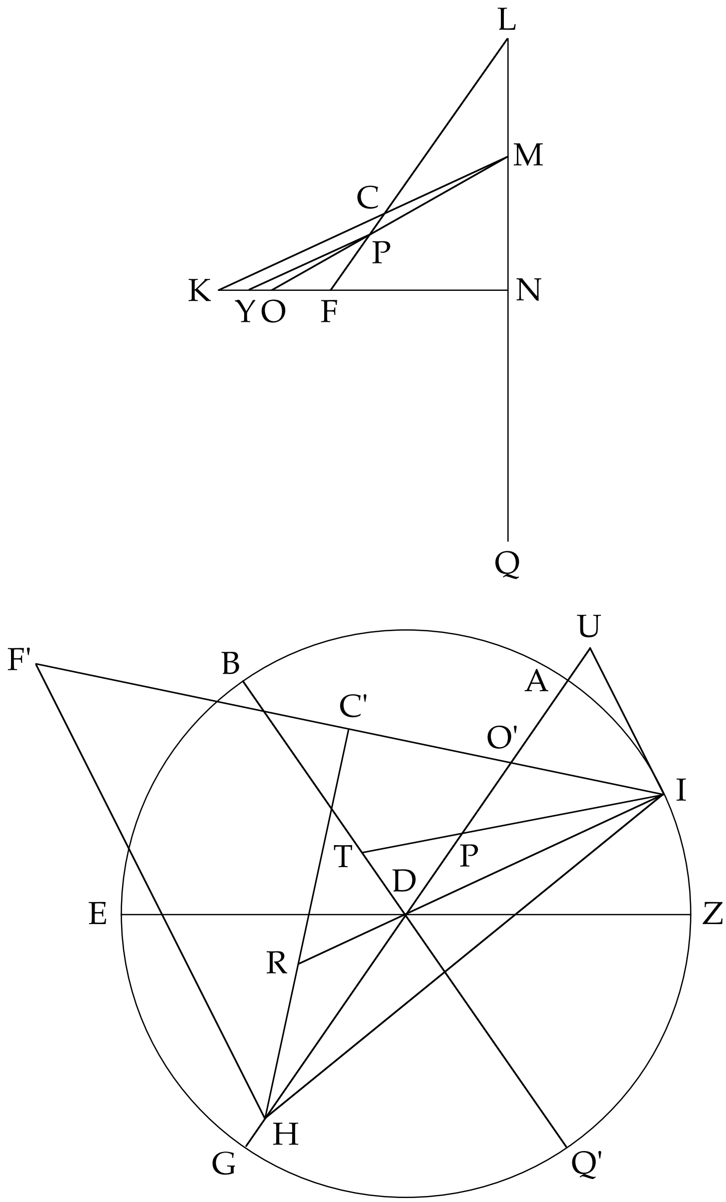


figure 5.2.38

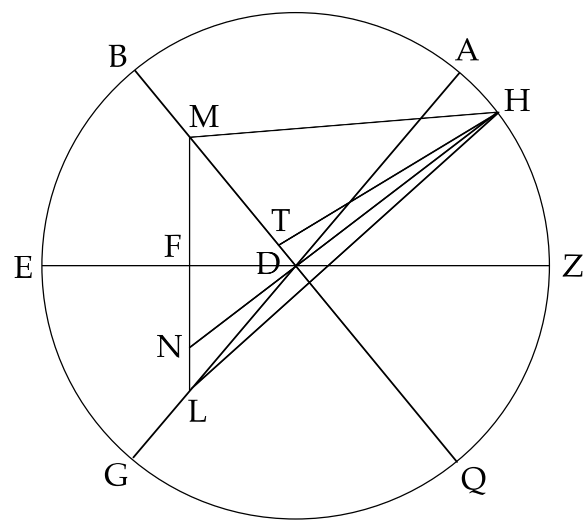


figure 5.2.39

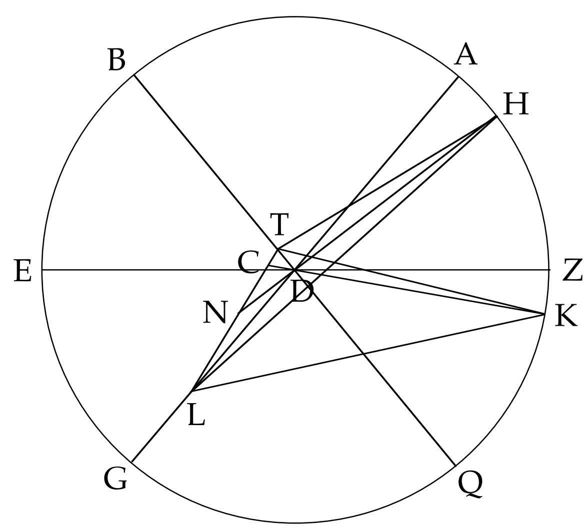


figure 5.2.40

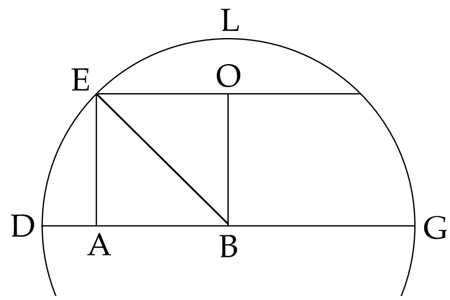


figure 5.2.41

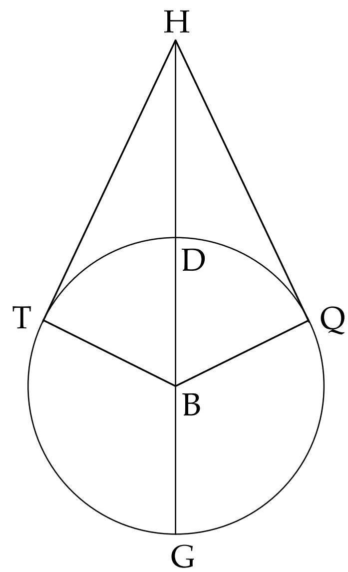


figure 5.2.41a

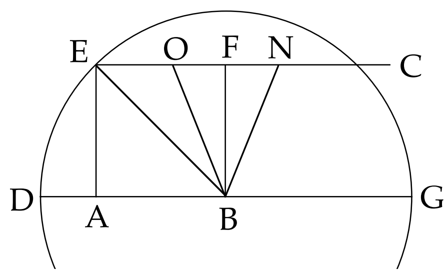


figure 5.2.42

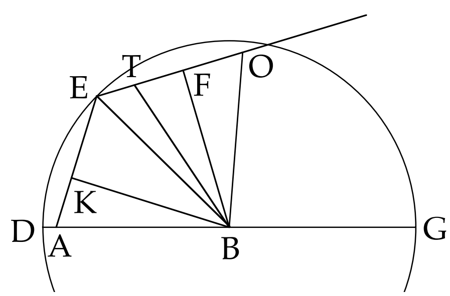


figure 5.2.42a

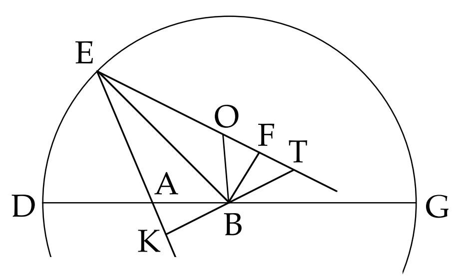


figure 5.2.42b

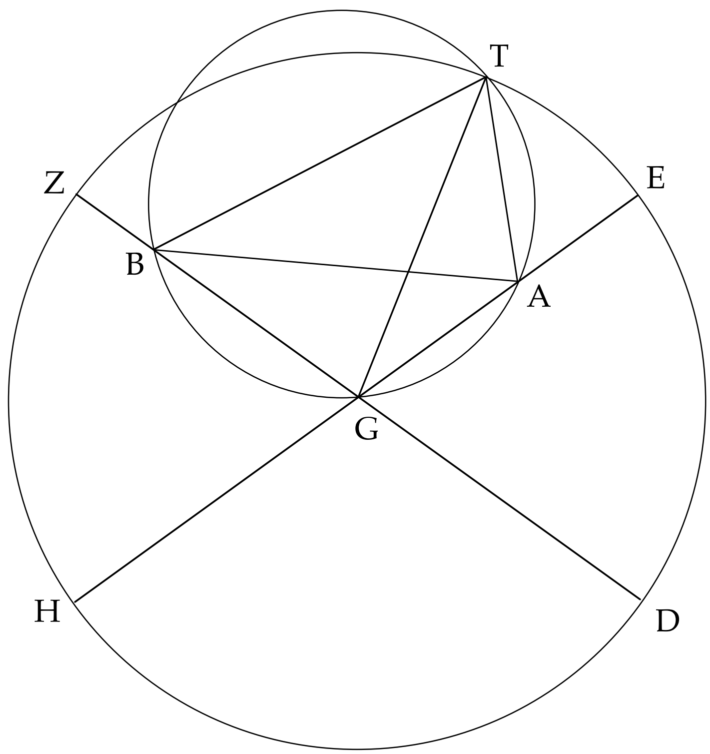


figure 5.2.43

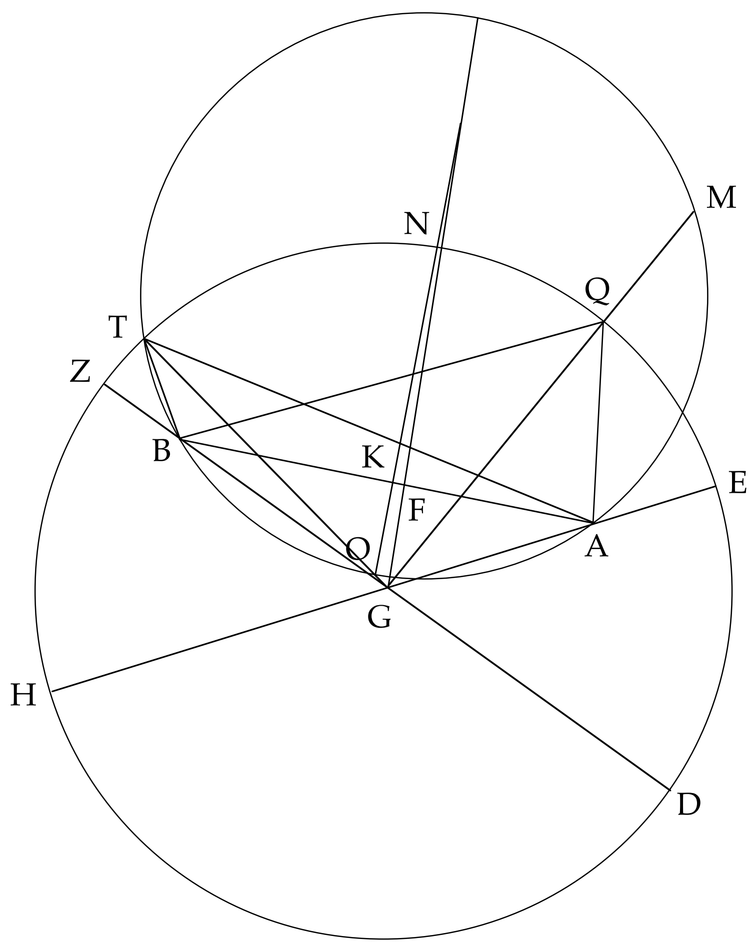


figure 5.2.44

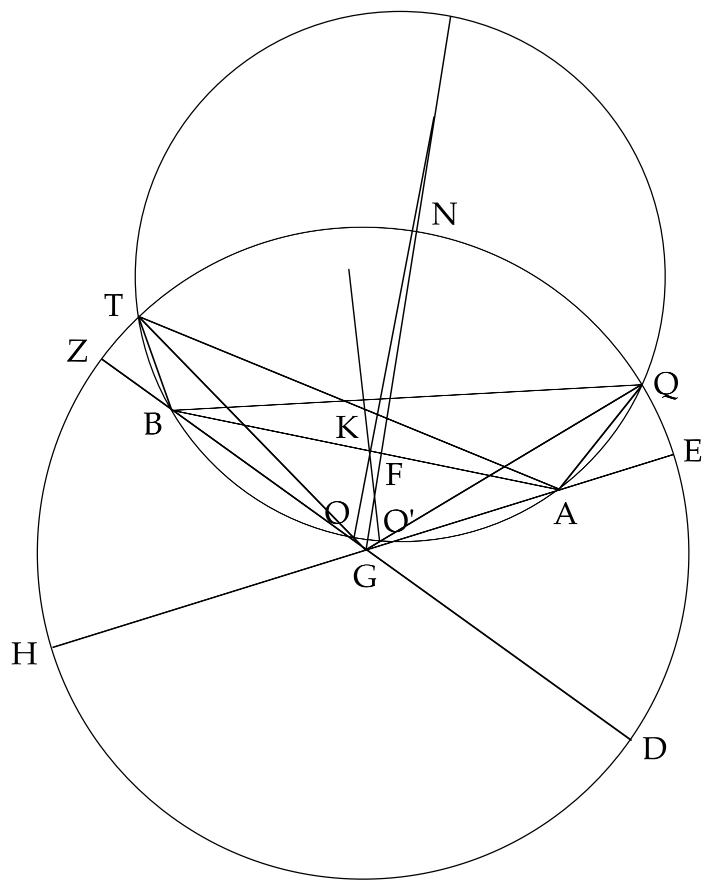


figure 5.2.44a

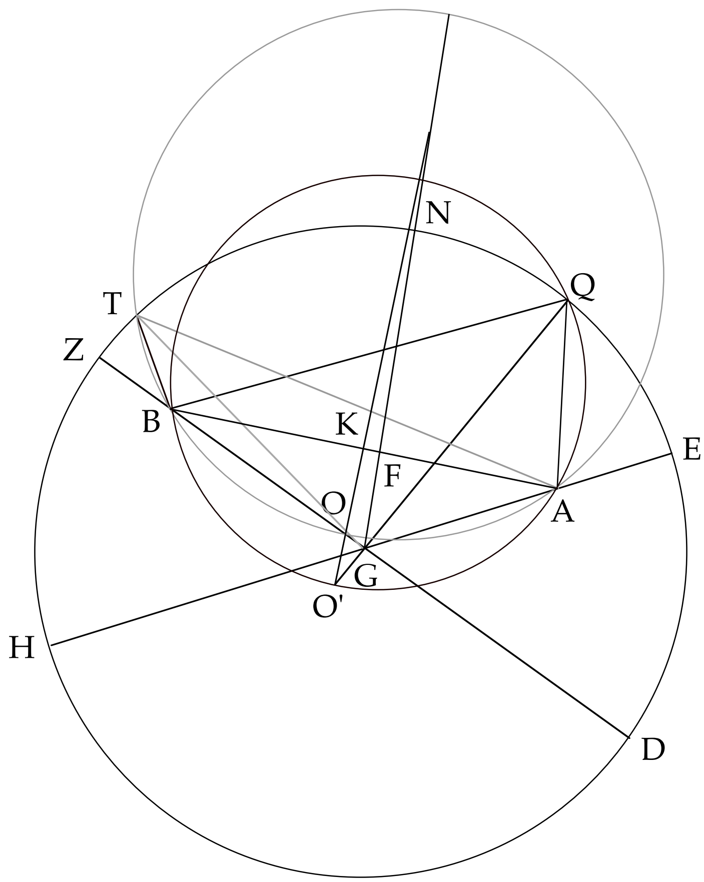


figure 5.2.44b

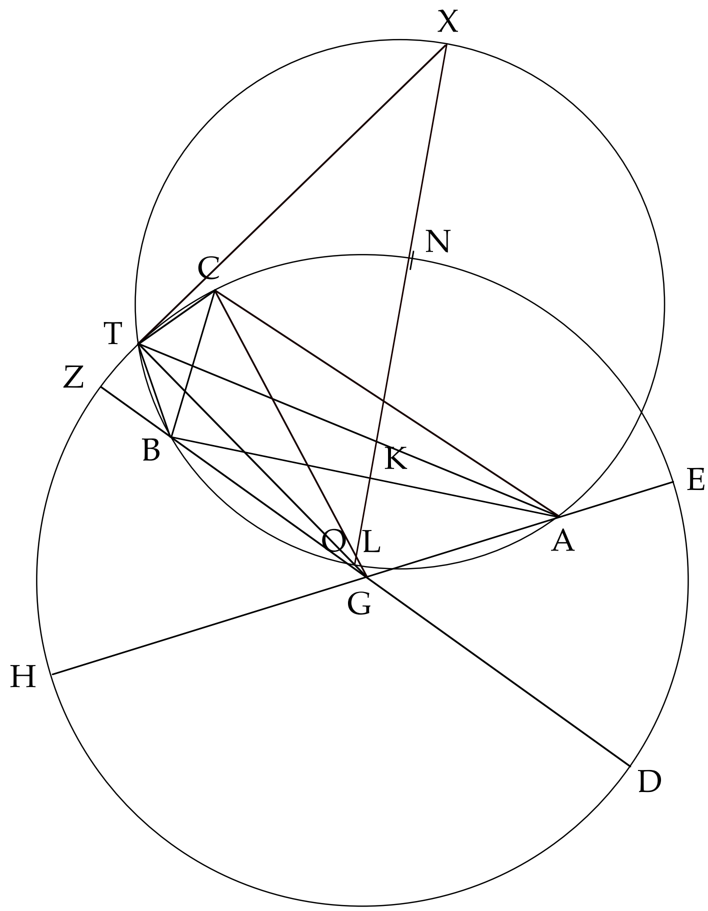


figure 5.2.44c

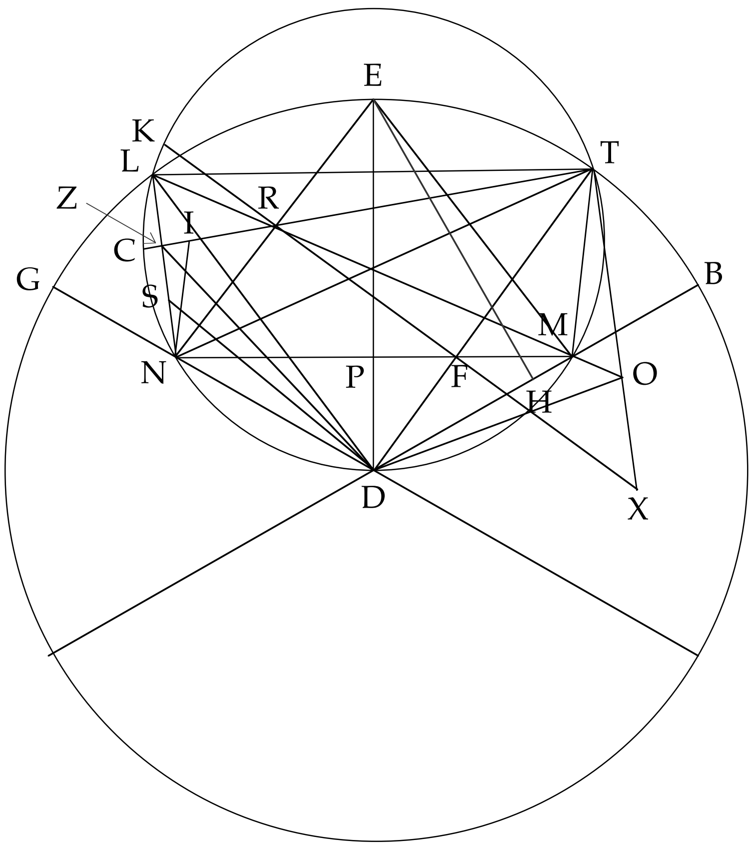


figure 5.2.45

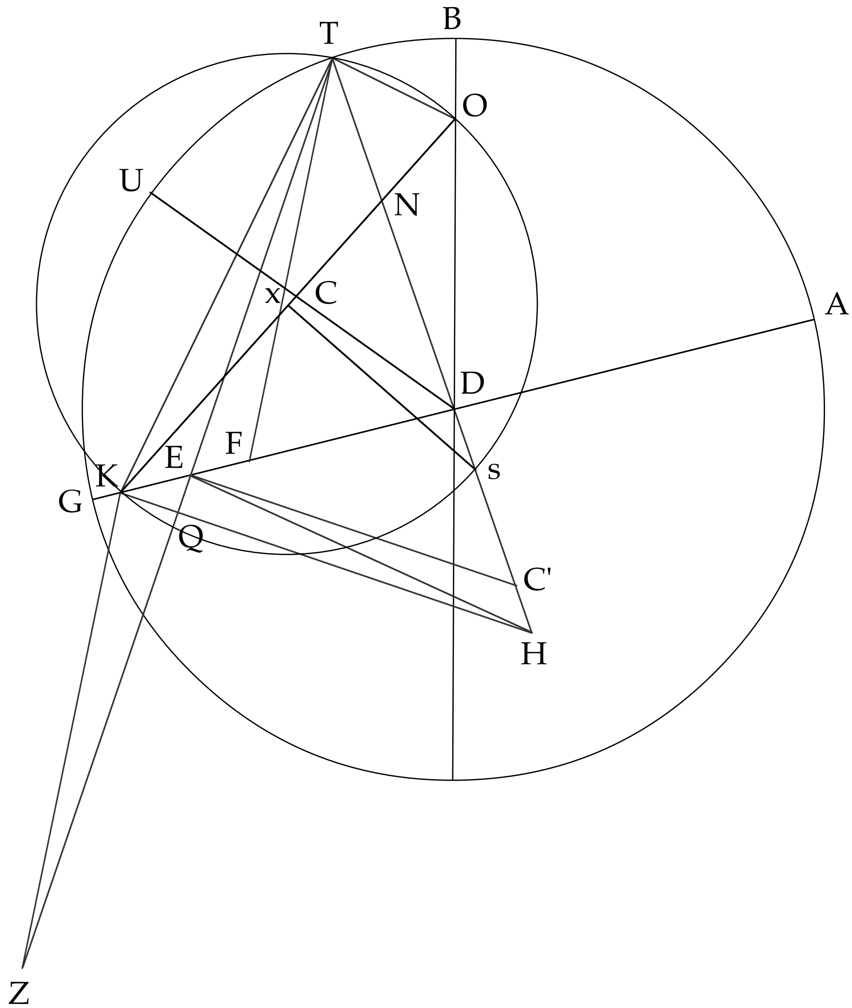


figure 5.2.46

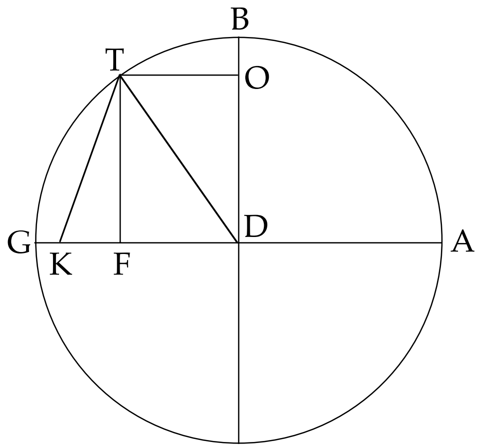


figure 5.2.46a

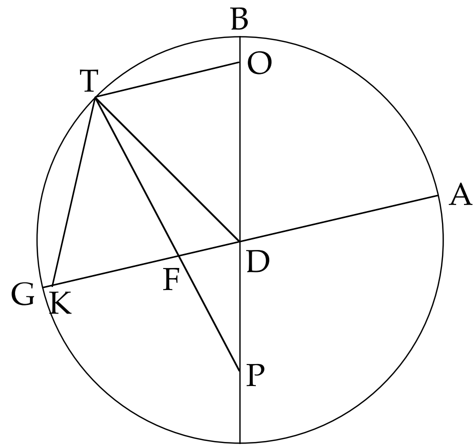


figure 5.2.46b

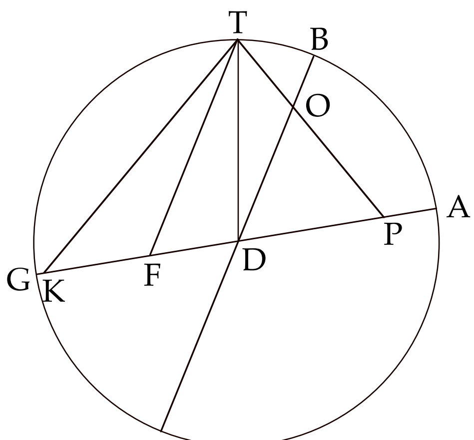


figure 5.2.46c

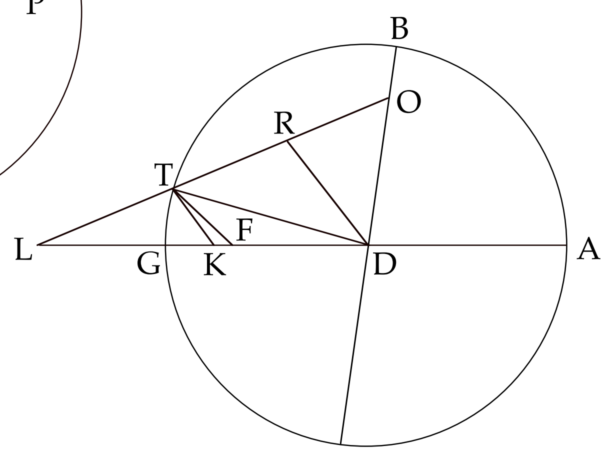


figure 5.2.46d

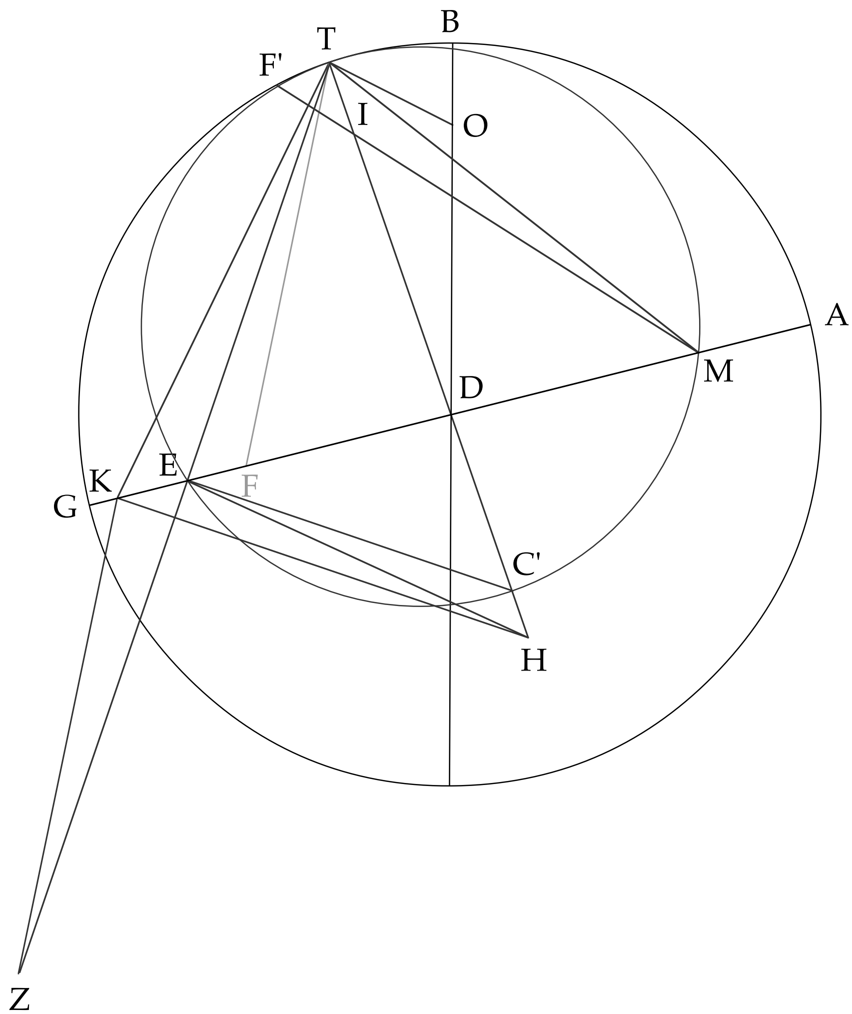


figure 5.2.46e

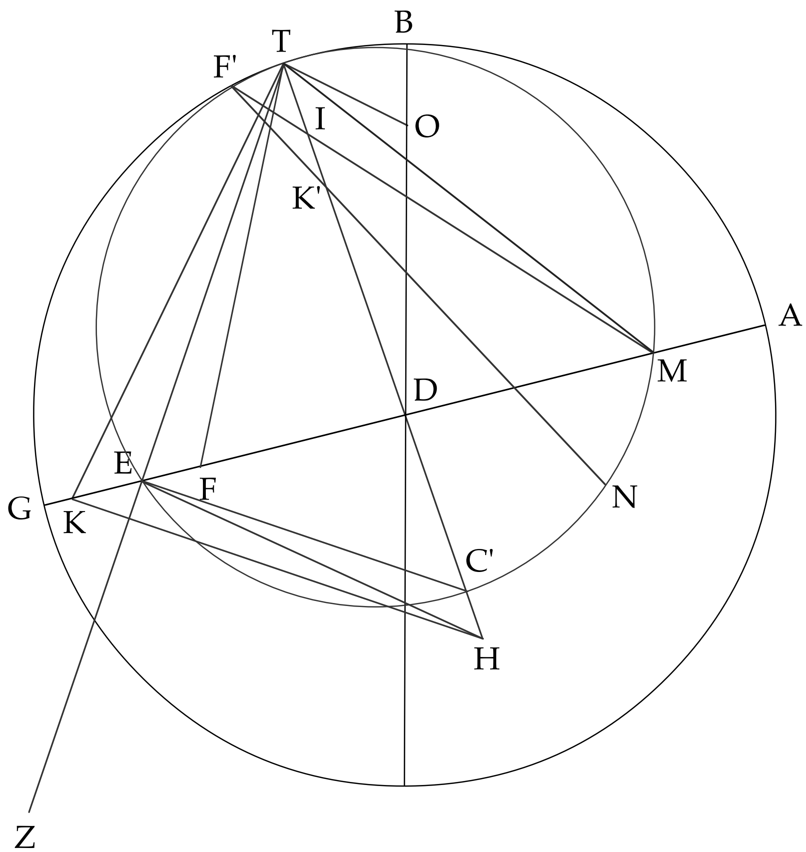


figure 5.2.46f

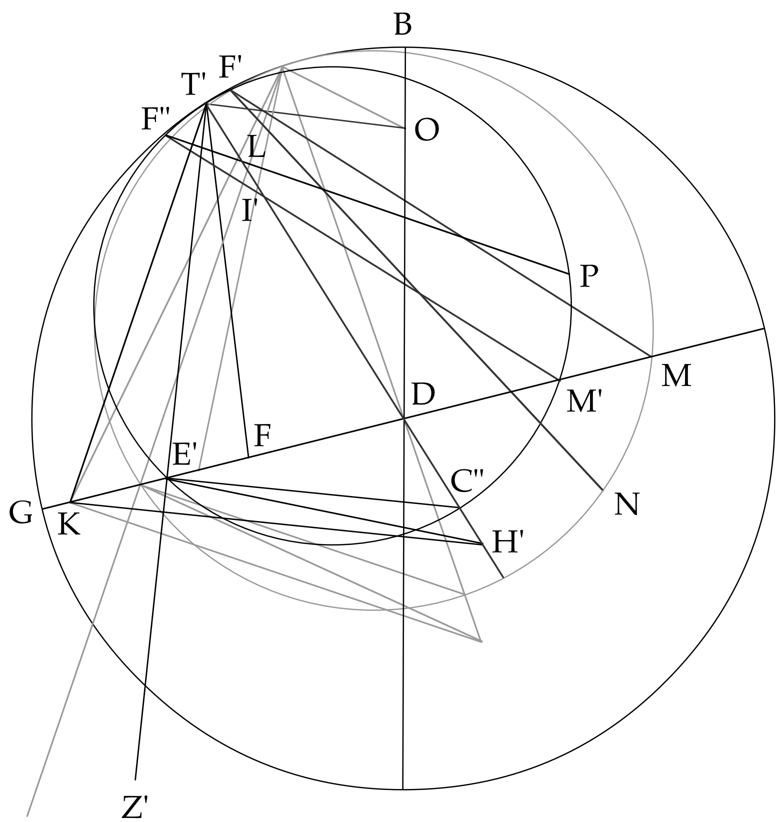


figure 5.2.46g

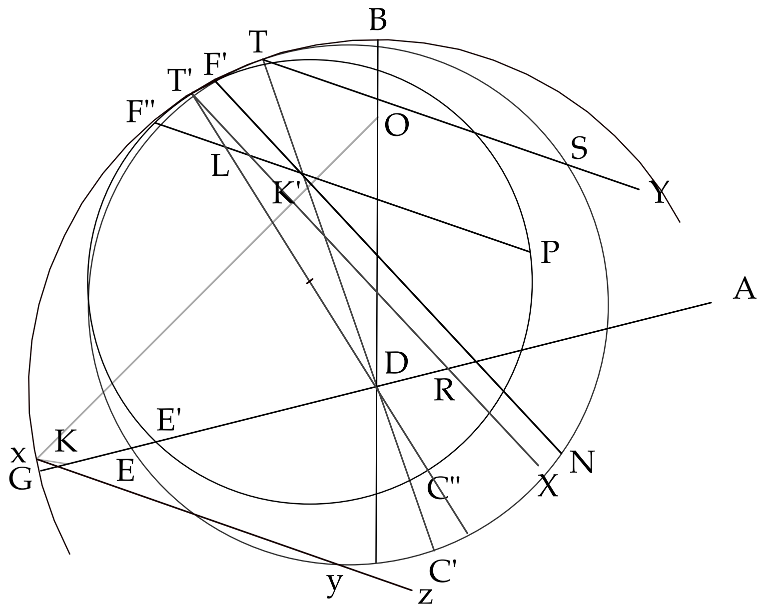


figure 5.2.46h

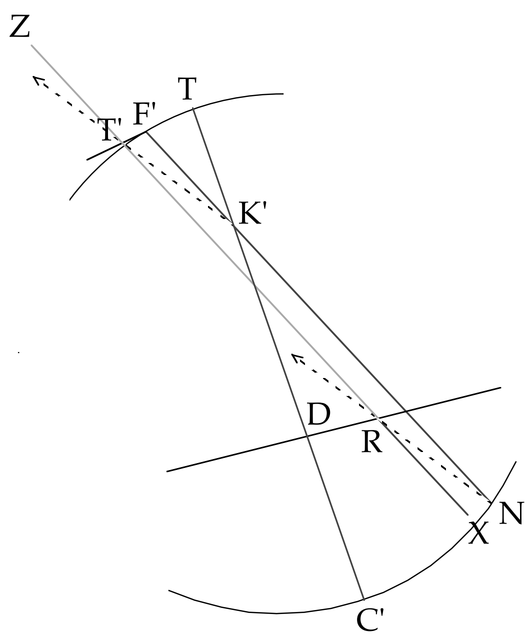


figure 5.2.46k

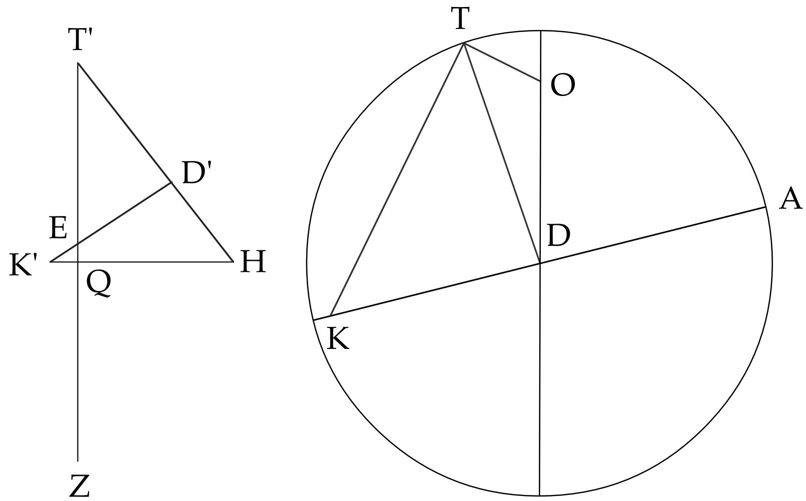


figure 5.2.47

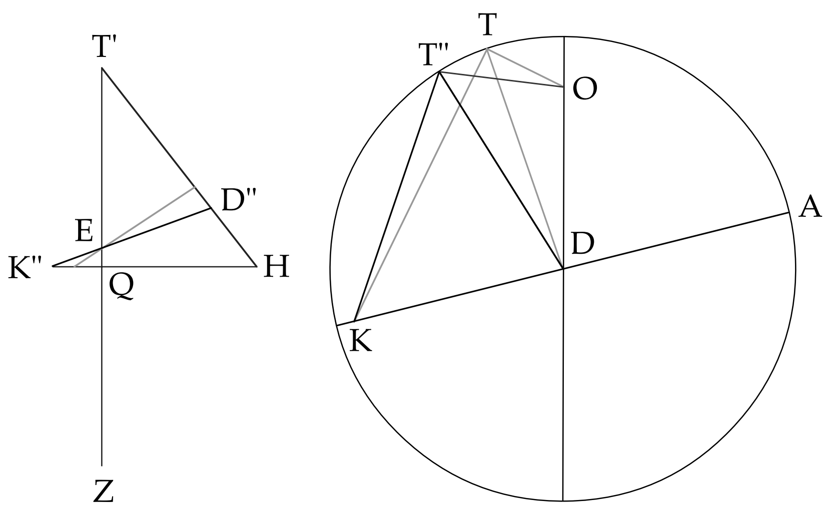


figure 5.2.47a

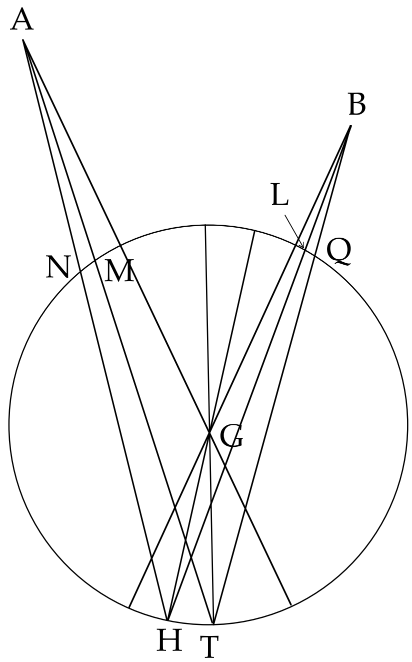


figure 5.2.48

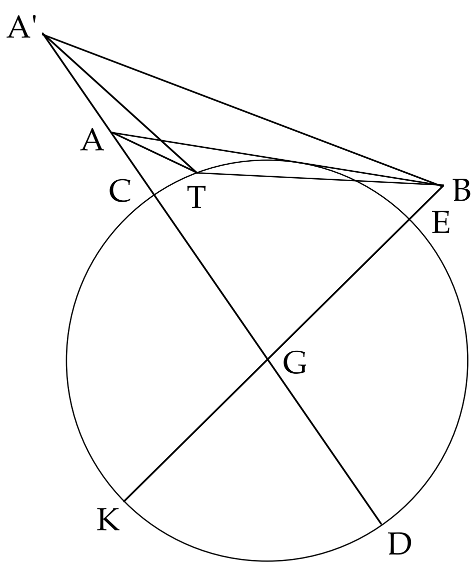


figure 5.2.48a

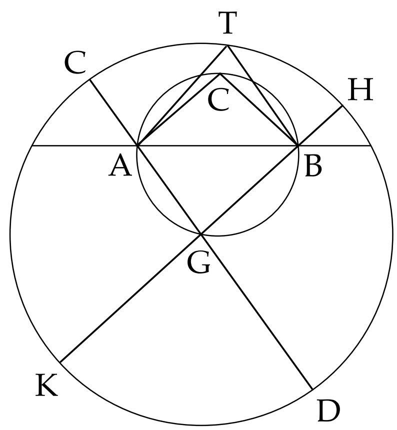


figure 5.2.49

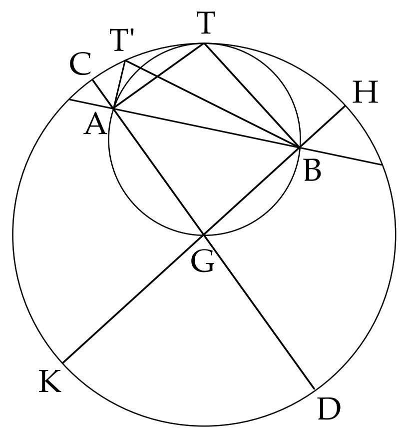


figure 5.2.49a

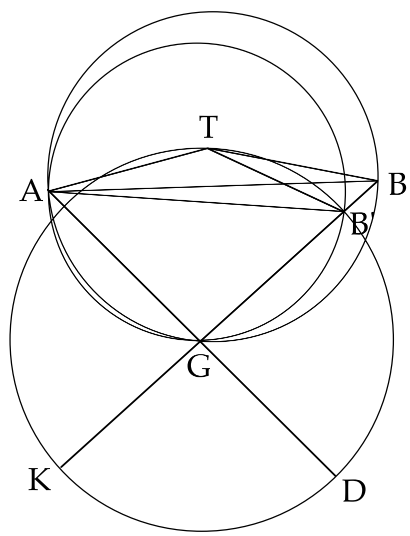


figure 5.2.49b

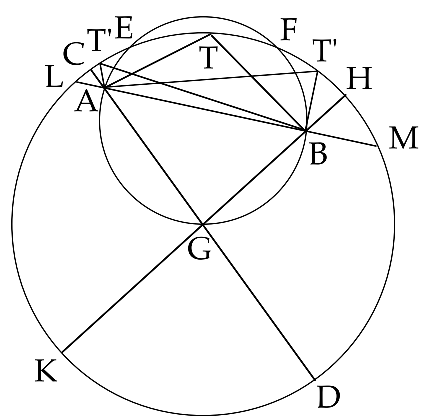


figure 5.2.49c

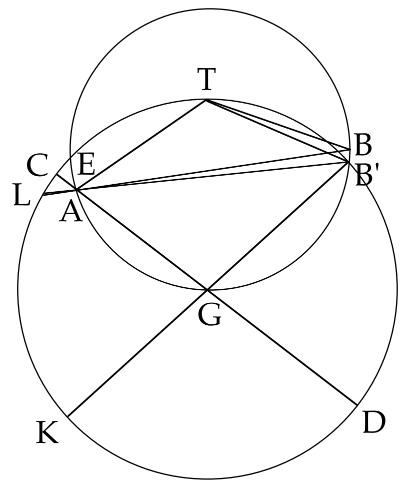


figure 5.2.49d

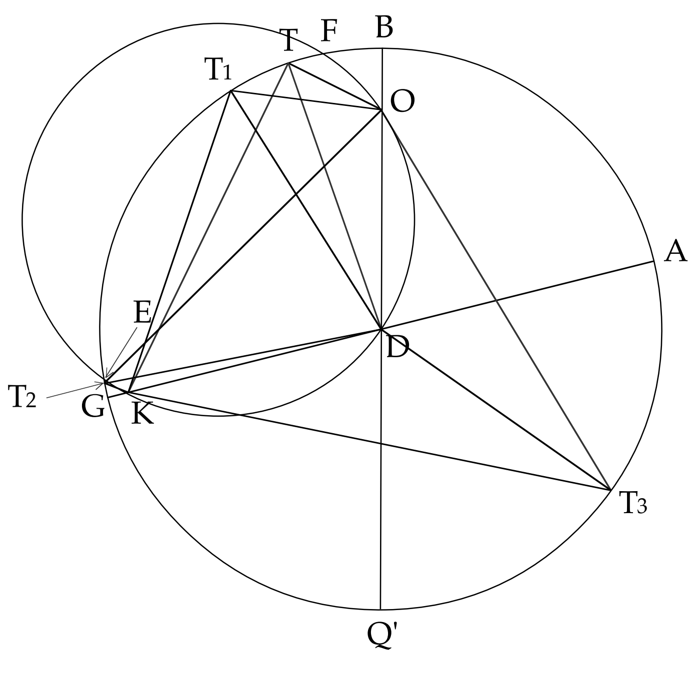


figure 5.2.49e

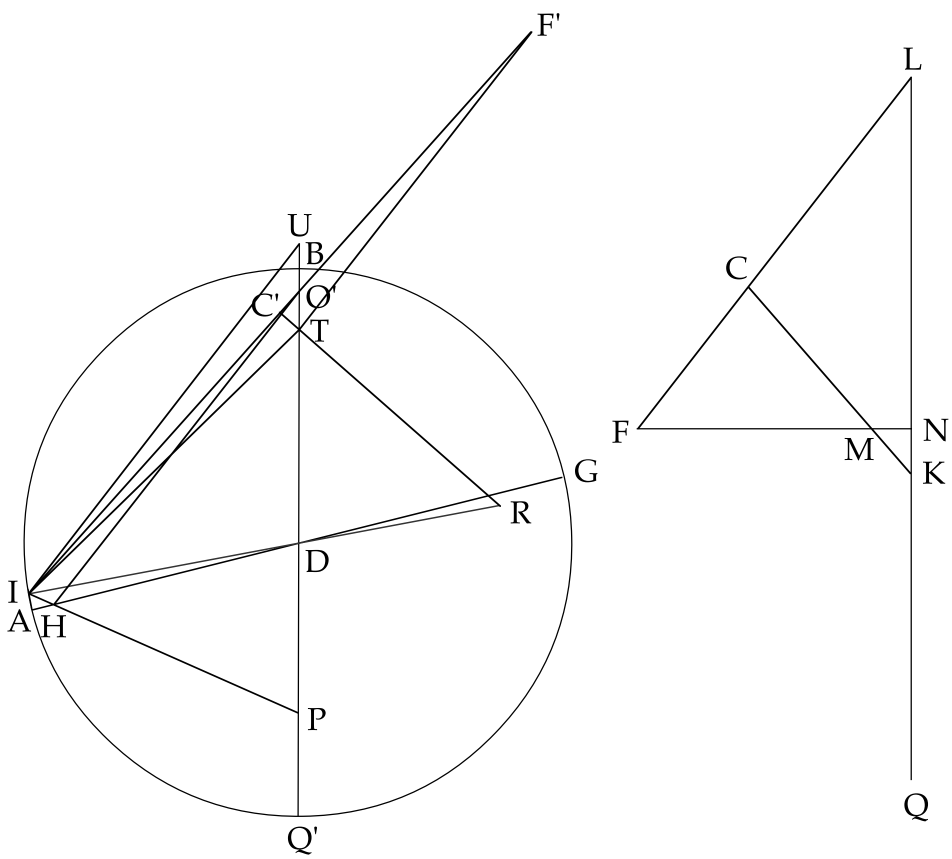


figure 5.2.49f

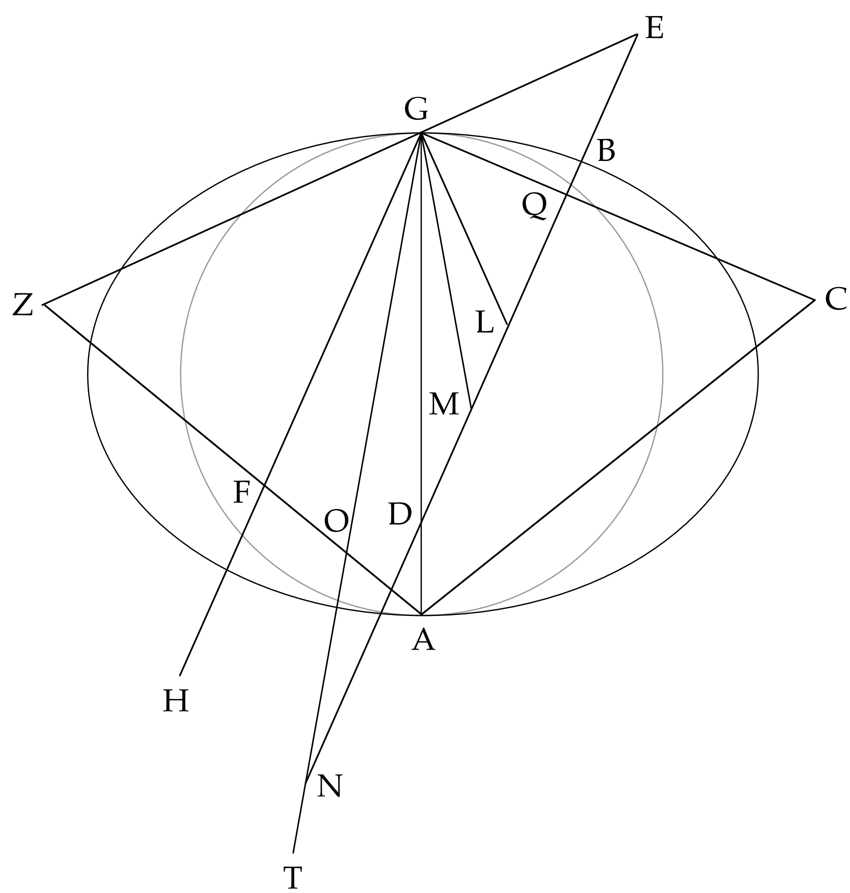


figure 5.2.50

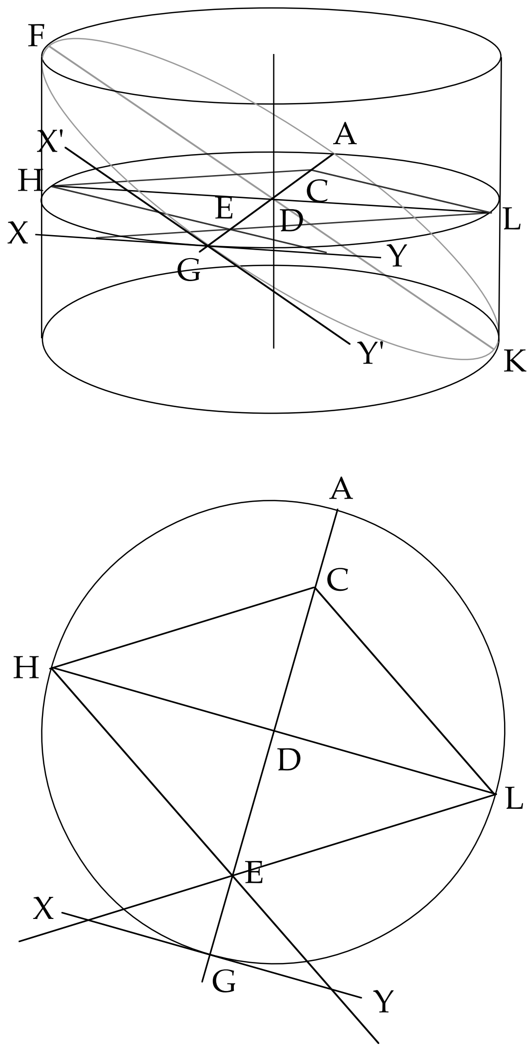


figure 5.2.50a

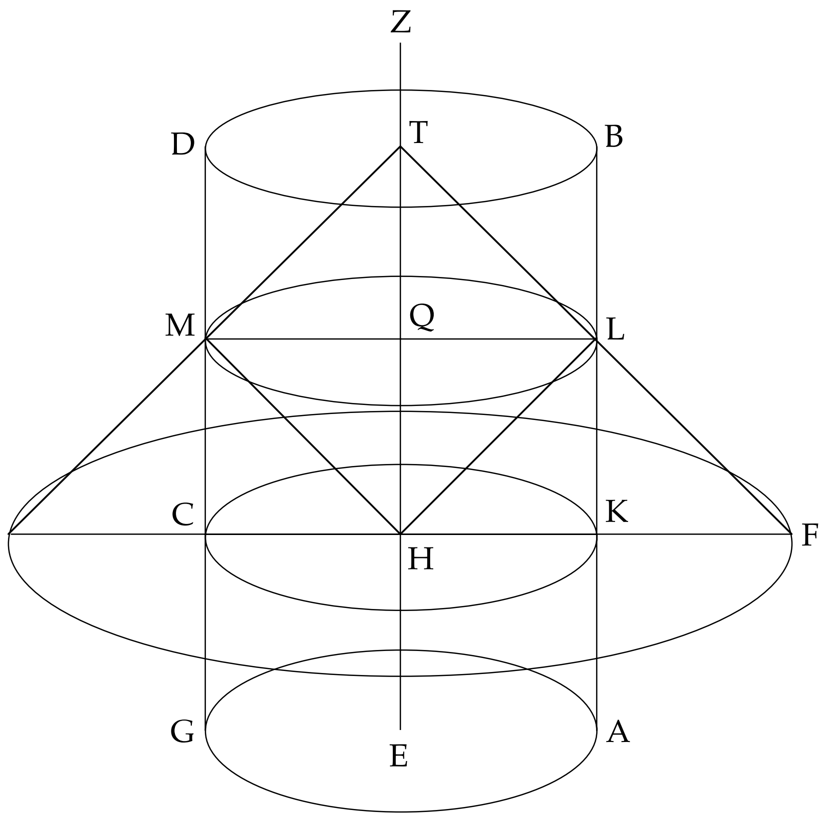


figure 5.2.51

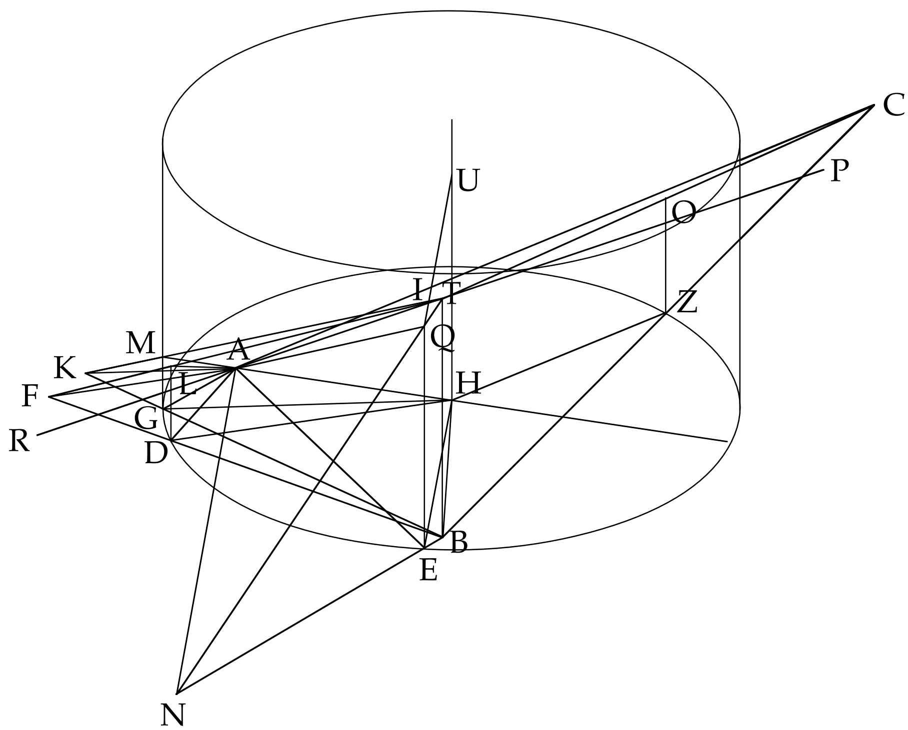


figure 5.2.52

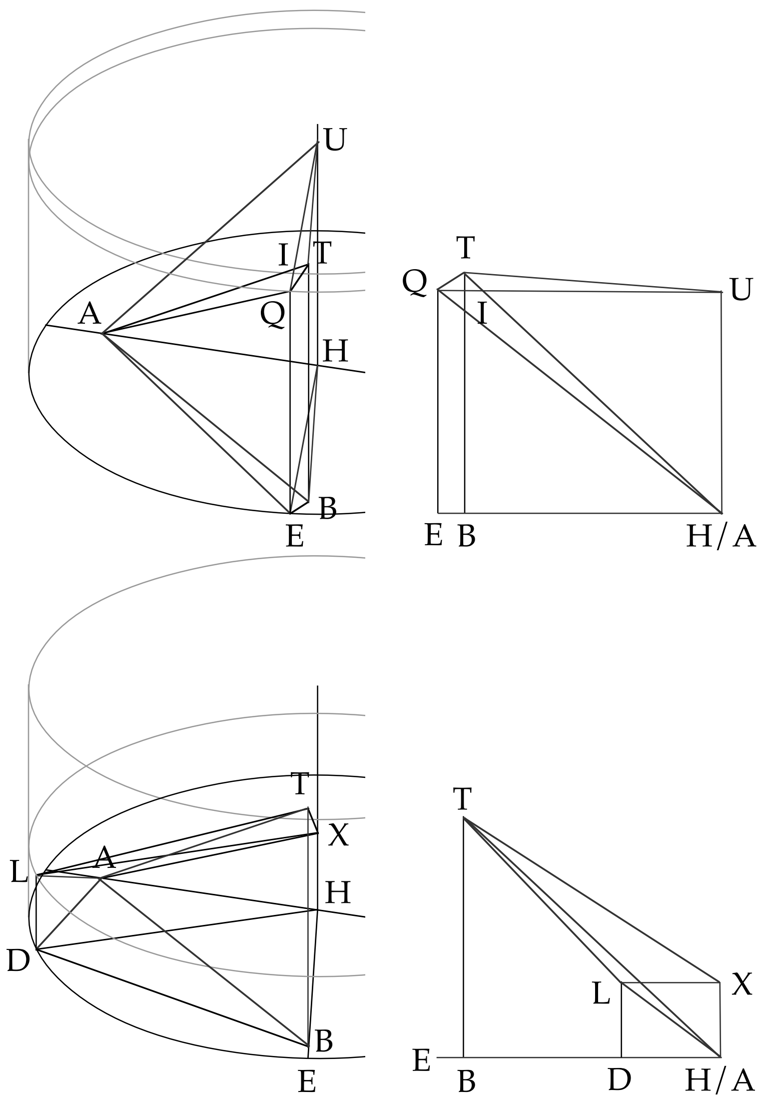


figure 5.2.52a

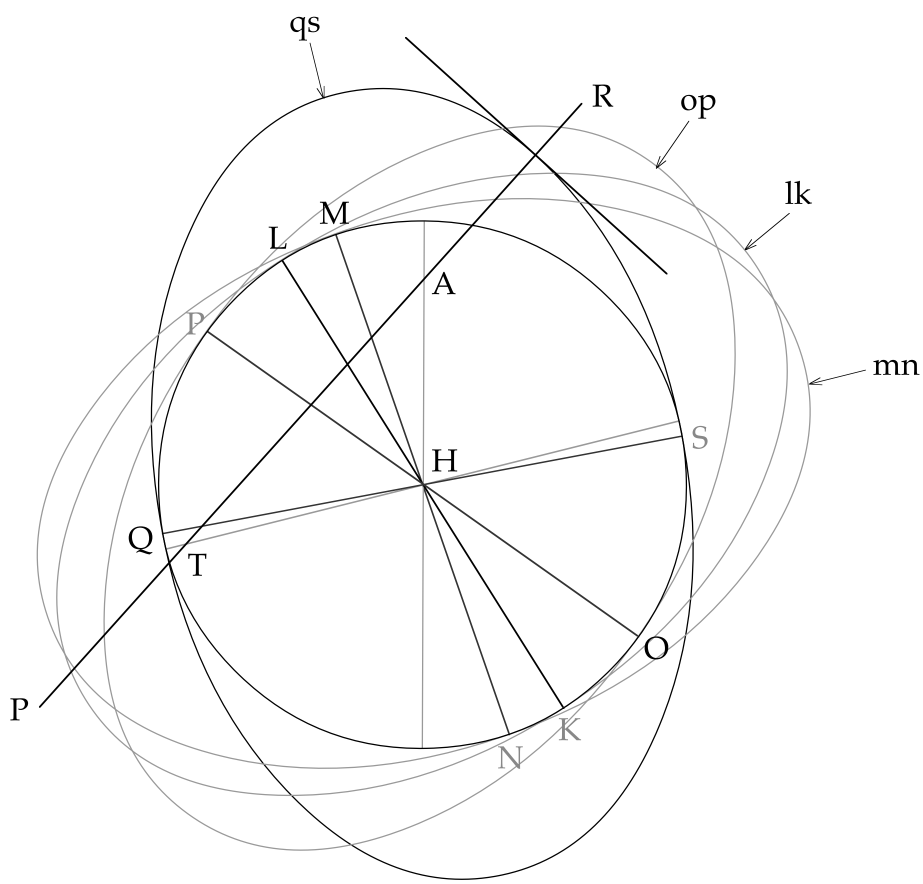


figure 5.2.52b

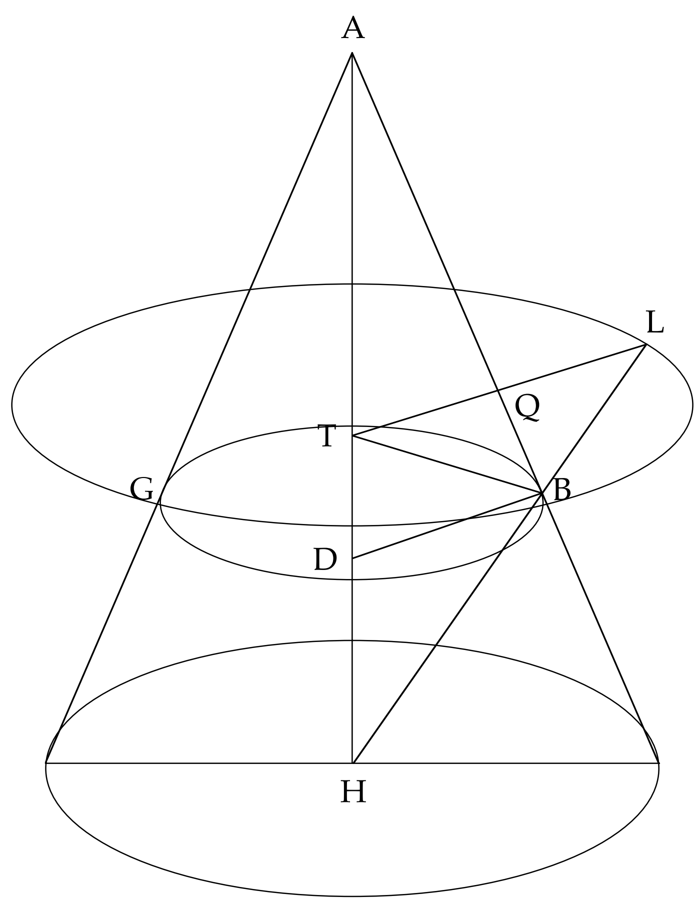


figure 5.2.53

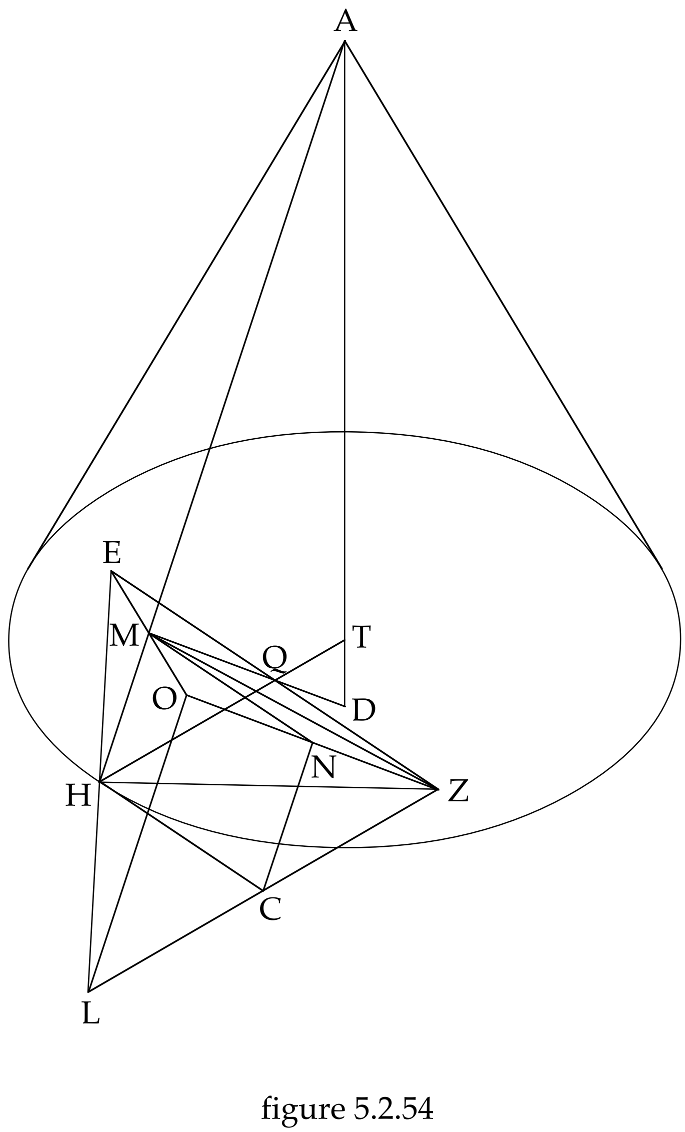


figure 5.2.54

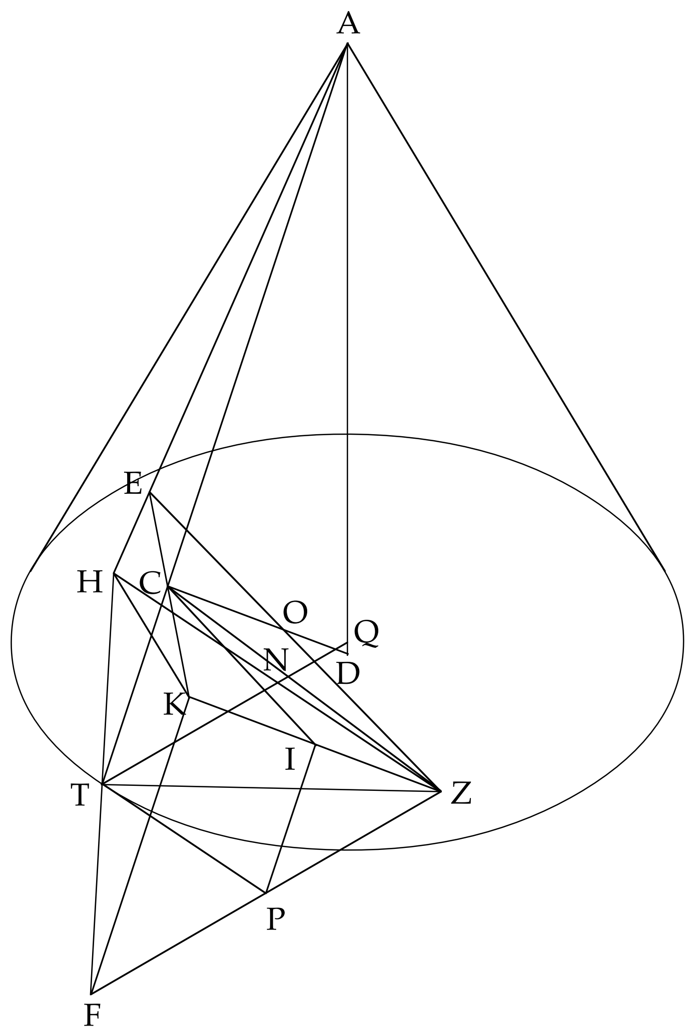


figure 5.2.54a

Transactions
of the
American Philosophical Society
Held at Philadelphia
For Promoting Useful Knowledge
Volume 96 Parts 2 & 3

ALHACEN ON THE PRINCIPLES OF REFLECTION

A Critical Edition, with English Translation
and Commentary, of Books 4 and 5
of Alhacen's *De aspectibus*

VOLUME ONE
Introduction and Latin Text

VOLUME TWO
English Translation

A. Mark Smith

American Philosophical Society
Independence Square • Philadelphia
2006

TRANSACTIONS

of the

American Philosophical Society

Held at Philadelphia for Promoting Useful Knowledge

VOLUME 96, Part 3

Alhacen on the Principles of Reflection:

A Critical Edition, with English Translation and Commentary, of Books 4 and 5 of Alhacen's *De aspectibus*

VOLUME TWO

COVER ILLUSTRATION: Paris, Bibliothèque Nationale MS Lat 7319, f 116v.
The author wishes to express his deep gratitude for permission to use the many figures from this manuscript that appear in this book.

Library of Congress Cataloging-in-Publication Data

Alhazen, 965–1039.

[Manazir. Book 4–5. English & Latin]

Alhacen on the priciples of reflection: a critical edition, with English translation and commentary, of books 4 and 5 of Alhacen's *De aspectibus*, the medieval Latin version of lbn al-Haytham's *Kitab al-Manazir*/[edited by] A. Mark Smith.

p. cm - (Transactions of the American Philosophical Society; v. 96, pts. 2 & 3)
Includes bibliographical references (p.) and indexes.

Contents: v. 1. Introduction and Latin text—v. 2. English translation.
1. Optics—Early works to 1800. I. Smith, A. Mark. II. Title. III. Transactions of the American Philosophical Society; v. 96, pt. 2–3.

ISBN-10: 0–87169–962–1 (pbk. © 2006)
ISBN-13: 978–0–87169–962–6
US ISSN 0065–9746

QC353
[.A32313 2001]
535'.09'021-dc21 2001041227

This digital edition is distributed online as volume 2 in *Interpretatio*: Sources and Studies in the History and Philosophy of Classical Science (http://www.ircps.org/interpretatio) published by the Institute for Research in Classical Philosophy and Science.

ISBN-13: 978–1–934846–01–8 (digital edition)
ISBN-10: 1–934846–01–5 (digital edition)

CONTENTS

VOLUME II

BOOK FOUR OF ALHACEN'S *DE ASPECTIBUS*

Topical Synopsis

[1.1-1.2] Visual perception occurs in only three ways: by direct, reflected, and refracted radiation.

[2.1] Light of every kind is reflected from polished bodies. **[2.2-6]** Reflected light of whatever kind is always reflected to a particular spot along particular lines. **[2.7-8]** Color reflects in the same way as light. **[2.9-10]** Reflected light and color are weaker than the original light and color producing them. **[2.11-13]** This weakening is not due solely to the mingling of the mirror's dark color with the light or color reflecting from it. **[2.14-16]** Nor is it due solely to increasing distance from the mirror. **[2.17-21]** Reflected light and color are stronger than secondary light radiated from the same sources.

[3.1] Polished bodies are reflective because of the extreme smoothness and continuity of their surfaces. **[3.2]** Reflection occurs from every point on a polished body, no matter its shape; the line of incidence, the normal to the point of reflection, and the line of reflection lie in the same plane, and the lines of incidence and reflection form equal angles with that normal. **[3.3]** The plane of reflection is perpendicular either to the plane of the mirror or to the plane tangent to the mirror at the point of reflection; this applies to plane mirrors, convex and concave cylindrical mirrors, convex and concave conical mirrors, and convex and concave spherical mirrors. **[3.4-41]** Construction of an experimental apparatus for testing the validity of the equal-angles law of reflection. **[3.4-3.8]** Formation of the bronze plaque that serves as the register. **[3.9-22]** Formation of the hollow wooden cylinder into which the bronze plaque is inserted. **[3.23-27]** Formation of the seven iron test-mirrors: plane, convex and concave cylindrical, convex and concave conical, and convex and concave spherical. **[3.28-41]** Insertion of the seven mirrors into their respective wooden panels. **[3.42-59]** Experiments with the panel containing the plane mirror to show that light entering one of the holes in the cylinder's wall reflects to the corresponding hole; light reaching the mirror orthogonally reflects back along the same line; light spreads out conically as it radiates through the hole to the mirror and is then reflected back to the hole; the narrower the beam of light striking the mirror, the less it spreads out in the course of radiation and reflection. **[3.60-61]** The same procedures will yield the same results for both primary and accidental, or secondary, light. **[3.62-64]** General conclusions to be drawn from these experiments. **[3.65-87]** Mathematical verification of these conclusions according to the structure of the apparatus. **[3.88-106]** Explanation of reflection according to the physics of light-radiation. **[3.88-89]** The radiation of light from a luminous source to a polished surface and the consequent reflection of that light from the polished surface to some facing body can be analytically resolved into an infinite number of intersecting cones with bases in the luminous source, the polished body, and the body to which reflection occurs. **[3.90-92]** Within such cones, all the constituent lines of incidence and reflection will follow the equal-angles law. **[3.93-96]** In reflection light is weakened by dispersal, distance, and the act of reflection itself; however, it is strengthened by concentration. **[3.97-102]** Analogy between the radiation and reflection of light and the action of physical projectiles. **[3.103-106]** The dynamics of reflection according to vertical and horizontal vectors. **[3.107-108]** Experimental verification that the reflection of color follows the rules that govern the reflection of light.

[4.1] The visual-ray explanation and the explanation of those who think such forms are actually impressed in the mirror and then seen there. **[4.2-6]** Empirical refutation of the impression-theory.

the point on the axis where the normal through the point of reflection falls. **[5.27-39]** Every point of the visible portion of a convex conical mirror will be a point of reflection for any given center of sight. **[5.28-29]** If the center of sight is placed lower than the vertex of a conical convex mirror so that the line of sight forms an acute angle with the axis of the mirror at its vertex, less than half the mirror's surface will be visible. **[5.30]** If the center of sight is placed so that the line of sight forms a right angle with the axis of a convex conical mirror at the vertex, half the mirror's surface will be visible. **[5.31]** If the center of sight is placed above the vertex of a convex conical mirror, so that the line of sight forms an obtuse angle with the axis of the mirror at the vertex, more than half the mirror's surface will be visible. **[5.32-33]** If the center of sight is placed so that the line of sight coincides with a line of longitude on the surface of a convex conical mirror, all of that surface except for the edge along the line of longitude will be visible. **[5.34-36]** If the center of sight is placed so that the line of sight enters the cone through the vertex, all of the mirror's surface will be visible. **[5.37-39]** Every point on that surface can therefore be a point of reflection for any center of sight. **[5.40-42]** The only two ways in which the plane of reflection can cut a convex conical mirror so that reflection can occur is along a line of longitude or along a conic section; if the plane of reflection cuts the cone along a circle, there can be no reflection in that plane. **[5.43-45]** If the plane of reflection cuts the cone according to a conic section, there can be at most two points of reflection in that plane for any given center of sight. **[5.46]** In spherical concave mirrors, if the center of sight lies within the sphere from which the mirror is composed, the entire surface of the mirror is visible; if the center of sight lies outside that sphere, more than half the mirror's surface is visible. **[5.47]** If the center of sight lies at the mirror's center of curvature, then the point on the surface of the eye through which the normal to the mirror's surface passes is all that will be seen. **[5.48-49]** If the center of sight lies outside the center of curvature, then the form of an object-point can reflect to the center of sight from some point on the mirror's surface, other than the point where the normal from the center of sight falls. **[5.50]** Accordingly, in every plane of reflection within a spherical concave mirror, there will be a center of sight, an object-point, a point of reflection, and the endpoint of the normal dropped from the center of sight through the center of curvature. **[5.51]** In concave cylindrical mirrors, if the eye lies within the cylinder from which the mirror is composed, the entire surface will be visible, whereas if it lies outside the cylinder, more than half the surface will be visible. **[5.52-53]** If the plane of reflection cuts a circle on the mirror's surface, then any point on that surface within the visible portion of the mirror can serve as a point of reflection for any center of sight within that plane of reflection. **[5.54]** If the plane of reflection cuts a line of longitude on the mirror's surface, then any point on that line of longitude will serve as a point of reflection for any center of sight within that plane of reflection. **[5.55]** If the plane of reflection cuts a circle on the mirror's surface, any point on that circle will serve as a point of reflection for any center of sight within that plane of reflection. **[5.56]** If the plane of reflection cuts a conic section on the mirror's surface, then there can only be two points of reflection for any given center of sight within that plane of reflection. **[5.57-58]** All, none, or more than half the surface of a concave conical mirror can be seen depending on where the center of sight is located; if the plane of reflection cuts a conic section on the visible portion of such a mirror, there can be two points of reflection within that plane. **[5.59-60]** If the conical mirror is integral, and if the center of sight is situated on the side of its base, nothing outside the mirror can be seen in it, but if the mirror is open, then outside objects can be seen in it. **[5.61-62]** Closing observations about reflection and image-formation in the various types of mirrors.

[BOOK FOUR]

This book is divided into five parts. The first part [constitutes] the book's prologue; the second [is devoted] to showing that the reflection of light occurs from polished bodies; the third [focuses] on how the [visible] form is reflected; the fourth part [is concerned] with showing that the perception of a form in polished bodies is due solely to reflection; the fifth part [deals] with how forms are perceived through reflection.

[CHAPTER 1]

[1.1] We have already explained in the [previous] three books how [visible] forms are perceived by sight when it is direct [and uninterrupted], and we have listed the particular characteristics of visible objects that the visual faculty perceives.[1] But visual perception occurs in three different ways: i.e., directly, as we [just] mentioned; or by means of [rays] reflecting from polished bodies; or by means of [rays] passing through transparent media, there being none more transparent than [pure] air. These three are the only ways in which vision can occur.[2]

[1.2] Furthermore, in the case of the latter two [types of vision], the visual faculty perceives all the visible characteristics of objects we discussed earlier when we showed how sight apprehends them in direct vision. And in these cases sight may be subject to error or it may be veridical.[3] In this book we shall focus upon how [visible] forms are apprehended by means of reflection, as well as how reflection [itself] will occur, and [we shall also discuss] the disposition of reflected rays. But first we should make some preliminary observations.

[CHAPTER 2]

[2.1] From the first book it is clear that light [emanating] from an illuminated body, whether that body is intrinsically or extrinsically illuminated, shines upon every object that faces it, and likewise color is propagated [toward all facing objects] as long as [that color] is illuminated.[4] Thus, when a polished body faces an illuminated body, the light from the illuminated body, which is mingled with its color, shines upon that polished body, and the resulting light is reflected along with the color, no matter whether it is bright or faint or whether it is primary or secondary.[5]

[2.2] That reflection may occur in the case of bright light can be demonstrated by directing an iron mirror[6] toward an intense light-source, having a wall face the mirror, and letting the light shine upon the mirror at a slant, not orthogonally. The bright light will [then] appear reflected upon the wall, but it will definitely not appear at the same spot if the mirror is removed, nor will it appear at the same spot if the mirror is moved; on the contrary, the spot at which the light is reflected upon the wall will shift with the motion of the mirror. So it is clear that reflection occurs in the case of bright light.

[2.3] This can be demonstrated as easily in the case of faint light. Within a room that has a single window that is not too high above the ground, let daylight, but not [direct] sunlight, shine through that window upon some object.[7] Place an iron mirror near that object, and put a white body near the mirror. On that second white body the light will appear brighter than [it did] when the mirror was absent, and the increase in that light is due solely to reflection from the mirror, for when the mirror is removed, only faint, secondary light will appear on the white body.

[2.4] Furthermore, if one carefully directs his sight along the lines according to which the light shines from the first object to the mirror, he cannot help but realize that the inclination of those lines upon the mirror is the same as the inclination of the lines of reflection to the [white body] itself. So it is characteristic of reflection that the inclination is the same and that the angle of the incident and reflected rays is the same. But if the white body is moved from the location of reflection to somewhere else, even [if that new location is] near the mirror, the increase in light will not appear in it, nor can it be detected except in that particular location [of reflection]. So it is clear that this location is peculiar to reflection.[8]

[2.5] You can observe this same thing in the case of secondary light, if the aforementioned mirror is silver and a third white body lies on the other side of the mirror. Secondary light will in fact appear on the third body, whereas on the second [white] body the light will be brighter than the secondary light [on the third body], and it is obvious that the only reason for this increased brightness is reflection.[9] Moreover, the reflection of light will be obvious everywhere strong light shines upon a body through any window when a mirror is disposed to face the light and a white body faces the mirror in the way that has just been described.

[2.6] But we shall describe the appropriate location of reflection and the disposition of the rays. It has already been demonstrated in the first book that reflected light follows straight lines, so reflection occurs from polished bodies to a definite spot according to a rectilinear course.[10]

[2.7] Now, it is evident from earlier discussions that secondary light emanating from a body that is extrinsically illuminated carries the body's color with it.[11] Hence, from every illuminated body or light-source, color mingled with light shines on facing polished bodies and, so mingled, is reflected in a determinate direction.

[2.8] The truth of this point can be ascertained if sunlight shines upon an intensely and beautifully colored body inside a room with only one opening. Stand an iron mirror near that body, and near the mirror place a vessel, in the shape of a goblet, with a white object in it, and set this vessel up at the point of reflection so that the reflected light strikes the white object [inside it]. The color of the body upon which the light shines will definitely appear on the surface of the white body, but that would not happen if the white object were placed outside the appropriate place of reflection.[12] And you will find this proven for such different kinds of colors as sky-blue, red, green, or [any other color] of this kind. So it is clear that the color is sent back [by the mirror] mingled with light, and the appearance of reflected color is [even] more obvious if the mirror is silver.

[2.9] Here, however, is why this demonstration may not be evident—i.e., [the demonstration] that reflected color may be perceived on every body that faces the mirror and presents a white surface to it. As has been asserted earlier, weak colors, even though they shine along with light, are not perceived because the forms that are reflected [in such cases] are weaker than the forms from which the reflection originates.[13] And this can be shown in the case of light, for when bright light strikes a mirror and is reflected to a wall, the light on the wall will appear weaker than it does in the mirror, and there is a noticeable difference [in intensity] between them.

[2.10] This will become apparent in the same way for weak light. In the case of the original room disposed as it initially was, if the third white body is placed at or near the [original] location of the iron mirror, the light will appear brighter upon that body than it does upon the second one, which would not happen unless reflection weakened the light.[14]

[2.11] But someone will argue that the reason for this phenomenon is the iron mirror's black coloring, which, having mingled with the light shining upon the mirror, darkens that light, so, when [the resulting light] is reflected to the second body, it appears faint and dark. On the other hand, in the case of the third [white] body placed at or near the [original] location of the mirror, the light shines [upon it] only from the first body with no mingling whatever of dark coloring. That this is actually not the case, however, is evident from the fact that, when a silver mirror replaces the iron one, the same thing will be demonstrated.

[2.12] In precisely the same way [as before], the reflected color will be fainter than the color from which the reflection originates. This can be shown for the reflection of color if the white body in the room is placed at or near the location of the mirror. The color will appear stronger on that body than [it does] on the white body placed inside the vessel.[15] The same will be evident if a silver mirror replaces the iron one. Therefore, reflection weakens both light and colors, but colors more than light, according to whichever mirror [is used]. And this happens because colors shine more weakly than light, so they are easily weakened in reflection.

[2.13] Now, when a weak color reaches a mirror, it does mingle with the mirror's color, so when it is reflected it appears faint and dark; and reflected forms are weaker than [the forms] at the point of reflection, and reflection is the cause of weakening.

[2.14] One might claim [in contradistinction] that the weakening of forms in reflection is due solely to their [increased] distance from the source [of radiation]. But it will be made clear that, even when direct light and reflected light are equidistant from their source, the reflected light will still be weaker.

[2.15] Let a shaft of sunlight enter a room through a window, and place an iron mirror that is smaller than the window in the air facing it. Let the rest of the light coming through the window shine upon a white body on the ground, and let the light that is reflected from the mirror shine upon a white body that is raised [above the ground]. When this is set up so that the elevated body and the body lying upon the ground lie the same distance [from the window], the [reflected] light shining upon the elevated body will definitely appear fainter than the light shining [directly] upon the body lying on the ground.[16] And this weakening [of light] can be imputed to no other cause than reflection. The same thing will happen if the mirror is silver.

[2.16] This very point can be demonstrated in the case of color, when sunlight passes through a window into the room [and shines] upon a brightly colored object facing a nearby mirror, and when another hollow object[17] with a white body inside it is positioned to catch the reflection. Within the room place another white body of the same kind as the body in the vessel, and let the distance of this white body from the colored body struck by the light shining through the opening be the same as the distance of the white body in the hollow vessel from that same [colored body] combined with the distance of the mirror from that same [colored body].[18] It can be determined [on this basis] that the color [appears] weaker on the white body in the vessel than [it does] on the body that lies outside it, even though they are [both] equidistant from their source—i.e., from the colored body. And the reason for this is the reflection that weakens color.

[2.17] Moreover, reflected light is brighter than secondary light, even when the [two kinds of light] lie the same distance from their source. For in fact, when reflected light shines on some object, if another object of the same kind is placed outside the location of reflection, and if it lies the same distance from the mirror [as the first object], the light on the second object will appear fainter than on the first one.

[2.18] The same thing will also be evident in the room [used in the previous experiments] if the mirror is laid upon the ground in a direct line with the opening so that it receives all the light [coming] from the window. The light will be brighter on an object lying at the location of reflection than on another object of the same kind lying outside that location at the same distance from the mirror.

[2.19] In the same way, if [the shaft of] light shining through the window is wider than the mirror, if the [excess] light around the mirror falls upon the ground or upon a white body, and if another body lies as far from the mirror as the object upon which the light is reflected, the light upon that body will appear fainter than the light upon the body [at the location] of reflection.[19]

[2.20] The same happens in the case of color. If some object lies as far from the mirror outside the location of reflection as another identical object lies from the mirror at the location [of reflection], the reflected color will definitely appear on the object lying at the location of reflection; [whereas] on the other object no color at all may appear. In fact, if the mirror is iron, almost no color, or none at all, will appear, but if the mirror is silver, some color, albeit very faint, will appear upon the object, but it will be far fainter than [the color] on the object that lies at the location of reflection.

[2.21] It is therefore now evident that the forms of lights and colors are reflected from polished bodies and that they are weakened in reflection. And a form [of light or color] that shines directly [upon an object] will be brighter than one that is reflected [to it] when they share the same source and lie the same distance from it. But a reflected form is brighter than a secondary one when they share the same source or [originate from] sources of equal [intensity] and lie the same distance [from their source].

[CHAPTER 3]

Part three: concerning the way forms are reflected from polished bodies

[3.1] A polished body has an extremely smooth surface, and smoothness consists in the parts of the surface being continuous without many pores. Extreme smoothness exists when there is considerable continuity of the parts of the surface and the pores are few and small; and perfect smoothness consists in the absence of pores and the absence of gaps between the segments [of the surface]. Hence, the polish in a polished [body] consists in the continuity of the parts of the surface with very few and very small pores, and perfect polish consists in absolute continuity of the parts along with the [complete] absence of pores.[20]

[3.2] Reflection will occur in the case of all polished bodies, even though they may be subdivided into different shapes, and [they are all subject to] the same mode of reflection and share the same specific characteristic[s]: i.e., [that] in every polished surface reflection occurs from every point; [that] whatever point on the surface from which reflection occurs is taken, the line of incidence for any form to that point and the line of reflection [extending from that point] will lie in the same plane as the normal dropped to that point; and [that] these lines will maintain an equivalent situation with respect to [that] normal and will form equal angles [with it]. Now, by "normal" I mean the perpendicular to the plane tangent to the polished body at that point [of reflection], and the two lines [of incidence and reflection] lie along with the perpendicular in the same plane, which falls orthogonally to the plane that is tangent to the polished body at the point from which reflection occurs.[21]

[3.3] If, however, the line along which the form reaches the mirror falls perpendicular to it, the form will reflect [back] along that same line and along no other, and this is characteristic of every reflection from every polished body. Thus, if the polished body is flat, the plane tangent [to it] at the point of reflection will be one and the same as the surface of the body. On the other hand, if the mirror is cylindrical and is polished on either the inside or outside [surface], the contact between the mirror's surface and the plane tangent to it will consist only of a line imagined along the length of the mirror.[22] The same holds for a conical mirror, [whether it is] polished on the inner or on the outer [surface]. In a spherical [mirror], be it polished on the concave inner [surface] or on the [convex] outer [surface], the plane of tangency touches at only one point.

[3.4] We will explain, moreover, how this account of reflection can be empirically demonstrated for all mirrors. Take a bronze plaque not less than 12 digits long that is thick [enough] to be quite rigid, and let it be 6 digits wide.[23] Draw a line right along the lengthwise edge [of the plaque] and parallel to it. Place the point of a compass on the midpoint of this line and draw a semicircle whose radius is the width of the plaque.

[3.5] From the centerpoint [of this semicircle] draw a line perpendicular to the diameter that has just been produced [lengthwise along the plaque]. This line will form a radius that divides the semicircle in half. Along this radius measure off 1 digit [below its upper endpoint] and, with the point of the compass placed on the centerpoint [of the semicircle], draw a semicircle according to the size of the remainder of the [original] radius, according, that is, to a radius of 5 digits.

[3.6] Divide the intermediate portions of the first semicircle into as many parts as you please so that they correspond in kind with the first—i.e., [so that] the first [division corresponds to] the first, the second to the second, and so forth—and draw [straight] lines from the centerpoint [of the semicircle] to the points of division.

[3.7] Then, mark off 1 digit on the [perpendicular] radius on the side of the centerpoint, and draw a line through the point [just] marked off parallel to the diameter of the semicircle, or to the [lower] edge of the plaque, which is the same thing.[24] Cut off from the plaque the portion bounded by this line and the radius [of the larger semicircle along the lower edge] to the centerpoint as well as by the first lines dividing the semicircle—i.e., to such lines as lie nearest the [semicircle's lower] radius.

[3.8] Afterwards, cut the plaque around the [circumference of] the larger semicircle so that all that remains is the semicircle. Then cut the plaque below the center, sharpening that spot at the center to a point in such a way that it lies on the same plane as the [larger] semicircle and all the other lines.[25]

[3.9] Next, take a flat, square block of wood that is broader than the bronze plaque by 2 digits [i.e., 14 by 14], and let its height, or thickness, be 7 digits. Then mark the midpoint [on the top surface] of this block, and from it draw a circle that is a full digit larger than the larger circular segment of the bronze plaque. From the same centerpoint draw a circle equal to the smaller circle on the bronze plaque.

[3.10] Then divide the larger circle into corresponding sections equal to the sections [marked off] on the semicircle of the bronze plaque—i.e., so that the first [section marked off on the bronze plaque's circle] corresponds to the first [section marked off on the circle drawn on the wooden block], the second to the second, and so forth.[26] Cut all around the [circumference of the larger circle] on the wooden block so that only the [part bounded by the] larger circle is left; this section will now serve as a template for cutting. Cut out the portion on the block bounded by the smaller circle as well, and the way to do this is to fit another block to this one so that the line passing through the centerpoint of the former and the centerpoint of the latter is perpendicular to [the top surface] of the latter. Then, fitting a lathe to their centerpoints, form the aforementioned [hollow] circular section. Moreover, the other block should fit firmly so that it remains rigidly in place during the cutting.

[3.11] What will remain of the wooden block, then, is a circular ring that is 2 digits thick, 14 digits across, and 7 digits high, and it should be rounded along its height to form a cylinder. But the lines that divide the circumference of this ring according to the divisions [marked off] on the bronze plaque's semicircular circumference are left [on the portion of the top surface that remains].[27]

[3.12] From the endpoints of these lines draw lines along the length of the outer surface [of the ring, and draw them] perpendicular to the [top] surface [of the ring], and this can be done as follows. Find a sharply pointed ruler to whose endpoint the lines [on the ring's top surface] are applied, move the ruler around until it touches the outer surface of the ring somewhere on the [ruler's] point. Mark [the points where] the endpoints [of the measuring line touch the surface of the ring], and draw the line [through those points], for that will be the perpendicular you seek.[28] The same procedure can be followed for each of the lines of division.

[3.13] Another way of doing this is as follows. Place the point of a compass at the endpoint of [a given] line of division, draw on the outer surface of the ring a semicircle whose radius is the height of the cylinder, and divide it in half. Draw a line from [one] point to [the other] point, and so on for each [line of division]. In the same way, draw the perpendiculars to the endpoints of the lines of division on the inner surface [of the ring].[29]

[3.14] To continue, on the inner surface [of the ring] mark off points on the [aforementioned] perpendiculars at a height of 2 digits above the [bottom] face [of the ring] that is not subdivided. Through these points draw a circle parallel to the [bottom] face [of the ring] in the following way. Form a flat circular template the same size as the smaller circle on the bronze plaque [i.e., 5 digits in radius], and, up to its centerpoint, cut out from it a triangular section of whatever size you please [whose sides are formed] from two radii and an arc on the circle, which allows you to insert the template manually [into the ring], and fit it up to the points that have been marked off. Place it in this way to those points so that it is parallel to the [bottom] face of the ring, and draw a circle [on the inner surface of the ring] according to [the circumference of] the inserted tablet.

[3.15] Then, at the height of this circle, mark off points at a level of half a grain of barley below it,[30] and through the points [just] marked off draw a circle using the [aforementioned] template as a guide. Along this latter circle scoop out a circular cavity that is 1 digit deep and of the same thickness as the bronze plaque.[31] And let this cavity lie within the [previously measured-off] altitude of 2 digits so that the latter circle [i.e., the upper one defining the top of the cavity] and the [top edge of the] cavity fall in the same plane.

[3.16] Now, insert the bronze plaque into this cavity, and it will fit into the cavity all the way to the smaller circle [drawn on its surface], since the [difference in] length between the [radius of] the smaller circle and [that of] the larger one is 1 digit, which is also the depth of the cavity.[32] Hence, the latter circle and the bronze plaque will lie in a common plane, and the perpendicular lines drawn along the height of the ring [along its inner surface] intersect the lines of division [drawn] on the bronze plaque, and they will fall orthogonally to the [surface of] the bronze plaque. Make sure, however, that the surface of the bronze plaque that is subdivided faces the [upper] face of the ring that is [equivalently] subdivided.

[3.17] Next, on the outer surface of the ring mark a point at a height of 2 digits [from the bottom], and, with the point of a compass placed on the point [just] marked, draw a circle with a diameter of a single grain of barley. With an iron drill whose diameter is likewise a single grain of barley, punch a cylindrical hole [through the ring's wall]. Insert a wooden peg into the hole so that it penetrates to the inner hollow [of the ring] and will [therefore] touch the surface of the bronze plaque. In the same way, drill holes of the same kind and size, at the same height, through each of the perpendicular lines on the outer surface [of the ring].[33]

[3.18] Now, take a square wooden block [each] of whose sides is equal to the diameter of the ring [i.e., 14 digits], and through the midpoint of its surface draw a line parallel to its sides and bisecting it. From one end [of that line] measure off a length of 2 digits and mark it off. Then [from that point] mark off a distance equal to the radius of the smaller circle on the bronze plaque [i.e., 5 digits], and placing the point of the compass [at that distance], draw a circle passing through the point [just marked off], that circle being the same size as both the smaller circle on the bronze plaque and the hollow of the [cylindrical] ring.

[3.19] Above the centerpoint of this circle measure off a distance of 2 digits, and do the same below that centerpoint, and mark off the points. From each point draw a line to both sides [of the block] that is parallel to the sides of the square, and on each of these lines mark off a distance of 2 digits on each side from each of the points [just] marked. Then, from the points marked on one of the lines draw parallel lines to the points marked on the other line, and [thus] a square 4 digits on a side will be produced.[34] Hollow out this square to a depth of 1 digit, and smooth the sides of the cavity to square them off while likewise making its bottom flat.[35]

[3.20] Next, attach this block to the [bottom] face of the ring so that the smaller circle [just drawn on it] coincides with the hollow of the ring and so that its edges meet the [outer] circumference [of the ring]. Make this attachment snug with nails so that the block remains perfectly fixed. Bear in mind that the measure of 2 digits applied in all the previous situations must be accurate and definite, so make sure to maintain that measure for each [relevant] edge so that no error arises from a change in measurements.[36]

[3.21] Next, make a hollow iron tube that is even [in size throughout] and [whose walls are] fairly thick so that it does not readily penetrate [the small holes drilled into the cylindrical ring's wall] and so that it cannot be squeezed out of shape, and let the diameter of its [outer] circumference be 1 grain of barley. Insert the tube into the holes so that when it reaches the inside of the [cylindrical] ring, it will touch the lines drawn on the bronze plaque. And this procedure will be carried out perfectly if the line on the bronze plaque touches the circumference of the tube at the point where the line [that is drawn] perpendicular to the bronze plaque along the height of the ring [on its inner surface] passes through the center of the circular section of the tube.

[3.22] Attach a ring or a bolt to the end of the tube so that the tube is allowed to penetrate [into the hollow of the ring] only to a determinate point. But make the tube long enough that, as it passes over the bronze plaque, it reaches the line that is parallel to the diameter of the plaque, these two lines defining the [lower] section [of the plaque]. This line is [thus] parallel to the base of the triangle [inscribed] on the bronze plaque.[37]

[3.23] Then, make seven iron mirrors: one of them plane; two spherical, of which one is concave and polished on its inner surface, the other [convex and polished] on its outer surface; two cylindrical, of which one is concave, the other polished on its [outer, convex] surface; and two conical, of which one is polished on its [outer, convex] surface, the other on its [inner] concave surface. Let the plane mirror be circular, and let it be 3 digits in diameter.

[3.24] Let the cylindrical mirror that is polished on its [outer] surface be clear and highly polished, and let the diameter of the circle at its base be 6 digits. Furthermore, let the cylinder be 3 digits high. At the base of the cylinder measure off a chord 3 digits in length. Likewise, at the opposite base of this same cylinder measure off a chord of equal length directly facing the other chord so that the lines drawn from the endpoints of one chord to the [corresponding] endpoints of the other are perpendicular [to both bases]. Cut the cylinder along these lines so that what we have left is a portion of the cylinder whose bases consist of the sections of those chords, or [to put it another way, so that] the height of the remaining portion's axis is less than half a digit. By "axis" I mean the line [extending] from the midpoint of the arc [at the base] to the midpoint produced on the chord.

[3.25] Let the height of the concave cylinder be 3 digits, let the diameter of its base be 6 digits, measure off a chord of 3 digits on that base, and cut off a section as [you did] in the first [cylinder]. So the height of the remaining portion's axis will be less than half a digit. Make sure that all these mirrors are exquisitely polished and even throughout.

[3.26] Find a conical mirror whose base circle is 6 digits in diameter, the chord at its base being 3 digits, and let it be 4.5 digits high along the longitude. Cut out the [appropriate] portion along the straight lines [connecting the endpoints of the chord to the vertex of the cone] so that the axis of the [base] of the remaining portion is less than half a digit long, and see that this [is done] in each cone [i.e., concave and convex].

[3.27] Let the [convex] spherical mirror be a portion of a sphere with a diameter of 6 digits, and let the base [of the cut-off portion] of the mirror be 3 digits in diameter, so the axis [of this portion] will be less than half a digit. The same should be done in the case of the concave spherical mirror.[38]

[3.28] Then you should produce seven flat wooden panels whose sides are parallel and orthogonal [at the corners] so that the edges are as parallel as possible, and the panels should be 6 digits long and 4 digits wide. Next, fit [one or] another of the panels into the hollow square [in the block attached at the base of the cylindrical ring] so that it stands perfectly erect on the bottom of the hollow square, and see to it that it fits easily into the [hollow] square without being squeezed out of shape.

[3.29] Accordingly, let the point on the bronze plaque touch the face [of the panel], mark off the point where it will reach to it, and from the point [just] marked to the [upper and lower] edges of the panel draw a line parallel to the sides of the panel so that this line forms a line of longitude on the panel.[39] Then, on the longer segment of that line, from the point just marked off, measure a distance of half a grain of barley, and mark the point. I say that this point lies at the midpoint of the panel and that it also lies directly in line with [each] centerpoint of the holes [in the wall of the ring].

[3.30] [Here is] the proof. Since the centerpoints of the holes lie half a grain of barley above the surface of the bronze plaque, and since those centerpoints lie 2 digits above the [bottom] surface of the ring, then that point lies 2 digits above that same surface. But the panel is sunk to a depth of 1 digit in the hollow square [in the block attached to the bottom of the ring]. Since the distance between the [top and bottom] edges of the panel and the point is 3 digits, then that point constitutes the midpoint. At this midpoint draw a line from side to side across the panel and parallel to the [top and bottom] edges. Then bisect the [two remaining sections of the] line of longitude to which this line is perpendicular with orthogonal lines that are parallel to the [top and bottom] edges. Hence the panel will be subdivided into four equal portions. Do the same thing with the [six] other panels.[40]

[3.31] When all this is finished, fit the plane mirror into one of the panels. In order to do this, scoop out a hollow [in the panel] as deep as the thickness of the mirror so that the surface of the mirror lies in the same plane as the surface of the panel, the midpoint of the mirror's surface lies directly upon the midpoint of the panel's surface, the line bisecting the surface of the panel also bisects the surface of the mirror, and the [top and bottom] points of the mirror coincide with that dividing-line. Be as careful as possible to follow this procedure accurately.[41]

[3.32] Then fit the cylindrical mirror that is polished on its [outer convex] surface into one of the panels so that its midpoint coincides with the midpoint of the panel, the line taken along the length of the mirror that bisects it coincides with the [remaining] segments of the line along the length of the panel that bisects its surface, and the midline along the length of the mirror lies in [the plane] of the panel's surface. And this can be accomplished if the arcs at both bases of the mirror are bisected, and a line is drawn from one point of bisection to the other. Match this line to the midline along the length of the panel so as to coincide with it.[42]

[3.33] Fit the concave cylindrical mirror into a panel so that the midline taken along its length, which bisects the arcs at the bases, is parallel to the midline along the length of the panel and so that the chord on each arc, along with the parallel lines along the mirror's edge, lies in [the plane of] the panel's surface.[43]

[3.34] Fit the conical mirror that is polished on its outer [convex] surface into a panel so that its vertex lies at the end of the midline along the length of the panel, and the line bisecting the section of the cone [from which the mirror is formed]—i.e., the line that extends from the vertex to the midpoint of the arc at its base—lies in [the plane of] the panel's surface and coincides with the remaining segment of the midline along the length of the panel.[44]

[3.35] Fit the concave conical mirror into a panel so that its vertex lies right at the midline along the length of the panel, and let the chord of the arc at its base lie in [the plane of] the panel's surface. The line drawn from the vertex to the midpoint of the arc at the base [of the mirror] should be parallel to the midline along the length of the panel. Furthermore, since the cone is 4.5 digits long, 1.5 digits will be left along the longitudinal midline of the panel [on its bottom half].[45]

[3.36] In order to fit the spherical mirror that is polished on its outside [convex] surface into a panel, draw a circle 3 digits in diameter on the panel. Let its centerpoint be the centerpoint of the panel. Then scoop out [a hollow according to that circle], and fit the mirror into it so that the midpoint on its surface lies in the plane of the panel['s surface] and coincides with the midpoint of the midline along the panel's length. One can make sure that this is done correctly by applying another sharp-edged ruler of the same length [as the panel] and identically subdivided upon the midline along the length of the panel so that the midpoint of this sharp-edged ruler touches the midpoint of the spherical mirror.[46]

[3.37] Having drawn a circle 3 digits in diameter on a panel, the centerpoint of that circle being the panel's centerpoint, and having scooped out this circular section, fit the concave spherical mirror into [the resulting cavity] so that the circle at the base of the mirror lies in the plane of the panel['s surface], and so that the midpoint of the concave surface of the mirror is directly opposite the panel's midpoint. To ensure that the diameter of the mirror's base coincides with the midline [along the length] of the plank, do the following. Mark a point on [the edge] of the sharp-edged ruler, and on each side of that point measure off a distance equal to the radius of the [circle forming the] mirror's base. Accordingly, apply this sharp-edged ruler to the midline of the panel so that the point marked on it lies directly opposite the midpoint of the concave surface of the mirror and so that the diameter marked off on it is the same as the diameter of the [mirror's] base.[47]

[3.38] Once all this is accomplished, measure off from the endpoint of the radius that bisects the triangle on the bronze plaque at its sharpened point a distance equal to the axis of this concave mirror [i.e., ca. .4 digits], and mark this point. The axis, moreover, can be determined as follows. Place a sharp-pointed ruler on the surface [of the panel bearing the mirror] so that its sharp point lies directly on the midline along the panel's length and directly above the midpoint on the concave surface [of the mirror]. Then, from that point on the ruler, lower a thin, sharp needle perpendicular to the mirror. It will, of course, reach to the centerpoint of the concave [surface of that mirror]. On the needle mark the point where, after it has reached [the mirror's surface], it touches the point of the ruler or a point [previously] marked [on the ruler], and slant the ruler slightly so that the mark can be made accurately on the needle. Then, from the vertex of the triangle on the bronze plaque and along the line that bisects that triangle, measure off the distance from the needle's point to the point [just] marked on it, and mark that point [on the triangle's line of bisection].[48]

[3.39] Next, you should insert the panel [containing the concave spherical mirror] into the hollow square [at the bottom of the cylindrical ring] so that the point of the bronze plaque touches the mirror; [against the face of the panel] apply the sharply pointed ruler [orthogonally] to the line that bisects the triangle [on the bronze plaque] so that a point may be marked on that line, which is touched by that sharply pointed ruler, since the vertex of the triangle reaches all the way to the surface of the concave mirror. Accordingly, mark that point.

[3.40] This second point, however, will lie a smaller distance from the vertex [of the triangle] than the first point, because the surface of the bronze plaque lies 2 digits minus half a grain of barley from the [bottom] surface of the [cylindrical] ring, or from the [top] surface of the block containing the square cavity. On the other hand, the midpoint of the panel [containing the mirror] is directly opposite the midpoint of the concave spherical mirror, that point of course lying 2 digits above the same surface [i.e., the top surface] of the block [at the base of the ring]. Consequently, since the vertex of the bronze plaque extends orthogonally [to the panel containing the mirror], it will not reach the midpoint of the concave [surface], which is the end of the [mirror's] axis, but to a point that is higher [with respect to the axial height of the mirror], so [we have established] what we set out [to show].[49]

[3.41] Mark the spot on the concave mirror where the vertex of the bronze plaque reaches, and, having bored a hole at that point, move the vertex orthogonally [into the hole] just far enough that the sharp edge of the ruler that is applied [to the face of the panel] touches the point first marked on the line bisecting the triangle. That being so, the point of the bronze plaque will lie in the same plane as the endpoint of the mirror's axis [on the convex surface of the mirror], and that plane is parallel to the surface of the panel [containing the mirror]. And the line drawn from the endpoint of the axis to the point [of the triangle] will be perpendicular to the surface of the bronze plaque.[50] Moreover, the axis of the mirror lies in the same plane as the centers of the holes, because they lie 2 digits from the [bottom] surface of the [cylindrical] ring, and so does the endpoint of the axis [at the] midpoint [of the mirror].

[3.42] When all of this is carefully done, what we predicted can be empirically verified. Insert the panel with the plane mirror upon it into the ring until the point of the bronze tablet touches the mirror, fix the panel into the hollow square [at the bottom], and apply something [adhesive] to the bottom of the panel in order to keep it firmly nested so it does not wobble. Then press [a piece] of parchment up to the holes [in the wall of the cylindrical ring], and make an impression [of each hole] with the finger in order to fill in the holes so that you can make out the impression. Then mark the impression of [each] hole on the parchment with red ink or something else [of that kind]. Leave one of the holes open, however, but not the one directly facing the middle of the tablet, and point the open hole at an [incoming] beam of sunlight. The result of this operation will be clearer if the apparatus is held up to a ray of sunlight entering a room through a window.

[3.43] Accordingly, when the beam passing into the hole reaches the mirror, you will see it reflected to the corresponding hole [on the wall of the cylindrical ring] along the line on the bronze plaque that forms with the line bisecting the triangle an angle equal to the angle formed by the line from the open hole with the same radius [that bisects the triangle]. On the other hand, if you uncover the hole to which the ray was previously reflected and shine the light through it, you will see the ray reflected to the [previously] open hole.[51]

[3.44] Furthermore, if into the hole you insert the hollow iron tube that we made earlier according to the size of the hole (apply a bit of wax around it to make it nest snugly), the light will pass through the hollow of the tube just as it passes through the hole. And it will be reflected to the corresponding hole, and the incidence and reflection will follow the [corresponding] lines on the bronze plaque as before. Also, if we transfer the tube to the second hole, we will see the light reflected to the first one. However, the light passing through the tube will be weaker than it was when it passed through the hole without the tube in it. Still, the same way of reflecting will be observed in the case of the weaker light [as in the case of the more intense light].

[3.45] Block the tube with wax in such a way that a tiny hole remains at its center, and the light will be seen to reflect to [the corresponding] hole, or, rather, to its centerpoint. Likewise, if you fill the hollow of the tube with wax in such a way as to leave [a tiny channel] virtually the size of the [tube's] axis, the light will pass along the axis of the tube and will be reflected to the centerpoint of the corresponding hole. By the same token, if the tube is inserted into the other hole, then when the light passes along the axis of the one hole, it will be reflected along the axis of the corresponding hole, for the center of the hole lies directly in line with the axis.[52] And since the reflected light falls at the center and only radiates along a straight line, it follows that it must proceed along the axial line.

[3.46] Next, after blocking each of the holes except the middle one [whose axis] continues directly along the [central radius of the] bronze plaque, make a cylindrical pointer the size of the hole, and sharpen its end so that it comes to a point at the end of its axis. Insert it through the hole and mark the spot on the mirror where it touches. Then let a beam of sunlight shine through that hole. It will in fact fall upon the marked point, and it will form a circle around it.

[3.47] Now, mark a point on the circumference of this circle of light, and draw a circle whose radius is the size of the line joining the points just marked off. This circle will in fact be larger than the circle of the hole, for light shining through a hole propagates in the form of a cone. But the light will be seen reflected to none of the [other] holes, so it is obvious that the light shining through the axial hole is reflected back to the same hole. Nevertheless, the circle of light formed at the inner base of the hole will appear larger than the ray itself [upon entering into the ring], and it will also appear larger than the circle of light [cast on the mirror] inside [the ring].[53]

[3.48] So it is obvious that the way this light appears is due to reflection, but not to the reflection of light passing along the axial line, a fact that can be demonstrated as follows. If both bases of the hole are blocked so that all that is left is a narrow opening along the axis, and if a ray of sunlight shines along the axial line, that light will not appear to form a circle around the interior base of the hole because it was not formed by light reflected along the axis.[54]

[3.49] Earlier, moreover, we set up [this] particular panel so that it stood orthogonally on the [bottom of the] hollow square. If the conditions [under which we set it up] are slightly changed so that the panel may be slanted in such a way that the end farther from the square is inclined down toward the ray passing through the middle hole, the ray will not fall orthogonally upon the mirror, so the light will appear reflected away from the middle hole. And the greater the slant, the farther away from the hole the reflected light will be. But if the panel is restored to an upright position, the reflected light will appear around the inner base of the hole as it did before.

[3.50] It is therefore evident that, when light falls orthogonally upon the mirror, it returns back to the hole through which it entered. When, however, the light falls along a line slanted to the axis, it does not reflect to the hole but will appear upon the line on the [inner] surface of the ring that is perpendicular to the bronze plaque and that passes through the centerpoint of the middle hole.[55]

[3.51] Furthermore, everything that has been said in the case of the first pair of holes that are [correspondingly] inclined you must understand [to hold] in the case of every [other pair of such holes]. Bear in mind, as well, that what has been claimed for the plane mirror [applies to] the other [types of] mirrors, whether the light passes through an inclined hole or through the middle hole, or whether the panel is [held] upright or at a slant.

[3.52] If, however, the panel containing the cylindrical mirror that is polished on its outer [convex] surface is held at a slant in the [hollow] square so that it does not stand upright in the square but is inclined to the right or left, the light will still appear to be reflected upon the hole that corresponds to the hole through which it enters, and the light [passing through] the middle [hole will be seen reflected back] to the middle hole, just as was seen when the panel was not slanted.[56]

[3.53] You should [next] insert the panel with the concave cylindrical mirror in it, and let the point on the bronze plaque approach it until it touches the surface of the mirror. You will slant this mirror to the side just as you slanted the one polished on its outer [convex] surface.

[3.54] You will follow the same procedure for concave conical mirrors.

[3.55] Set up the concave spherical mirror so that the point of the bronze plaque enters the hole in the mirror that was made according to the insertion of that point.[57]

[3.56] The spherical mirror that is polished on its outer [convex] surface should be inserted so that the point of the bronze plaque reaches the plane of the panel and lies in the same plane as the midpoint of the mirror. This can be done as follows. Apply the sharp-pointed ruler to the panel so as to touch the midpoint of the mirror, and move its point down to the bronze plaque until that point is directly in line with the point of the ruler. Then bring them together.[58]

[3.57] In the case of cylindrical mirrors, you will observe [what happens in] reflection as follows. Set up the mirror as described before, and pass the cylindrical pointer through the middle hole, just as was done in the case of plane mirrors. Of course the pointer will fall upon the midline of the mirror along its length, and its point will lie in the plane of the panel. On this midline mark the point where it falls, and from this point measure off on the surface of [the mirror inserted in] the panel the distance of the radius of the circle drawn [earlier] on the panel for the purpose of observing how the impinging light forms a circle [on the mirror's surface]. Measure off the same distance on the other side of the point, and a line equal to the diameter of the aforementioned circle will be determined. Moreover, the impinging light will be observed to extend only upon the aforementioned line, and it is reflected to the middle hole. And the circle of light at its inner base will appear larger than the circle of light [on the mirror] inside [the ring], just as was observed in the case of plane mirrors.

[3.58] You can observe the same thing in the case of conical mirrors.

[3.59] Likewise, in the case of spherical mirrors, when light passes [to them] through the middle hole, draw a circle on the surface of the [the mirror inserted in the] panel the same size as the aforementioned circle. The light will be observed to fill this circle and then reflect to the middle hole in the way already described. In all these straight-on reflections the perpendicular line [of longitude] drawn along the inner surface of the [cylindrical] ring will appear to transect the circle of reflected light and divide it in half.

[3.60] What has been described for natural [or primary] light can be observed for accidental [or secondary] light. In a room with one window place a screen facing [the window] so that a beam of sunlight shines upon it, and set up the apparatus with respect to the window so that the accidental light [radiating from the screen] passes through one of the holes, but not the middle one. The light will be seen to reflect to the [corresponding] hole opposite it, and if the instrument is set up so that the light enters through two holes, it will be reflected to the two corresponding holes.

[3.61] Moreover, in order to be able to determine that the light enters directly and passes straight through, place the aforementioned piece of parchment [inside the cylindrical ring], and turn the apparatus until you see the light shining on the parchment. In fact, the shining of accidental light on mirrors is not clearly perceived because of its faintness. However, in this kind of light the same thing will be evident as was evident in natural [i.e., primary] light, for there is no difference in their nature except that the one is bright, the other faint.

[3.62] Thus, it is clear that lights reaching mirrors along various lines are reflected along various [corresponding] lines. And if light reaches the mirror from the same direction, it continues [from it] in an equivalent direction, and the inclination of the reflected rays will be equal to the inclination of the incident rays. And it is evident that the incident and reflected rays of light lie in the same plane, which is perpendicular to the polished surface or [the surface] tangent to the point from which reflection occurs. Moreover, if the light arrives along the perpendicular, it will reflect along the perpendicular, and no matter what point it strikes, it is reflected in a plane that is perpendicular to the plane tangent to that point.

[3.63] In addition, the reflected ray invariably forms with the normal to that point an angle equal to the angle formed by the incident ray with that same normal. And the proof of this point is evident from the fact that, if any light shines through any hole [in the ring], it is reflected to the corresponding hole. And if the hole is narrowed so that all that is left is virtually equivalent to the axial line, the light is reflected along the axis of the corresponding hole. And if the light shines through the other hole, it is reflected along the lines according to which it was incident before. And it is obvious that the corresponding holes are equivalently disposed with respect to the middle hole, and since light only propagates along straight lines, it is evident that it is reflected along lines that are equivalently disposed with respect to the middle hole as the lines of incidence.

[3.64] Hence, when it arrives along the orthogonal, it is reflected along that line alone, because the lines of reflection invariably maintain the same disposition as the lines of incidence with respect to the plane that is tangent to the point of reflection. And this is [an] essential characteristic [of light], whether the light be essential or accidental, or whether it be intense or faint, and it applies universally in every case.[59]

[3.65] Now we shall demonstrate the equivalent disposition [of the lines of incidence and reflection]. We already know that the surface of the panel stands orthogonal to the [bottom of the] block [at the base of the ring] in which we dug out the [hollow] square. Hence, the midline of the base-block is perpendicular to the common section formed by the [top surface of the] base-block and the panel, and it is [therefore perpendicular] to the line across the width of the panel [formed by this common section]. Moreover, the [top surface of] the base-block is parallel to the [top surface of the] bronze plaque, and its midline is parallel to the midline of the bronze plaque, that is, to the line drawn from the center of the bronze plaque [at the point of its triangle] that bisects its arc.

[3.66] Furthermore, the common section of the [top surface of the] bronze plaque and the panel, which is a line across the width of the panel, is parallel to the common section of the base-block and the panel, so the midline of the bronze plaque falls orthogonally to the common section of the panel and the bronze plaque. And the panel stands perpendicular to the [bottom] surface of the square [hollowed out of the base-block], and the [bottom] surface of the square [hollowed out of the base-block] is parallel to the [top] surface of the base-block [itself], so the [top] surface of the base-block is orthogonal to the surface of the panel.

[3.67] Likewise, the surface of the bronze plaque stands perpendicular to that same surface [i.e., of the upright panel], and the midline along the panel's length is perpendicular to the line across its width, so the midline of the base-block [at the bottom of the ring] will be perpendicular to the midline along the length of the panel, since it stands [upright] upon it; and, by the same token, the midline of the bronze plaque will be perpendicular to that same midline. Accordingly, the midline of the bronze plaque is perpendicular to the surface of the panel as well as to the midline along its length, and so it is perpendicular to the surface of the plane mirror as well as to the midline along its length.[60]

[3.68] In addition, the surface of the bronze plaque is parallel to the plane passing through the centers of the holes [in the ring], for the centers of [all] the holes lie the same distance from the surface of the bronze plaque—i.e., half a grain of barley—and the diameter of [each] hole is 1 grain of barley. Likewise, the diameter of the surface of the [iron] tube is 1 grain [of barley], and the plane passing through the centers of the holes bisects the tube. Hence, the axis of the tube lies in that plane, and, as it extends [into the hollow of the ring], the tube touches a line [drawn] on the [surface of the] bronze plaque, that line of course being parallel to the [tube's] axis, for that axis is parallel to any line [of longitude] on the surface of the tube.

[3.69] Moreover, the axis of the tube falls to a point on the surface of the panel, and the line drawn from that point to the center of the bronze plaque is perpendicular to the bronze plaque, because, no matter what hole the tube extends through, its axis falls upon the midline along the length of the panel, and all of these perpendiculars are equal [in length].

[3.70] Also, the line extended from the point where the axis falls on the panel through the center[s] of the holes is parallel to the line extended from the center of the bronze plaque to the endpoint of the hole's diameter. For the line [drawn] between that point [on the panel's midline] and the [bronze plaque's] center is perpendicular to the surface of the bronze plaque, since it is a segment of the midline along the length of the panel, and [it is] also [perpendicular] to the axis [of the hole]. And [every] line on the [inner surface] of the ring that passes through the centers of the holes and falls perpendicular to the surface of the bronze plaque is parallel to this line, which extends between the centerpoint of the bronze plaque and that point [where the axes of the holes intersect the midline along the length of the panel]; hence, the lines extending from the [respective] endpoints of the [segment of the] line on the [inner surface of the] ring and [the endpoints of the segment of the midline] along the length of the panel will be parallel [since they connect the endpoints] of equal and parallel [lines].

[3.71] The same holds for every one of the holes, because the lines drawn from the point on the panel where the axis falls to the centers of [any] pair of corresponding holes are parallel to the two lines drawn from the center of the bronze plaque to the endpoints of the diameters of the same holes, so the angle formed by these two lines is equal to that formed by the other two.

[3.72] If a line is erected from the endpoint of the axis [where it intersects the midline along the length of the panel] to the center of the [middle] hole, it will lie in the plane passing through the centers [of all the holes], and it will be parallel to the midline of the bronze plaque. For the line connecting the inner ends of these lines is perpendicular to the bronze plaque, and it is equal to the line connecting their outer ends, which is perpendicular to the bronze plaque. And [this inner line] is parallel to that [outer one], so the line from the center of the middle hole to the endpoint of the tube's axis [where it intersects the midline along the length of the panel] is parallel to the midline of the bronze plaque, and that line is perpendicular to the panel, so the other is too.[61] Therefore, this line [extending from the center of the middle hole to the midline along the length of the panel] and the sides forming alternate angles [with it] are parallel respectively to the midline of the bronze plaque and to each of the [corresponding] lines on the [surface of the] bronze plaque forming the [same] angle, so the corresponding segments [of the entire angle formed by those respective sides] are equal.

[3.73] Hence, the midline of the bronze plaque bisects the [entire] angle in its plane, so the [axial] line [drawn] from the center of the middle hole bisects the angle in its plane.[62] And since there is no doubt that the light entering one of the inclined holes proceeds along lines that form an angle, it is evident that all light is reflected along lines that lie on the same plane as the lines of incidence, that plane being orthogonal to the reflecting surface [formed by the panel's face], and such lines [of reflection] form an angle with the normal that is equal to the angle formed by the lines of incidence with that normal.

[3.74] Furthermore, light that falls along the normal is reflected along the normal. And this holds universally for all light.

[3.75] However, if the panel is slanted not sideways but [lengthwise] from the top so that the axis of the middle hole is not perpendicular to the panel, the light is [still] reflected, and it will be seen on the line drawn lengthwise [on the inner surface] of the ring, this line being perpendicular [to the bronze plaque] and passing through the center of the hole. And the greater the slant, the farther the reflected light will fall from the hole or from its axis. However, if the slant is decreased, the distance [between the center of the hole and where the reflected light falls] will decrease, so when the panel is restored to a perfectly upright position, the light is reflected along the normal.

[3.76] Furthermore, it is evident from the following that in the case of such an orientation the axis of the middle hole and the line of reflection lie in the same plane, which is orthogonal to the reflecting surface [formed by the panel's face]. For the axis of the middle hole is perpendicular to the [line along the] width of the panel—i.e., to the common section of the panel's surface and the plane passing through the centers of the holes—and the midline of the block [at the bottom] of the ring is parallel to this axis and parallel to the midline of the bronze plaque.

[3.77] Also, the midline of the bronze plaque is perpendicular to the [plane] of the panel along its width, and [so] it is perpendicular to the common section of the panel's surface and the surface of the bronze plaque, so the plane in which the midline of the bronze plaque and the axis of the middle hole lie is orthogonal to the panel's surface. Moreover, the line drawn lengthwise [and perpendicular to the bronze plaque on the inner surface] of the ring lies in this same plane, for it passes through the endpoints of parallel lines—i.e., the midline of the bronze plaque and the axis of the middle hole.

[3.78] It follows, therefore, that the reflected light appearing on the perpendicular drawn lengthwise [on the inner surface] of the ring is reflected along a line that lies in the same plane as the axial line along which it is incident, and that plane is orthogonal to the surface of the panel. Hence, when light is incident upon a plane mirror, its reflection occurs along lines that are identically inclined to the surface of the mirror, and those lines lie in the same plane as the normal, that plane being orthogonal to the surface of the mirror.[63]

[3.79] In the case of the cylindrical mirror polished on its outer [convex] surface, the proof is precisely the same as in the case of the plane mirror—i.e., because the point of the bronze plaque falls orthogonally to the [mid]line along the length of the mirror, and so does the [axial line through the iron] tube when it reaches it. And the segment of the [mid]line [of the mirror] that passes through those [points] is orthogonal to the bronze plaque. Invariably, then, whether the light shines through the middle hole or through one of the inclined holes, its [line of] reflection will lie in the same plane as its [line of] incidence, that plane being orthogonal to the plane that is tangent to the [mid]line along the length of the mirror.

[3.80] In the case of the conical mirror [polished] on its outer [convex surface], since the surface of the panel lies in the same plane as the midline along the length of the conical mirror, as it does in the case of the [convex] cylindrical mirror, the lines in the planes [of the axial lines and the bronze plaque] will be equivalently disposed, so reflection will occur in the same way as it does in the plane mirror, and the demonstration will be precisely the same.

[3.81] In the case of the concave cylindrical mirror, as well, the point of the bronze plaque falls on the midline along its length, and the axis of each hole falls on that same line. And the line-segment between these [two points] to which [the two lines] fall is orthogonal to the surface of the bronze plaque, and the axis of the [middle] hole and the midline of the bronze plaque are orthogonal to the plane tangent to that mirror upon the [mid]line along its length, which is where reflection occurs, and they are parallel to the surface of the [bronze] plaque.

[3.82] Thus, as before, [we follow] the same method for proving that [the lines of] reflection and [the lines of] incidence lie in the same plane, which is perpendicular to the reflecting surface [formed by the panel's face], that those lines are equivalently slanted [with respect to the normal dropped to the point of reflection], and that [light] incident through the central hole is reflected back to that hole. Also, when the top edge of the panel is slanted [backward], the light will be reflected to the perpendicular [drawn lengthwise on the inner surface] of the ring, just as was pointed out in the case of the plane mirror.

[3.83] In the case of the concave conical mirror, the same proof holds in all respects.

[3.84] In the case of the [convex] spherical mirror [polished] on its outer surface, it is clear that its midpoint lies in the plane of the panel, while its axis falls on that point, so the lines [of incidence and reflection] will have the same disposition in that case. And what holds for the plane mirror holds in all respects for the other mirrors, and the demonstration is the same [for all].

[3.85] In the case of the spherical concave mirror it has already been determined that the axis of the hole passes to the mirror's midpoint, and the point of the bronze plaque passes into the hole in the mirror, which we made earlier, until it lies in the same plane as that midpoint. And the line drawn from that point to the point [of the bronze plaque] is parallel to the midline along the length of the panel, and so [the lines of] incidence and reflection lie in a plane that is orthogonal to the plane tangent to the mirror at that midpoint, [that latter plane being] parallel to the surface of the [bronze] plaque. And in this case the demonstration is precisely the same as it is for the other mirrors.

[3.86] Hence, it is evident that if any light shines on any of those mirrors [just discussed], the [lines of] reflection and incidence lie in the same plane, which is orthogonal [to the surface of the mirror or the plane tangent to the point of reflection on the mirror]. But the reason reflection follows this rule is not specific to the axis, or to the point on which the light shines, or to the hole through which it shines, or to the mirror. Indeed, it holds for any hole, no matter what kind of light [shines through it], and it holds for any line of incidence as well as for any point on the mirror to which the light may fall. For, no matter what point on the mirror is taken, if light shines on it, since its disposition is the same with regard to the [midline along the] length of the mirror, and since any of the others will be equivalently disposed with regard to the lines extended from it, those lines are identically inclined with respect to the lines understood [to extend] from that point, just as from the point taken earlier, as well as from any other point.

[3.87] So it is invariably the case that the disposition of each [such line] at the point to which the light falls, which is the same as the previously taken point, is equivalent with respect both to the axis and to the point of the bronze plaque. And the same proof and the same demonstration hold for all cases, so it is certain that this is due not to a particular kind of light nor to the shape of any particular mirror, but it is a characteristic that is common to every polished body and to any kind of light. Moreover, when the light shines through various holes to a given point, the variation in reflection and in the angles of reflection will be observed to conform to the way the light is incident, and the same holds in all cases.

[3.88] It is obvious from what we have established above that, if a polished body faces a luminous body, light from any point on the luminous body falls to any point on the [exposed surface of the] polished body, so at any point on the polished body there stands a cone whose vertex lies on that point and whose base is formed by the surface of the luminous body. Also, from any point on the luminous body there extends a cone whose vertex lies at that point and whose base is formed by the [exposed] surface of the polished body.

[3.89] Moreover, if some point is imagined between the luminous and polished bodies, the light from the luminous body will reach that point in the form of a cone whose vertex lies at that point, and when the edges of that cone are extended until they reach the surface of the polished body, they form a cone. Hence, the vertices of two cones will lie at the point [so] imagined, their bases consisting of the surface of the luminous body and the surface of the polished body, and if a cone is imagined at some intermediate point with its base formed by the polished surface, and if the lines [contained] by this sort of cone are extended, the portion of the luminous body's surface that they will envelop forms the part from which the light has radiated to the polished body according to two cones whose vertices lie at the [intermediate] point [just] imagined.

[3.90] And the light that propagates within these two cones radiates within and is contained by the first pair of cones, and the light propagates from the luminous source to the mirror along parallel lines, but these lines are included within the first pair of cones. Moreover, whatever lines the light follows as it radiates to the mirror, the lines of reflection maintain precisely the same relative disposition as the lines of incidence the light originally followed, so if the light radiates along parallel lines, it is reflected along parallel lines, and light falling within the extent of the polished body in the form of a cone is reflected in such a way as to form an equivalent cone.[64]

[3.91] When light shines from a luminous body to a polished body through some hole, and if a point is imagined in the plane of the hole facing the luminous source, and from that point two cones are imagined, one with its base on the luminous body, the other on the polished body, light reaches the polished body through that point only from the base of a cone whose base is on the luminous body. Likewise, if a point is imagined in the plane of the hole facing the polished body, that point forming the vertex of two cones, one based on the mirror, the other on the luminous body, the light reaches the polished body through that point only from the base of a cone whose base is on the luminous body.

[3.92] So it is from the direction of the luminous body, which forms one common [base] of these two cones, that light shines toward the mirror, [which forms the other] common [base] of the two cones. Also, light reaches the mirror from the luminous source along parallel lines, and no matter what lines it follows, it is reflected in the way described earlier. And all of the lines of reflection maintain an equivalent disposition with respect to the corresponding lines of the light's incidence, and in every reflection the light maintains the same precise form as it had in the polished body, and in what follows we will explain this in a clear manner.[65]

[3.93] Now it has been established that the farther light extends from its source, the weaker it gets. It has also been established that concentrated light is more intense than dispersed light.[66] Thus, since light shines from any point on a luminous source to the surface of a mirror in the form of a cone, the farther it extends from that point, the weaker it gets for two reasons: because of its [increasing] distance from its source and because of its dispersal. Furthermore, when light is reflected from a point on a mirror, it is weakened in three ways: because of the reflection [itself], which weakens [the light]; because of its [increasing] distance from the point of reflection; and because of its dispersal.

[3.94] If, however, the light that is reflected from the mirror is concentrated at some point, it will be strengthened by that concentration, although it is weakened by the reflection [itself] as well as by its [increasing] distance [from its source].[67] Therefore, if the concentration of the light intensifies it as much as the fact of reflection and the distance [it lies from its source] weaken it, the concentrated reflected light will be as intense as it is at the mirror's surface. On the other hand, if the increase in intensity due to concentration is outweighed by the weakening due to the other two factors, the [reflected] light will be weaker [than it is at the mirror's surface], whereas if that increase outweighs [the weakening], the light will be more intense.[68]

[3.95] Likewise, if the cone [of radiation] extends from the luminous surface [which forms its base] to some point on the mirror [which forms its vertex], the light radiating in the form of such a cone will be weakened by [increasing] distance but intensified by concentration. If, moreover, the [intensification caused by] concentration outweighs [the weakening caused by increasing] distance, the light concentrated at the point on the mirror will be more intense than a single [point of] light that radiates from the luminous source [to the mirror] along a single ray. I say "single," because a cone radiates from the luminous source to any point on the line chosen among [all those radiating from the given point on the luminous source], but such a cone is excluded from this consideration along with others of its kind.

[3.96] But if [the weakening due to] distance outweighs [the intensification due to] concentration, the light at the point [of incidence] on the polished surface will be weaker than the light taken along a single line of radiation, whereas if [the intensification due to] concentration outweighs [the weakening due to] distance, the light [at the point of incidence] will be stronger [than the light taken along a single line of radiation].[69] Moreover, light that extends from the luminous source to the mirror along parallel lines will be weaker than that extending in the other way [i.e., along a cone with its base in the luminous object and its vertex at the mirror], for such light does not become concentrated on the mirror to counterbalance the weakening due to distance [from its source], and it proceeds along parallel lines after reflection. Hence it is weakened both by reflection [itself] and by [increasing] distance. But if it is concentrated upon reflection, that concentration will intensify it to the degree that it was intense [when it shone] upon the mirror, insofar as such concentration can balance out [the weakening due to both] reflection [itself] and [increasing] distance.[70]

[3.97] Now, every line along which light radiates from a luminous body to a body facing it is a sensible line, not one without breadth, for light proceeds only from bodies, since it can subsist only in bodies. But in the least light that can be imagined there is some breadth, and [so] there is breadth in the line [or shaft] along which it radiates. And in the middle of that sensible shaft there is an imaginary line, and [all] the other [such] lines within that sensible shaft are parallel to it. So if the least [possible amount of visible] light is divided, neither part of it will constitute [actual, sensible] light; rather, both will be [effectively] extinguished and will [therefore] not be visible. On the other hand, if the least [possible amount of visible] light is doubled, or further multiplied according to equal increments, and if the resulting compound [light] is divided [equally], both of its portions will constitute [visible] light. But if [that compounded light] is divided unequally, one portion of it will constitute [effective] light, and the other [will be diminished] to the smallest [possible amount of visible light].[71]

[3.98] Now the smallest [possible amount of visible] light radiates upon the smallest portion of the [surface of a] body that light can occupy, and it radiates along an imaginary line centered within the sensible shaft [of radiation] whose edges are parallel to it. So the least quantum of light falls not upon an imaginary point on a [facing] body, but upon a sensible spot, and it is reflected along a sensible shaft that is the same breadth as the sensible shaft along which it reached [that body]. So if one conceives of an imaginary line in the center of the sensible shaft of reflection, it maintains the same disposition with regard to the point of reflection as the imaginary line in the center of the sensible shaft of incidence, and every line imagined in the sensible shaft of reflection maintains precisely the same disposition with respect to the imaginary line corresponding to it in the other sensible shaft [of incidence]. Thus, in the case of any light, the reflection of lines and of imaginary points follows this rule, even though light does not [actually] proceed from such [points] along such [lines], and this is how the reflection of light will occur.[72]

[3.99] Moreover, the reason that reflection occurs from polished rather than from rough bodies is because, as we have [already] pointed out, light approaches [any given] body with only the swiftest of motions, and when it reaches a polished body, that polished body causes it to rebound from it.[73] A rough body, on the other hand, cannot cause it to rebound because there are pores in the rough body into which the light enters; in polished bodies, though, it encounters no pores. But this rebound is not due to the [physical] resistance or hardness of the body, for we see reflection [occur] in water; on the contrary, this [kind of] repulsion is a function of polish, just as happens in nature when some heavy object falls from high above onto a hard stone [surface] and bounces back up, and the less hard the stone [surface] upon which it falls, the weaker the rebound of the falling object will be.[74] Furthermore, the falling object will invariably rebound in the opposite direction from that along which it [originally] traveled. The rebound that occurs in the case of a hard body does not, however, occur in the case of sand because of its softness.

[3.100] Moreover, [even] if there is some polish in the pores of a rough body, the light entering into those pores is nevertheless unreflected, and if it does happen to reflect, it is scattered and is not perceived by sight because of this scattering. By the same token, if the elevated portions of the rough body are polished, they will cause the reflected light to scatter, and on that account the reflection will be unnoticeable to sight. If, however, the height of these portions is slight, so that it is roughly the same as that of the lower portions, then the light reflecting from it will be perceived as if it came from a polished rather than from a rough body, even though [it will be perceived] less clearly [than light reflected from a perfectly polished surface].[75]

[3.101] Now, the reason that the reflection of light will occur along a line that has an equivalent disposition with the line along which that same light will reach the mirror is because light moves extremely swiftly, so when it strikes the mirror it is not absorbed by the mirror; instead, its being trapped in that body is prevented. And since, on that account, it conserves the force and nature of its previous motion, it is reflected back in the direction along which it arrived and along lines that have the same [relative] disposition as the original lines [of incidence].

[3.102] We can observe the same thing in the case of natural as well as accidental motions.[76] If we drop a heavy spherical body perpendicular to a polished body from some height, we will see it reflected back along the perpendicular it followed in dropping. In the case of accidental motion, if the mirror is raised to the height of a man and is attached firmly to a wall, and if a spherical body is attached to the point of an arrow, and the arrow is shot from a bow at the mirror in such a way that the arrow is at the same height as the mirror (and the arrow should be parallel to the horizon), it is obvious that the arrow reaches the mirror orthogonally, and you will see it rebound along the same orthogonal. If, on the other hand, the arrow flies along an oblique line with respect to the mirror, it will be seen to reflect not along the line according to which it arrived but along another one that is not parallel to the horizon, as was the case with the earlier path, but it will maintain the same disposition with respect to the mirror as that [original path] and with respect to the normal to the mirror.[77] Moreover, that the motion of light in reflection is due to the resistance of the polished body is evident from the fact that the more intense the repulsion or resistance [of that body], the more intense the reflection of the light will be.[78]

[3.103] Here is why the motions of incidence and reflection turn out to be identical. When a heavy object falls orthogonally, the rebound from the polished body and the motion of the heavy body's incidence are perfectly opposed, so in this case there is motion only along the perpendicular. And the resistance occurs along the perpendicular, which is why the body is repelled along the perpendicular, so it rebounds orthogonally. When, however, that body impinges along an inclined path, the path of incidence lies between the normal that passes through the surface of that polished body and the line on its surface that is orthogonal to that normal.

[3.104] But if the motion [of the body] were to penetrate through the point [on the polished body's surface] that it strikes so as to find free passage, then the resulting line [of transit] would fall between the normal passing [through the surface of the polished body at the point of incidence] and the line on [the polished body's] surface that is orthogonal to that normal. Furthermore, it would maintain the same degree of orientation with respect to the normal passing [through the surface] as well as with respect to the other line that is orthogonal to that normal. For the measure of the disposition of this motion depends on the disposition [of its motion with respect] to the normal and the disposition [of its motion with respect] to the orthogonal [to that normal].

[3.105] However, since the resistance along the normal cannot balk the motion according to its measure along the line that passes orthogonally through the normal, because the impinging body does not penetrate [the reflecting surface], the polished body repels it according to the measure of its disposition with respect to the normal that is perpendicular to the orthogonal [that is parallel to the mirror's surface]. And since the motion of rebound has the same degree of orientation with respect to the orthogonal [to the normal] that it had with respect to that same orthogonal on the other side, then the degree of its orientation with respect to the normal [after penetration] will be the same as it was before.[79]

[3.106] But when the force of resistance that pushes a heavy, rebounding body upward is exhausted, that body, by its very nature, tends to drop to the center [of the universe]. Light, too, has the same intrinsic tendency to rebound, although it is not in its nature to rise or fall, so, in the case of reflection, it follows its original path until it reaches an obstacle that causes the [incident] motion to stop, and this is what causes reflection.[80]

[3.107] It is also clear from [what we have established] above that colors radiate in tandem with light, so the reflection of color will be like that of light. And if you want to determine this fact observationally in the way described in chapter 2, you can do so by again using the [ringlike] apparatus. In attempting to observe such reflection accurately, you will not see [the relevant phenomena] clearly because of the weakness of color, for color is weakened by distance, by reflection [itself], and by the [restriction posed by the] hole through which it passes. That the [restriction posed by the] hole has a weakening effect is evident from the fact that light appears more intense after [passing through] a large hole than [after passing through] a small one. By the same token, when the holes [through which the color passes] are narrow, no color at all will be seen after reflection, or it will appear very faint. Nevertheless, if you wish to observe [the reflection of color] in the aforementioned apparatus, then you should make the mirror out of silver, for color appears very weak in an iron mirror, because it would be mingled with the reflected light, which is composed of the incident light and the dim light in the iron mirror, and the color of the iron mingled with the reflected color would weaken it.

[3.108] Once again, place the aforementioned apparatus in the room with only one window facing a white wall inside the room. Set the apparatus toward the room's window, which is narrow enough that [the space between] two of the holes in the apparatus can block it, [and let it be set up] so that the white wall facing it in the room can be viewed [in the inserted mirror] through either of the holes. Place a brightly colored body in front of the visible portion of the wall, and look at that portion of the wall through either hole in the apparatus. Accordingly, when the [colored] light enters through the holes in the apparatus, the color will be seen reflected through the hole corresponding to the one facing the colored body, [but it will be] invisible through the other hole. And this will happen with any [colored] body facing the hole, and what has been claimed about the reflection of light can be extended to the reflection of color. Moreover, the opening in the wall was as wide as the two holes in the apparatus exposed to it so that the light might shine more intensely upon the mirror, and the reflected color is [thereby] more apparent. And because color is weakened when it shines directly through an opening and [is] likewise [weakened when] reflected, then, when it shines on a body facing the eye, it will be perceived as secondary, so if, after reflection, it shines on a white body that is exposed to the color at the opening, it may not be perceived by the eye because of the [resulting] weakening. When, however, the color at the second hole is exposed to the eye, it may be perceived because it will be seen as primary rather than secondary.[81]

[CHAPTER 4]

Part four, [showing] that the forms in bodies are perceived through reflection

[4.1] A number [of authorities] disagree about how the [visible] form is perceived in polished bodies. Accordingly, some of them [suppose] that rays emanate from the eye to the mirror, return from the mirror, and perceive the form of an object [seen in the mirror] upon its return.[82] Others claim that the form of the object is impressed upon a facing mirror, so it is seen in the mirror the same way that natural forms of objects are perceived in objects.[83]

[4.2] That in fact [the perception of images in mirrors] happens in another way [than this] is clear from the following. If someone sees himself moving in a given direction in some portion of a mirror, he will no longer see himself in the original place, but in a subsequent place, and this would not happen if his form were impressed on the original portion [of the mirror]. By the same token, if he moves to a third location, the place where his form appears [in the mirror] will shift, and his image will not be seen in the first or second location.

[4.3] Moreover, when some object is viewed [in a mirror] and the viewer draws away from it, it may happen that the object cannot be seen in that mirror, even though he may see the entire surface of that mirror, and this would certainly not be the case if the form were impressed on the mirror['s surface] while the mirror remains in view and does not change its location, and while the object, too, remains stationary as its form is impressed on the mirror, just as before.

[4.4] In order to make it obvious that this is not how the form is perceived, block up half of the holes of the [ringlike] apparatus,[84] and place something with a letter on it at one of the blocked holes. If [the plane] mirror [inserted] in [its] panel is viewed through the hole corresponding to the hole where the letter lies, the letter will be perceived in the mirror, but it will not be [perceived] through any other hole. If the form of the letter were impressed on the [surface of the] mirror, though, it could be perceived through any hole in the apparatus. Likewise, in the cylindrical mirrors, the situation of the letter will be perceived only through the corresponding hole. However, in the conical and spherical mirrors, both the situation and the size of the writing will be changed.[85]

[4.5] Furthermore, if the cylindrical mirror is taken out [of the apparatus] and the panel [in which it is inserted] is placed [sideways] with its top and bottom edges standing vertical, the face of a person will appear frontal in it. If, however, the panel is placed upright or is placed at a considerable slant, [the viewer's face] will appear distorted.[86] So it is obvious that what is perceived will not be due to the form's being impressed on [the surface of the] mirror, since the visible object is not perceived in mirrors unless the eye is at the location of reflection. It is also obvious that the apparent distortion of the face is not due to the [actual] form of the object but to the disposition of the mirror [within which that form is seen].

[4.6] Moreover, if an object is viewed in a mirror and then drawn away, that object will be perceived [to lie] farther behind the mirror than before, and this will not happen if the form of the object lies on the surface of the mirror and is perceived there. Hence, reflection causes the form to be perceived [as it is] in a mirror.[87]

[CHAPTER 5]

Part five, [showing] how forms are perceived in polished bodies

[5.1] It has just been shown in [the third] chapter above that, when a luminous colored body faces a mirror, light, along with color, radiates from any point on it to the entire surface of the mirror and is reflected along the appropriate lines of reflection. Thus, from any given point on a body facing the mirror, light, along with color, radiates to the mirror in the form of a continuous cone with its base on the surface of the mirror, and its form is reflected along lines whose disposition is equivalent to that of the lines of incidence, so the [cone of radiation] after reflection will be continuous with the cone of incidence. And if the form reaches the surface of an object [that faces the mirror as it radiates] along reflected lines, then, because those lines form a continuum, the entire form will fill the [facing] surface [of that object], leaving no area on it untouched. Accordingly, if the form of that body were to radiate to the mirror along these [same] lines (i.e., the [original] lines of reflection), and if it were to reach the base of the cone [on the mirror's surface], then, since the lines forming this same cone [of incidence] are disposed in the same way as [subsequent] lines of reflection, the form is reflected along lines that form an [equivalent] cone, and the entire form will be concentrated on the point [previously] chosen [on the original luminous body].[88]

[5.2] So, whenever the form of a given body reaches the mirror along some given lines, then, since those lines are disposed in the same way as the lines forming a cone that can be imagined projected to the mirror from a given point [on the other side], and since those lines correspond, the form [of the given body] will reach the given point according to that [imagined] cone. And if the eye lies at that given point, it will see the body that is represented by that form. And it has been established earlier that a form is apprehended in a mirror at a specific location.[89] Hence, the appropriate location and the natural way in which sight perceives [an object] in reflection are such that the lines [of radiation] along which the form reaches the mirror are disposed in the same way as the lines comprising the cone that is imagined to extend from the center of sight to the endpoints of each of those corresponding lines, and the reflected form cannot be perceived anywhere else but at this point.

[5.3] It is thus evident that the perception of forms [in reflection] occurs according to this arrangement of lines alone. It is also evident that light, along with color, radiates to the mirror from a luminous colored body and is [then] reflected [from it], and nothing but color and light radiate from that body. So it follows that the form of such a body is perceived on the basis of light and color alone, and since the form consisting of light and color radiates in such a way as to maintain the disposition described above, it is superfluous to claim, as many authorities have done, that rays pass from the eye to the mirror and then reflect to the place just described. So this [account of] how reflection occurs does not contradict, but accords with, the theory proposed by the geometers, for it preserves the disposition of the visual rays posited in the geometrical theory, and up to now this way [of accounting for reflection] has been evident only to me.[90]

[5.4] However, since the form radiates from the luminous body to the mirror according to various dispositions along lines that are imagined [to extend] from each point on the body to the entire surface of the mirror, the same form will be reflected according to various cones with their vertices at various points and their bases on the surface of the mirror, [and they will be reflected in such a way as] to maintain a [corresponding] disposition with the lines along which the form radiates. From this it follows that at any given instant, with the mirror fixed in place, the form of a body is perceived from various viewpoints where the vertices of the cones of reflection lie. By the same token, when a given center of sight passes over the vertices of such cones while the mirror remains stationary, the same form will appear at different locations in the mirror. But when the same form is perceived at different places in the mirror, the lines of sight are directed to different spots on the mirror, for they cannot grasp the forms of different points on the body identically at the same spot on the mirror.

[5.5] It has already been asserted that light radiates from any point on a [luminous] body to any point on a mirror, so at any point on the body lies the vertex of a cone whose base is the surface of the mirror. Also, any point on the surface of the mirror constitutes the vertex of a cone whose base is the surface of the body. Thus, the entire form of the body will be [radiated] to each point on the mirror along lines that go in different directions, and they cannot possibly intersect [before that point]. In addition, the form reaching from the [luminous] body to any point on the mirror in the form of a cone will be reflected in the form of a cone. And the same process by which the form reaches the mirror can be multiplied indefinitely, for the entire form overlaps with each part and at each point [on the mirror]. And there is no interruption in such forms; instead, there is an absolute continuity in reflection. Still, since the entire form does not fall at disparate spots on the mirror, it is directed with the same disposition to different spots where the viewer perceives it.[91]

[5.6] Thus, if the shape of the mirror is the same as the shape of the object [seen in it], the complement of the form and shape of the body will be [seen] in the mirror. For in the case of a mirror that is the same shape as the object, the form of the first point [on that object] is directed to the first point on the mirror, the second to the second, and so forth among all the corresponding points. Hence, the shape of the entire object will be [seen] on the mirror's surface, which does not happen in the case of a mirror with a different shape.[92] Likewise, if some portion of the mirror is taken, and if it is the same shape as the object [seen in it], the complement of the shape of the object will be [seen] in it. Moreover, since such parts on the mirror are infinite, the forms of the body that are reflected will be infinite, but they will radiate to different points where the viewer perceives the form.

[5.7] Thus, since the form is perceived according to this arrangement of lines, the form radiating from the body will not be impressed on the surface of the mirror. And this is the way reflection occurs for all mirrors, but most evidently in plane mirrors; in the other [sorts of mirror], however, some variation occurs because of visual error, which occurs in the way previously mentioned.[93] So, according to the way just described, any [given] center of sight perceives only one of the object's points at one point on the mirror, and the same point on the object is not perceived by two centers of sight at the same point on the mirror.

[5.8] Furthermore, suppose that a mirror faces the eye, and imagine a cone [extending] from the center of sight to the surface of the mirror. Let a point be taken on the base of this cone, and imagine a line [extending] within this cone from the center of sight to that point. Then, given that an infinite number of lines can be extended from that point, since each of them maintains an equivalent disposition with respect to its cone's edge and forms an equal angle with the normal, and since this holds for every point taken on the mirror, it is clear that reflection can occur from every point on the mirror. I say, therefore, that among [each of] the lines extended from the given point there is a line that is equivalently disposed with respect to the edge of the cone, and it forms an equal angle with the normal to that point. And this [line] forms the edge of a cone imagined to extend from that point to the surface of the [visible] object, and whatever lies at the endpoint of that line, since [its form] reaches the given point [on the mirror] along that line, [that form] will be reflected to the center of sight along the edge of the cone just described. Furthermore, this edge of the cone will lie in the same plane as the line projected from that point, and that plane is perpendicular to the plane tangent to the mirror at that point. And I say [that] this [is so] when the edge of the cone is oblique to that point. If, in fact, the [line along the] edge of the cone extending from the center of sight falls orthogonally to the plane tangent to the surface of the mirror at the given point, it will be reflected back onto itself and will return to the source of its propagation in the eye.

[5.9] What we have claimed is obvious in the case of a plane mirror, for at whatever point on the [mirror's] plane surface the ray from that point [on the given object] falls, a normal can be erected to that surface. And from the center of sight a line can be imagined to fall orthogonally to the surface continuous with the aforementioned surface, or to the surface itself. And these two perpendiculars will lie in the same plane, because they are parallel, and the line extending in that plane surface from the endpoint of one to the endpoint of the other will form an acute angle with both and will lie in the same plane with both. So the ray projected according to that line will form an acute angle with the normal to the mirror [at the point of reflection], and likewise [it will form an acute angle] with the perpendicular drawn from the center of sight.[94] Then imagine a line drawn on the other side of the plane surface passing orthogonally through the endpoints of the perpendiculars. On that other side it will form a right angle with the normal to the mirror, so from that right angle an acute angle can be cut that is equal to the acute angle the ray forms with that same normal. These two angles [will lie] in the same plane, so the incident and reflected rays lie in the same plane as the two aforementioned perpendiculars. Moreover, if some other point [on the visible object] is viewed [in the mirror], the [two] rays will be equivalently disposed with respect to the perpendiculars, one of those rays extending from the point viewed, the other from the center of sight.

[5.10] Hence, in every plane of reflection four points occur in a related group—i.e., the center of sight, the point [on the visible object] that is perceived, the endpoint of the perpendicular [dropped] from the center of sight [to the mirror's surface], and the point of reflection.[95] And all the planes of reflection [for the given point of reflection] intersect along the normal imagined to extend to the point of reflection, and this line is common to all the planes of reflection. And since the same thing happens when any point on the plane surface is viewed, there will also be an identical reflection from all points, and it will occur in the same way.

[5.11] Furthermore, what we have claimed will be clear in the case of [convex] spherical mirrors. If a [convex] spherical mirror faces the eye—and such a facing situation means that the eye does not lie on the surface of the spherical mirror itself or on the spherical surface continuous [with that surface][96]—and if one looks into this mirror, the portion of it that is perceived will be the portion of the sphere defined by the circle cut off by a ray tangent to the surface of the sphere that rotates [axially] about that circle of tangency until it returns to the original point where the rotation began. And if one imagines planes intersecting along the diameter that is imagined to extend [through the sphere] from the pole of the aforesaid circle, any of the arcs upon the surface of the sphere that are imagined to be the common sections formed by these planes on the surface of the sphere between the pole and the [defining] circle will be less than a quarter of the great circle [of the sphere], because the line from the center of the sphere to the endpoint of [any] ray drawn tangent to the sphere—that is, to the aforementioned [defining] circle—forms a right angle with the ray on account of its tangency. Hence, that tangent ray forms an acute angle with the radius drawn from the circle's pole, and this angle corresponds to the arc between the circle's pole and the [defining] circle itself, so every one of these arcs is less than a quarter of the [great] circle.[97]

[5.12] I say, then, that reflection can occur from any point on this portion [of the sphere], for, if any point on it is taken, the diameter of the sphere imagined [to extend] from this point will be perpendicular to a plane tangent to the sphere at that point. The proof of this claim is as follows. If two planes are imagined to cut the sphere along the diameter imagined [to extend] through the given point, the common sections of the sphere's surface and these planes form [great] circles on the sphere that pass through the given point. And if two lines are imagined tangent to these circles at the given point, the diameter will be orthogonal to both of the lines, so [it is orthogonal] to the plane in which they lie. And when a ray is incident to the given point, it will lie in the same plane as the sphere's diameter, whose endpoint is the given point, and [it also lies in the same plane as] the line imagined [to extend] from the center of sight to the center of the sphere, which of course passes through the circle's pole, and this radial line also falls orthogonally to the surface of the sphere. So a triangle will be formed by these three lines, and the ray that is incident to the given point forms an acute angle with the diameter of the sphere [extended] beyond [its surface], for, since this ray lies above the ray that is tangent to the sphere [at the given point], it will cut the sphere when it is imagined to extend [into the sphere]. And the plane tangent to the sphere at the given point will lie below this ray, and it will extend between the sphere and the visual axis—i.e., the line that is imagined to pass from the center of sight to the center of the sphere through the pole of the [defining] circle [formed by the rotating tangent].

[5.13] Therefore, since the diameter of the sphere is perpendicular to the plane that is tangent [to the sphere] at the [given] point, it will form an angle greater than a right angle with the [incident] ray inside the circle at the point of incidence, so outside the circle it will form with it an angle less than a right angle. And when [this diameter is] extended beyond [the sphere's surface], it will be orthogonal to the plane tangent [to that surface at the given point], so within the right angle that it will form with that plane on the other side of the ray an acute angle can be cut off equal to that [original acute angle] formed by a ray with that diameter. And the three lines forming these two angles will lie in the same plane, so a line can be drawn from the point on that portion [of the surface] that lies in the same plane as the ray that is incident to that point, as well as in the same plane as the normal to the plane tangent to the surface at that point, and it can form an equal angle with that normal. And the form of a point propagated to the mirror's surface along that ray will reach that line. Thus, it is equivalently disposed with respect to the line along which it can be reflected, and the plane in which these lines lie is orthogonal to the surface of the sphere at the point of tangency, and this should be understood to hold for every point on the [visible] portion [of the spherical mirror].[98]

[5.14] Therefore, the center of sight, the center of the sphere, a point of reflection, and a point [whose form is] reflected will lie in every plane of reflection, and all such planes will intersect along the line that extends from the center of sight to the center of the sphere. And the common section for each plane of reflection and the surface of the sphere will be a [great] circle on that sphere, and all [these] circles will intersect at the point on the [surface of the] sphere where the visual axis falls, and this [point] lies on the pole of the circular section [of visibility defined by the tangent rays].[99] Moreover, when the ray strikes the mirror along the perpendicular to the surface where the plane tangent to the sphere touches the sphere's surface at the point where the ray strikes—and this ray constitutes the visual axis [that passes] along the pole of the circular segment to the center of the sphere—the reflection will take place along the same radial line to the eye according to the source of the ray's propagation.

[5.15] Now, what we have said [above] will also be clear in the case of [convex] cylindrical mirrors. Place a cylindrical mirror that is polished on the outer [convex] surface directly facing the eye—and such a facing situation means that the eye does not lie on the surface of the cylindrical mirror itself or on the continuation of that surface—and we shall imagine a plane [passing] through the center of sight that cuts the surface of the cylinder along a circle that is parallel to the bases of the cylinder. Then, in this plane take two lines that are tangent to the circle of intersection at two opposite points. From each of those points extend a line along the length of the cylinder, and imagine two planes that contain these two lines [extending] along the [cylinder's] length as well as the two lines that are drawn from the center of sight to the circle of intersection and tangent to it. I say that these [two] planes are tangent to the cylinder.

[5.16] For if it is claimed that either of them cuts it, then it is obvious that an intersection will occur upon a line that extends along the length of the cylinder where the plane falls on its surface, and likewise an intersection will occur along a line of longitude on the opposite side. And the circle of intersection passes through these two lines of longitude. In addition, since the line tangent to the circle of intersection lies in some plane, it cuts the cylinder according to some [given] lines along the length that are parallel to one another, and if it intersects one of them it will intersect the other, both cuts being at equal angles. Therefore, if it passes through the point where the circle of intersection touches the first line along the [cylinder's] length, it will also pass through the point where the other line along its length touches that circle. Accordingly, it cuts the circle, so it will not be tangent to it, and this is counter to what we supposed. It is clear, then, that these two planes are tangent to the mirror and that whatever lies between them upon the surface of the mirror is visible to sight.[100]

[5.17] Furthermore, since these two planes meet at the center of sight, they will intersect, and their common section will pass through the center of sight, and it is parallel to the axis of the cylinder, because the cylinder's axis is perpendicular to the circle of intersection. The lines along the length of the cylinder are also perpendicular to that circle, and the planes tangent to the cylinder along these lines are perpendicular to that same circle. So they are perpendicular to the plane cutting the cylinder along that circle, and the common section of these planes is perpendicular to that same plane, so they are parallel to the cylinder's axis.[101]

[5.18] I say, then, that whatever point is taken on the visible portion of the mirror, the line extended from the center of sight to that point will cut the mirror. For if the line extended from that point along the length of the cylinder is imagined, it will pass through the circle of intersection and will touch it at a point where a line drawn from the center of sight will intersect the portion of the mirror that lies between the lines tangent to this circle. Also, the plane passing through the center [of sight] that contains this line will cut the cylinder. Therefore, since this line lies in the same plane as the line drawn from the center of sight to the point chosen [on the mirror's surface], it will intersect the mirror, and so, any line that is imagined [to extend] from the center of sight to the [visible] portion of the mirror cuts the mirror. By the same token, any line that is imagined to extend from the common section [of the two tangent planes passing] through the center of sight to that portion [of the mirror] cuts the mirror, so whatever plane is tangent to the mirror along any line on the visible portion cuts the planes that are tangent at that portion's edges. And not one of all the planes tangent to that portion passes through the center of sight; instead, it will extend between the eye and the mirror.

[5.19] Accordingly, I say that light can be reflected from any point on this portion [of the mirror]. For, given such a point, let a circle parallel to the bases of the cylinder be drawn through it. Hence, if the plane passing through the center of sight cuts the cylinder along a plane parallel to the bases, it should cut the cylinder along that circle, and the line extended from the center of sight to the center of the circle should pass through the given point. The form of that point will be reflected back along the same line to its origin-point, because that line is the visual axis [which is] perpendicular to the axis of the cylinder. Furthermore, if any point through which the [visual] axis passes orthogonally to the axis of the cylinder is taken, the reflection of that point will take place along that very [visual] axis.

[5.20] But if the chosen point lies outside this axis, then any line extended to that point from the center of the circle drawn parallel to the bases [of the cylinder]—that line also being extended to the plane tangent to the line along the length of the cylinder passing through that point—will be orthogonal to the [cylinder's] axis, so it is orthogonal to the line along the length [of the cylinder] passing through that point. And since the center of sight lies above the plane tangent [to the cylinder] at that point, the line drawn from the center of sight to the chosen point will form an acute angle with the normal extended to that point from the center of the circle. This is the case outside [the cylinder], so [the angle formed] inside [the cylinder] is obtuse. And from the right angle that the normal forms with the line on the plane that is tangent to the circle [of intersection] an acute angle the same size [as the aforementioned angle] can be cut. And that normal lies in the same plane as the center of sight, so it lies in the same plane as the line drawn from the center [of sight] to the [chosen] point. So the reflected ray will lie in the same plane [as this line], and this plane will be orthogonal to the plane tangent to the mirror at that [chosen] point, for the normal falls orthogonally to this plane. And this is how the plane of reflection will be defined.[102]

[5.21] Furthermore, the common sections of the planes of reflection and the surface of the cylinder vary among each other, for, when reflection occurs [back] along the same ray [along which the form reaches the mirror], that same ray will fall orthogonally to the axis [of the mirror]. And the common section of the cylinder's surface and the plane of reflection will be a straight line—i.e., an edge of the cylinder—since the diameter of the cylinder lies in the plane of reflection. And this is clear because a cylinder is generated by the rotation of a plane with parallel sides about one of those sides, which remains stationary [to form the axis of rotation]. Accordingly, the common section of a plane that cuts the cylinder through its axis—i.e., the stationary side [of the generating surface]—and the surface of the cylinder will be the side that moves [in the process of generation]. And I claim that only one among all the planes of reflection forms a rectilinear common section with the surface of the cylinder, for only one plane, and no more, can be imagined to contain the axis of the cylinder and the center of sight.

[5.22] On the other hand, if the plane of reflection is parallel to the bases of the cylinder, the common section will be a circle, and this is the only plane that can form a circular common section with the surface of the cylinder, for in the case of every reflection, the normal to the plane that is tangent to the point of reflection forms the diameter of a circle parallel to the bases of the cylinder. And on the surface of the cylinder there can be but one circle that is parallel to the bases and that lies in the same plane as the center of sight. All the remaining planes of reflection cut the [surface of the] cylinder and the cylinder's axis, for the normal dropped to the point of reflection cuts the cylinder's axis, but the common sections of these planes and the surface of the cylinder form the sections that geometers attribute to cylinders and cones [i.e., ellipses].[103]

[5.23] When the common section of the surface of the cylinder and the plane of reflection is a straight line [along the length of the cylinder], then, to whatever point on that line one directs his sight, reflection occurs in the same plane as the axis, for there is only one plane tangent to the cylinder along that line of longitude. So, given some point on this line, the perpendicular extended from it to the axis will lie in the same plane as the axis, and this extended line is perpendicular to the plane that is tangent to the surface of the cylinder [along that line of longitude]. But the center of sight lies in a plane orthogonal [to the plane of tangency], and so it lies in the same plane [as just described], and the axis of the cylinder and the common section lie in that plane, and there is only one plane orthogonal to that surface along the same common section, so all reflections that take place from points on this line lie in the same plane of reflection.

[5.24] However, when the common section of the plane of reflection and the [surface of the] cylinder forms a circle, then, if any point on that circle is viewed, reflection will occur in one and the same plane, because any perpendicular extended from the point that is viewed will form a diameter of this circle, so it lies in the plane of that circle, as does the center of sight. And the plane of this circle is orthogonal to the plane tangent to the circle at whatever point is chosen on it, so in this single plane reflection will occur from whatever point [is chosen] on the aforementioned circle. If any other common section is taken, however, reflection will occur in the same plane of reflection at only one point on this line [of section], for the perpendicular extended from the point of reflection is orthogonal to the line along the length of the cylinder that passes through that point, so it is also perpendicular to the axis. But that perpendicular forms the diameter of a circle parallel to the bases of the cylinder. So the plane of reflection and that circle intersect, and their common section forms a diameter of that circle, and it is perpendicular to [the cylinder's axis], and the plane of reflection cuts [that axis] but is oblique to it. But in a plane that is oblique to a given line, only one line can be imagined orthogonal to that line. If reflection were to take place from two points within the same plane of reflection, two lines within that plane would be perpendicular to the axis, and that cannot be, since the plane is oblique to the axis.[104]

[5.25] Moreover, the normal through the point of reflection falls on a circle that is parallel to the bases of the cylinder, and [it falls on] a point on the axis that is common to the circle and the plane of reflection. If, therefore, reflection were to occur from any other point on the common section [of the cylinder and the plane of reflection] within the same plane, there would be another normal dropped from the other point, and it would form the diameter of another circle on the cylinder parallel to the first one, and it would fall on a point on the axis where the plane of reflection does not fall. And so it must be understood that in every [oblique] plane of reflection, reflection occurs from only one point on the common section [of the cylinder and the plane of reflection] in the same plane with respect to the same center of sight. Hence, with respect to both eyes [reflection] can take place from the two endpoints of the circle's diameter, which constitutes the normal. Yet with respect to one eye this does not occur, because those two points cannot be perceived at the same time by the same eye, for [with one eye] it is necessarily the case that less than half of the cylinder is seen.[105]

[5.26] From the foregoing it is evident that the perpendicular imagined to extend to the point of reflection from outside [the cylinder] forms the diameter of the circle as it passes inside [the cylinder], for, if such were not the case, then, since it is agreed that the diameter of the circle, which passes to that point, is perpendicular to the plane tangent to the [surface of the] cylinder at that point, and [since it is agreed that it is] also perpendicular [to that plane] on the outside [surface of the cylinder], there will be continuity between these perpendiculars, and they will form one line. For if the diameter is not perpendicular to that plane [of tangency] when it is extended beyond [the cylinder's surface], it will follow that two perpendiculars could be drawn to the same point on a [given] plane. Hence, it is clear that in every plane of reflections four points occur in a related group: the center of sight, the point on the axis where the normal falls, the point that is seen on the mirror [i.e., the point of reflection], and the point from which the form of the [visible] object radiates.

[5.27] In the case of conical mirrors that are perpendicular to their bases [i.e., formed from right cones] and that are polished on their outer [convex] surfaces, the eye is considered to face them if it does not lie on the surface of the mirror or on the continuation of that surface, and how much of [the surface of the] mirror is seen will depend on how the eye is disposed with respect to the mirror.

[5.28] Accordingly, if the ray from the center of sight to the endpoint of the cone's axis—i.e., [the ray that is] understood [to extend to] the vertex—forms an acute angle with the axis on the [visible] side of the cone, we will imagine a plane [passing] through the center of sight and cutting the cone's surface along a circle that is parallel to the base of the cone. And we will imagine two lines from the center of sight that are tangent to that circle at opposite points, and from those points we will draw lines along the length of the cone. Hence, the plane containing either of these lines along the length [of the cone] as well as the [corresponding line that is] tangent to the circle will be tangent to the cone, for if it were to cut the cone, it would touch some other point besides the point of tangency on the circle. On that [other] point draw a line along the length of the cone, and that point and the vertex of the cone lie in this plane, so that line will lie in this plane and will pass through some point on the circle. That point, therefore, lies in this plane and on the circle, so it lies on a line that is common to the circle and to the surface [of the cone]. But it is [also] tangent to the circle, so this tangent passes through two points on the circle that it touches, which is impossible.[106] It follows therefore that this plane is tangent to the [surface of the] cone.

[5.29] And it is invariably the case that any plane of reflection in which the line that is tangent to some point on the cone and the line of longitude that passes through that point intersect is tangent to the cone along the line of longitude. Hence, we have two planes passing through the center of the eye and tangent to the [surface of the] cone, and in this situation the portion of the cone between them is visible to sight, and it constitutes less than half the cone, because the lines that are tangent to the circle encompass less than half of it.[107]

[5.30] If, on the other hand, the line drawn from the center [of sight] to the vertex of the cone forms a right angle with the axis, then imagine a circle that cuts the cone so as to be parallel to the base. The common section of this circle and the plane that contains the axis of the cone and the center of sight will be [a line] perpendicular to the axis of the cone, because the axis is perpendicular to the plane of the circle. To this common section draw a diameter through the center of the circle perpendicular to this line [that forms the common section], and from the endpoints of this diameter draw two [lines] tangent to the circle, and also draw two lines to the vertex of the cone. The two planes in which these latter two lines lie, along with the lines tangent [to the circle], are tangent to the cone in the way we described before. And since the common section of the circle and the plane in which the center of sight and the cone's axis lie is parallel to the line drawn from the center of sight to the endpoint of the axis, and since the lines drawn tangent to the circle at the aforementioned points are parallel to this common section, those tangents will be parallel to the line drawn from the center of sight to the endpoint of the axis, so they will [each] lie in the same plane as that line. Hence, both planes that are tangent to the circle pass through the center of sight, and the common section of those planes is the line drawn from the center of sight to the endpoint of the axis. So whatever lies on the cone between those planes is visible to sight, and it constitutes half the cone, because half the circle lies between those lines of tangency. And so it is evident that in this situation half the conical mirror is visible.[108]

[5.31] However, if the line extended from the center of sight to the endpoint of the cone's axis forms an obtuse angle with the axis from a point of view above the cone, and if a circle is formed to cut the cone parallel to the base, the [line forming the] common section of this circle and the plane containing the center of sight and the axis is perpendicular to the cone's axis. When this common section is extended beyond [the back edge of the cone], it will intersect the line drawn from the center of sight to the endpoint of the axis according to an acute angle, [and] an acute angle is [also] formed by this line [of sight] with the axis below [the point of intersection]. From the point of intersection of these lines, draw two lines tangent to the circle at two opposite points, and draw the lines [of longitude] from these points to the vertex of the cone. The planes containing the lines tangent [to the circle] and these lines along the length [of the cone] are tangent to the [surface of the] cone, and in both of these planes two points on the line extended from the center of sight to the endpoint of the axis lie—i.e., the endpoint of the axis and the endpoint of the perpendicular where that line [drawn from the center of sight to the endpoint of the axis] and the perpendicular intersect—so that line lies in both planes. Therefore, both planes pass through the center of sight, and on the inner, lower side [of the cone] these planes encompass a portion of the cone that is less than half, because the lines tangent to the circle encompass a portion [of the circle] that is less [than half]. Hence, from [the point of view] above [the cone], more than half [the cone] lies between the planes that are tangent to [the surface of the] cone, and that is what is visible to the eye, so in this situation the eye perceives more than half the [surface of the] cone.[109]

[5.32] On the other hand, if the line extended from the center of sight to the endpoint of the axis falls on an edge of the cone so that this line and the edge form a continuous line, I say that none of the cone will be hidden from sight except for the given line imagined [to extend along the edge], for every plane that contains the line extended from the center of sight to the endpoint of the axis and a line extended along the length of the cone's edge cuts the cone except where it touches the cone along the edge that forms part of the line [drawn from the center of sight to the axis]. And [of all the lines] imagined to lie on the surface of the cone in this situation, this edge-line alone passes [unseen] by the center of sight.

[5.33] The truth of this point is evident from the fact that, whatever point is chosen on the surface of the cone, if a line is drawn to it from the center of sight, and if from that same point a line is drawn along the length of the cone to the endpoint of the axis, these two lines will form a triangle with the line drawn [from the center of sight] that coincides with the edge [of the cone]. And this triangle lies in a plane that is imagined to pass through the center [of sight] so as to cut the cone, and among [all] the lines lying in this plane only two fall on the surface of the cone—i.e., the line [that coincides with the line along] the edge and the line along the length [that extends] from the chosen point to the vertex of the cone. But the line extended from the center [of sight] to the chosen point cuts the line along the length [of the cone] at the chosen point of reflection, and it cuts the line that coincides with the edge of the cone at the center of sight, so none of these lines [of longitude] will intersect that line [along the edge of the cone] outside the center of sight. Therefore, since no other point can be chosen to which the line from the center of sight reaches and through which it passes, this point is not blocked by any other point. And so it is visible to the eye, since there is no solid object lying between it and the eye. The same can be demonstrated for any [other] point on the surface of the cone.[110]

[5.34] Moreover, if the line [extended] from the center of sight to the endpoint of the axis enters the cone, I say that no point on the entire surface of the cone is hidden from the eye. For, given some point on the surface of the cone, imagine a line drawn to it from the center [of sight], and imagine another drawn from it to the vertex of the cone. These two lines enclose a triangular surface with the line extending from the center of sight to the endpoint of the axis that enters the cone, and this triangle lies in a plane that cuts the cone's surface. Since every plane containing a line that enters the cone cuts the cone, then the line extended from the center [of sight] to the chosen point cuts at that point the line along the length of the cone that extends from it to the cone's vertex. And among [all] the lines lying in the plane that contains these two lines, there are only two that lie on the cone's surface—i.e., the one line that is extended along the length of the cone from the [chosen] point to the [cone's] vertex and the other line that is [directly] opposite, which cuts the angle formed by the first line with the line that enters the cone. Thus, when it is extended beyond the cone, the line on the opposite side cuts the line extended from the center [of sight] to the chosen point; hence, within the plane [formed by these lines] the latter line [extending from the center of sight to the chosen point] cuts only two lines that lie on the surface of the cone, [cutting] one beyond the cone and the other at the chosen point, so if it is extended to infinity it will intersect none of the other lines [along the length of the cone]. According to the earlier account, then, the chosen point is not blocked from sight.

[5.35] Therefore, in this situation none of the planes tangent to the cone will pass through the center of sight; rather, each of them will intersect the line of sight that enters the cone through the endpoint of the cone's axis between the center of sight and the cone, and the [point of intersection] lies at the endpoint of the axis. But when the line of sight coincides with a line along the length of the cone, none of the planes tangent to the cone will reach the center of sight except the one that is tangent to the cone along the aforementioned line. So all the tangent planes will cut that line between the center of sight and the cone.[111]

[5.36] Likewise, in the case when two planes that are tangent to the cone pass through the center of sight, any plane that is tangent to the cone in the visible portion of the cone that lies between the two tangent planes misses the center of sight. No matter what point of that portion the line of sight reaches, that line will cut the cone, because that portion lies between the two lines of sight that are tangent [to the cone]. And the plane containing this line of sight [i.e., the one reaching the given point on the cone's surface] and the line along the cone's length will cut the cone, and this will be the visual plane for whatever place on the cone's surface it touches within that portion, and so [that place on the cone's surface] is also visible.[112]

[5.37] I maintain, therefore, that in any [such] situation reflection can occur from any given point [on the visible portion of the mirror]. Choose a point, and imagine a circle passing through that point and parallel to the base of the cone. The diameter originating at that point on this circle will be perpendicular to the axis, since the axis is perpendicular to the plane of the circle, so the line along the length [of the cone] extended from the [chosen] point to the vertex of the cone forms an acute angle with the diameter as well as with the endpoint of the axis within the same plane [i.e., the plane containing the diameter, the axis, and the line of longitude]. Let a line of sight fall to the [chosen] point within the plane containing the line along the length [of the cone] and the axis, and in that plane draw a normal to the line along the length [of the cone] at that point. This normal will of course intersect the axis, and a triangle will be formed by this line with the axis and the line along the length [of the cone]. Imagine a line tangent [to the circle] at that [chosen] point, and imagine another diameter in the circle we drew that is orthogonal to the original one and orthogonal to the axis as well, so it will be orthogonal to the plane containing the axis and the first diameter. This second diameter is parallel to the line of tangency [just drawn], because that line of tangency is perpendicular to the first diameter. Hence, the line of tangency is perpendicular to the plane containing the axis and the first diameter, so it will be orthogonal to the normal that we just drew. And so that normal falls orthogonally to the plane tangent to the cone at the point we chose.

[5.38] Accordingly, if the line of sight that falls upon the chosen point passes to it along the normal, it will be orthogonal to the plane tangent to the cone at the chosen point, and the [visible] form will be reflected back along that same line of sight. On the other hand, if the line of sight does not pass along the normal, it will form an acute angle with that normal at the chosen point. And in the plane of this line of sight another line can be drawn from that point to form an angle equal to that which it forms with the normal, because that normal is orthogonal to the plane of tangency. Moreover, any line that falls orthogonally to the plane of tangency at the chosen point passes to the axis. And if a perpendicular is drawn to that plane from the axis, the inner and outer segments of that perpendicular will form a single line. For, if the interior segment of the perpendicular is not also perpendicular to the plane when it is extended outside [the cone], it will follow that from the same point on that [same] plane two perpendiculars can be erected on the same side [which is impossible].

[5.39] So it is evident that reflection can take place at equal angles from any point that is viewed on the surface of the cone. And when the [visible] form reaches the line of reflection, it will arrive at the mirror along that [original] line and will be reflected to the eye along the other, and these two lines lie in the same plane, which is orthogonal to the plane that is tangent to the cone at the point of reflection. This is the plane of reflection, and in it four points are invariably grouped together: i.e., the center of sight, the point that is seen, the point of reflection, and the endpoint of the normal.[113]

[5.40] The common sections of the planes of reflection and the surface of the cone [can] vary, however. For when the line of sight coincides with the axis of the cone—i.e., when the entire axis of the cone and the normal [to the cone's surface] that passes to the axis lie in every plane of reflection—then in this case the common section of each plane of reflection and the surface of the cone will be a line along the length [of the cone]. For every plane that contains the entire axis [of the cone] has this [particular] common section with the surface of the cone.

[5.41] But in every other case only one line along the length of the cone will form the common section—i.e., the common section lying in the plane that contains the center of sight and the axis. And because the center of sight does not lie in a direct line with the axis, there will be only one such plane [of section], and every other common section will be a conic section, not a circle. For if it were a circle, the plane of that circle would lie in the plane of reflection, and since the [cone's] axis is orthogonal to that circle (because any circle [of section] on the cone is parallel to its base), the edges of the cone will be inclined to the circle, as well as to the plane of reflection. In that plane [of the circle], therefore, a normal cannot be drawn to the line along the length of the cone. But the normal extended to the plane that is tangent [to the cone] at the point of reflection lies in the plane of reflection, and it is perpendicular to the line along the length [of the cone], because any plane that is tangent to the cone is tangent [to it] upon a line along its length.

[5.42] It therefore follows [that it is] impossible [for the plane of reflection to form a circular section with the cone], so all the other common sections of [the plane of] reflection [and the surface of the cone] are conic sections, and when the common section is a line along the length [of the cone], reflection can occur from any point on that line in the same plane as reflection from any other point of reflection. For the perpendicular extended from any point on that line will meet the axis; and the center of sight, the point of reflection, and the point on the axis [where the normal intersects it] will lie in the plane of reflection; and the line along the length [of the cone] and the plane within which this reflection occurs is the one formed by the line along the length [of the cone] and the axis, so reflection takes place from any point in this plane.[114]

[5.43] If, however, the common section is not a line along the length [of the cone], I say that reflection may take place from either one point, or at most two points, on the common section within the same plane. For, if a normal is extended from the point of reflection, it will reach the axis and will fall on one of its points. When a circle is imagined [projected] through the point of reflection, that circle will cut the axis orthogonally, and since the perpendicular [in the plane of the circle] cuts the axis [in a plane] parallel to the [cone's] base, the normal [to the cone's edge] will be inclined to the [plane of the] circle. And when it is rotated about [this circle] that normal will always be equal[ly disposed with respect to the circle], so it will form a cone whose base is the circle and whose vertex is the point at which the normal intersects the axis. Therefore, the plane of reflection will either be tangent to, or will cut, this [second] cone.

[5.44] If it is tangent to it, I say that reflection can occur in the same plane from only one given point of reflection. It is evident that the plane of reflection will be tangent to this [second] cone along the normal, which is an orthogonal line in the plane of reflection, and if lines are drawn from the vertex of the entire [second] cone to the common section of the plane of reflection and the larger, original cone, they will fall on the circle that forms the base of the imagined cone, but only one falls on that common section at the point of reflection. Therefore, if reflection were to take place from another point on this common section, the line imagined to extend from that point to the [second cone's] vertex would be perpendicular to the line along the length passing through that point on the [original] cone. But the line [extending] from the vertex of the imagined cone to the point on the circle through which the line along the length of the [mirror's] cone passes is certainly perpendicular to this latter line, so any other line forms an acute angle, not a right angle, with it.[115]

[5.45] On the other hand, if the plane of reflection cuts the imagined cone, it will cut the circle at its base in two points. I say that these are the only points on the entire common section [formed by the cutting plane] from which reflection can take place in the same plane, for the line extended from either of these points to the vertex of the imagined cone is perpendicular to the line along the length [of the mirror's cone] at the point through which it passes. From any other point on the section [formed by the cutting plane], the line extended to the vertex of this [imagined] cone will form an acute angle with the line along the length [of the mirror's cone] through which it passes, since the normal to the same line along the length [of the mirror's cone] forms a right angle in the circle. So the lines drawn from the vertex of the imagined cone to points on the section lying between the vertex of the mirror and the circle will form obtuse angles with the lines along the cone's length on the side opposite the vertex of the entire cone [of the mirror]. And the lines drawn to the points lying between the circle and the base of the mirror form acute angles with the line along the length [of the cone] on the side opposite the vertex while forming obtuse angles on the side opposite the base.[116]

[5.46] In the case of spherical concave mirrors, if the center of sight is located within the hollow of the mirror, the entire surface of the mirror will be visible to it. However, if the eye lies outside [that hollow], it can perceive more than half of the mirror, that portion being defined by the circle on the sphere upon which two rays drawn from the center of sight are tangent.[117]

[5.47] Now, if the eye lies at the center of this mirror, reflection will not take place from any point other than [that according to which the form of the eye reflects back] to itself, for any line extended from the center of the sphere to the [surface of the] sphere is perpendicular to the plane that is tangent to the sphere at that point. Hence, in this situation the eye will see nothing but itself through reflection.

[5.48] If, however, the eye is located outside the center of the sphere, reflection can occur to another body from any point on the mirror other than the point where the diameter drawn from the center of sight to the sphere passes through the sphere's center, because the diameter falls perpendicularly to a plane that is tangent to the sphere. Now, having chosen another point [on the surface of that sphere], draw a diameter [normal] to it from the center of the sphere, and [draw] a line from the center of sight [to it]. Accordingly, an acute angle will be formed by these lines, for the line of sight lies between the diameter [that forms the normal] and the plane tangent to that point, and that plane lies [on the] outside [surface of] the sphere. And whether the center of sight lies inside or outside the sphere, this line of sight falls inside the mirror because it falls between the lines of sight that are tangent to the circle [defining] the [visible] portion of the sphere when the center of sight lies outside [the sphere].

[5.49] If the center of sight lies inside the sphere, it is obvious that the line of sight falls inside the mirror. Accordingly, since the diameter [that forms the normal] forms a right angle with the tangent [to the chosen point], cut off from that [right angle] an acute angle equal to the aforementioned acute angle in the same plane. I say, then, that the line of reflection falls inside the mirror, because the common section of the mirror's surface and the plane of reflection is a circle that forms with the diameter [constituting the normal] an acute angle greater than any acute rectilinear angle, and in each such point there will be reflection of this kind.

[5.50] It is evident from these observations that in every plane of reflection there will be a center of sight, the center of the mirror, a point of reflection, a point that is seen, and the endpoint of the diameter extended from the center of sight to the [surface of the] sphere through the sphere's center. Furthermore, the common section of all these planes with the surface of the mirror is a circle, and reflection can occur in the same plane from any point on this common section.[118]

[5.51] In the case of concave cylindrical mirrors the entire mirror can be perceived if the center of sight lies inside it. However, if it lies outside, more than half the mirror will be visible, the portion, that is, that lies between the two planes projected from the center of sight and tangent to [the surface of] the cylinder.[119]

[5.52] We will imagine a plane projected from the center of sight and parallel to the bases of the cylinder. This plane will either intersect the cylinder or it will not. If it intersects, then the common section of this plane and the cylinder will be a circle, and the line of sight passing through the center of this circle will fall orthogonally to the plane tangent to the cylinder at the point where the line of sight intersects it. So reflection will take place along the same line to its source.

[5.53] Let some other point be chosen. The line extended normal [to the cylinder's surface] from this point will intersect the axis, and the line of sight dropped to this point will form an acute angle with the [aforementioned] normal, since it lies between that normal and the [line] tangent [to the cylinder's surface at that point]. And that this line will fall inside the mirror is obvious from the fact that it falls between the planes that are tangent to the visible portion [of the cylinder]. Hence, in the same plane of reflection we can cut off from the angle formed by the normal and the tangent an acute angle equal to the aforementioned acute angle. Moreover, the line of reflection forming this angle will fall inside the cylinder, because it will fall between the normal and the line along the length [of the cylinder] passing orthogonally through the endpoint [of the flanking, tangent line of sight]. Thus, the center of sight, the point of reflection, the point that is seen, and the point on the axis to which the normal falls will [all] lie in the plane of reflection.[120]

[5.54] Moreover, if the center of sight is located in such a way that the common section of the plane of reflection and the surface of the cylinder constitutes a line along [the cylinder's] length, reflection may occur from any point on that common section. The plane common to all these reflections will be in one specific plane—i.e., the one in which the center of sight and the entire axis of the cylinder lie—as was said earlier in the case of the non-concave cylindrical mirror.

[5.55] Likewise, if the common section is a circle, all of the reflections occurring from points on that circle will proceed in the same plane, as was shown earlier for the other circles.

[5.56] But if the common section is a cylindric section [i.e., an ellipse], reflection will occur from only two of its points within the same plane, although in the case of [convex] cylinders discussed above, [when the points of reflection lie at the intersection of the ellipse and a] circle, reflection would occur from only one point in a single plane when one eye is looking, because in the previous case the corresponding points of intersection through which the circle parallel to the bases passes were invisible to the eye. For when one [such point] was seen [with one eye], the other was invisible because a portion smaller [than half] of the cylinder is visible, but in these cases [when the cylinder is concave], a greater portion [than half] of the cylinder is visible, so the [corresponding] points [of intersection] of the circle parallel to the bases and the common section are perceived by one eye.[121]

[5.57] In the case of concave conical mirrors, if the center of sight lies inside the mirror, it will see all of it. If, however, it lies outside, and if the line extended from a center of sight [located above the mirror] to the vertex of the cone enters the cone or coincides with a line along the length of the cone, nothing of the mirror will be seen. For any other line extended from the eye to the cone will fall on the cone's outer surface, so the inner surface will be hidden.[122]

[5.58] If, however, a segment is removed from the cone, the portion of the cone lying between the planes projected from the center of sight and tangent to [the cone's] surface—i.e., more than half—can be seen, and if the line from the center of sight is perpendicular to the plane tangent to the cone and reaches to the axis, then, as was said in the case of the other cones [i.e., convex cones], the common sections [of the plane of reflection and the mirror's surface] will be either [straight] lines along the length of the cone or [conic] sections.[123] Moreover, in these cases [i.e., when the plane of reflection is a conic section and more than half the surface of the mirror is visible], reflection can occur from two points on the [conic] section in the same plane with respect to the same center of sight; and the center of sight, the point seen, the point of reflection, and the point on the axis will lie in the same plane of reflection.

[5.59] However, if an entire conical mirror faces the center of sight, and the center of sight lies on the side of the base, only what lies inside the mirror will be perceived, because the normal forms an acute angle with the line drawn to it from the center of sight on the side of the base. Thus, reflection occurs on the side of the vertex, and all of the reflected lines will fall within the cone, and [only] what is placed within the cone can be seen.[124]

[5.60] On the other hand, if a segment of the cone is cut off lengthwise, things outside the mirror can be perceived, since what lies beyond may be open to the lines of reflection. Likewise, if the cone is truncated in the form of a ring with the vertex removed, lines [of incidence] will have free entry [through that upper opening], and things outside the mirror will be visible. And if the center of sight lies inside the concave [inner surface of the mirror], more things that lie outside can be seen than [if the the center of sight lies] on the side of the base, because it provides a wider scope for the lines of reflection to project outward.[125]

[5.61] Furthermore, given a point on any mirror, it is possible for the form of only one point to be perceived at it by the same center of sight. For only one plane passes through the normal and the center of sight, and there exists only one [straight] line between the center of sight and the [given] point, and there exists only one acute angle formed by that line and the normal, and in the same plane there is only one acute angle equal to this one, so there is only one line that forms an angle with the normal that is equal to this one. So when the line [of incidence] is extended to a point on a [visible] object, the form of another point [on that object] cannot be conveyed along it, since the [form of the] first point passing along it will block any following it. But two point-forms can be perceived at the same point on the mirror by both eyes, because an infinite number of cutting planes can be imagined along the normal, and in each of them two equal acute angles can be imagined with respect to the normal.

[5.62] We have therefore now explained the character of reflection as well as the character of each [type of] mirror. When sight perceives forms by means of reflection, it is unaware that this perception is due to reflection. For reflection is not a function of sight, because, when the eye is removed, the form nonetheless radiates from the [visible] object to the mirror, and it will be reflected in the way just described. But if the eye happens to lie where the lines of reflection come together, then sight will perceive that form at the endpoints of these lines, and that form [appears to] exist in the mirror, not as if it were adventitious but as if it were a natural form [actually] in the mirror. Moreover, sight sometimes perceives forms in mirrors on the surface alone, sometimes in front of the mirror, sometimes beyond it. And the place where the form appears will depend on the shape of the mirror and the location of the visible object, and the form is always perceived in its appropriate place when the [relative] situation of the center of sight and the mirror changes. And the distance of the form's location from the mirror will vary with variation in the shape of the mirror. And the location of the form is called image-location, and the form is called an image.[126] Moreover, the eye perceives the visible object at the image-location, and we shall discuss this location and its essential characteristic for each [type of] mirror, and we will enumerate and explain the reasons visible objects are perceived at that location, and we will do so in the following book, God willing.

NOTES TO BOOK FOUR

[1]The characteristics to which Alhacen is referring here are the twenty-two visible intentions that he deals with in detail in book 2 of the *De aspectibus*; see Smith, *Alhacen's Theory*, 438-512.

[2]This tripartite categorization of vision is clearly articulated in Ptolemy's *Optics*, although its wellsprings lie much earlier in Greek optics, particularly with the study of refraction by Archimedes. Note the emphasis on *vision* rather than on the action of the rays (whether visual or light) in this threefold breakdown of the science of optics.

[3]That is, aside from all the visual deceptions pertaining to direct vision discussed in book 3, sight is subject to particular deceptions in reflection and refraction. In both cases, for instance, the image and the object will lie at different locations (e.g., behind the mirror), and the object may appear larger or smaller than it should (e.g., magnified or diminished). In reflection, morever, the object may appear distorted in shape and orientation depending on the type of mirror. As far as reflection is concerned, these distortions are dealt with in book 6.

[4]The fact that light and color radiate from any luminous or illuminated body to any facing body is discussed in detail by Alhacen in chapter 3 of the first book of the original Arabic version of the *De aspectibus*, where he establishes the punctiform and rectilinear radiation of light and color; see Sabra, *Optics*, vol. 1, 13-51. Along with the first two chapters of the Arabic original, this chapter is missing from the Latin version; see Sabra, *Optics*, vol. 2, lxxvi-lxxvii, and Smith, *Alhacen's Theory*, ix. However, the basic point that Alhacen makes in this chapter—that light and color radiate along straight lines in all possible directions from any point on a luminous or illuminated surface—is clearly implicit in his subsequent analysis of light and vision in books 1-3 of the Latin version of *De aspectibus.*

[5]The luminosity inherent to an illuminating object is primary, whereas the light in a body illuminated by that object is secondary. Thus, the light of the sun is primary with respect to the light of the moon, but the light of the moon is primary with respect to the light on objects illuminated by moonshine. As is clear from these examples, secondary light is fainter than primary light. For Alhacen's discussion of these two types of light, see Sabra, *Optics*, vol. 1, 13-40.

[6]Throughout book 4, Alhacen describes mirrors made of only two materials, iron and silver. Most of his experiments specify iron mirrors, although he suggests silver ones when greater reflectivity is needed. Later on in book 4, Alhacen describes the formation of convex and concave spherical, cylindrical, and conical iron mirrors with very short focal lengths. Given the extreme difficulty of molding and working iron, Alhacen's reliance on such mirrors indicates an extraordinarily high—improbably high, in fact—level of technical proficiency among iron workers in early eleventh-century Cairo, where Alhacen presumably carried out the experiments described in *De aspectibus*; see pp. xxv-xxiv above for some discussion of this point.

[7]Daylight, according to Alhacen, is the secondary light in the atmosphere that results from its being illuminated by the sun. See Sabra, *Optics*, vol. 1, 23-29.

[8]In this experiment, as well as in subsequent ones carried out in the same room, Alhacen's description of the requisite controls is less precise than we might wish. Granted, the room has one window, but how large should it be? How far from the window should the mirror be placed? How far from the mirror should the directly illuminated object be placed? The size of the window is perhaps the most pressing issue, and it is reasonable to suppose that Alhacen intended it to be quite small so that the light shining through it would be narrowly channeled into the room, leaving most of it in shadow. Accordingly, figure 4.2.1, p. 191, offers a schematic reconstruction of the experiment. C and D represent two objects placed in a room with one window through which daylight shines. Since Alhacen does not specify the shape of these objects, I have represented them as spheres to stand generically for whatever objects he may have in mind. C, which is presumably white (although Alhacen makes no such specification), is directly exposed to the incoming daylight. D is blocked from such exposure, either because it lies outside the area of effective illumination or because it is shielded by the mirror at F, which is represented as plane, although again Alhacen makes no specification. This mirror is placed in such a way that the secondary light radiating from C is reflected to D, thus faintly illuminating the part of its surface exposed to the reflected light. If D is moved from its original location while kept from direct exposure to the daylight shining through the window, it will lose the illumination previously imparted to it, although it will do so progressively rather than suddenly, since the light from C is reflected from the entire mirror.

[9]Under the same conditions specified in the previous experiment, a third white object, E in figure 4.2.2, p. 191, is placed on the other side of mirror F out of the way of direct illumination through the window. To ensure that E receives no illumination through the window, one might put a screen in front of the object, although Alhacen makes no mention of this expedient. The presumed point of this experiment is to show by contrast with the relative darkness of E that D is indeed illuminated by C's reflected light. A silver mirror is suggested in this case to heighten the effect of the reflected illumination on D.

[10]For this demonstration, see I, 3.99-103, in the Arabic original of the *De aspectibus*, which is missing in the Latin version; see Sabra, *Optics*, vol. 1, 40-41.

[11]There is no distinction between "accidental" and "secondary" light insofar as both are extrinsic rather than intrinsic sources of illumination in the given body; see, e.g., I, 3.88, in Sabra, *Optics*, vol. 1, 38.

[12]The experiment suggested here is the same as that illustrated in figure 4.2.1, p. 191, except that object C is brightly colored rather than white, and it is exposed to direct sunlight rather than daylight. The goblet Alhacen has in mind here is no doubt the clear glass one he describes earlier in I, 3.105 (Sabra, *Optics*, 1, 41) and 1, 4.22 (Smith, *Alhacen's Theory*, 347). Why object D, to which the luminous color on the surface of object C is reflected, is placed inside such a goblet is left unexplained, but Alhacen's intent in placing the object there may be to have it suspended in air without the interference of a hand or some other opaque body to hold it in place.

[13]Alhacen's point here follows from his discussion of color-radiation in 1.6.95-99, in Smith, *Alhacen's Theory*, 381-382. When a colored body is exposed to light so that the color becomes illuminated, the resulting luminous color radiates. But the illumination in that body is already secondary, so its resulting illumination, along with the color, is essentially tertiary. Accordingly, the radiated form of the color is commensurately weaker than the illuminated color serving as its source, and if the color is already weak (i.e, pale or muddy), its radiated form will be all the weaker when it reaches the mirror.

[14]As illustrated in figure 4.2.3, p. 192, the setup for this experiment is essentially the same as that for the earlier ones carried out in the room with one narrow window. White object C is exposed to light radiating through the window, and white object D is placed toward mirror F to catch the reflected light from C. White object E is placed near the mirror in such a way that it is exposed to the secondary light radiating directly from C. The illumination on the face of E will be brighter than that on the face of D. To ensure that no direct light from the window reaches E, one might place a screen in front of E. Likewise, to ensure that secondary light from C does not reach D, one might place a screen between the two bodies. This second screen would seem to be crucial to the success of the experiment because, if secondary illumination were allowed to reach D directly from C, that illumination would be added to the illumination reflected to D. Alhacen, however, makes no mention of either expedient, which is somewhat surprising given the punctilio with which he describes the experimental conditions for his empirical analysis of reflection later in the third chapter.

[15]In this case, object C in figure 4.2.3, p. 192, is brightly colored, rather than white. Accordingly, the luminous color from object C will radiate directly to object E while reflecting to object D, the coloring effect on object E being brighter than that on object D. The vessel mentioned here is presumably the (glass) goblet mentioned earlier. Alhacen does not mean to deny that the color of the iron mirror has a darkening effect on the light or luminous color reflected from it; in fact, he discusses this effect in the next paragraph. What he is denying is that this effect is the sole, or even primary, cause of the weakening of light and color in reflection.

[16]Clearly, Alhacen understands the distance between the window, as source of illumination, and the elevated body to which light is reflected as a composite of the lengths of the incident and reflected rays. Hence, that distance, overall, should be equal to the uninterrupted distance between the window and the body lying on the ground.

[17]That is, the clear glass vessel mentioned earlier.

[18]Here Alhacen makes explicit that the distance between the source (i.e., the brightly colored object illuminated by sunlight) and the body to which light is reflected is composed of the lengths of the incident and reflected rays.

[19]In specifying that the object illuminated by secondary light should lie the same distance from the mirror as the object to which the light shining directly through the window onto the mirror is reflected, Alhacen seems to have in mind that the object be placed behind the mirror at the appropriate distance so as to be blocked from direct illumination. In addition, although Alhacen does not stipulate

whether the light coming through the window should be direct sunlight or daylight, the experiment as described would seem to be more suited to direct sunlight. Accordingly, in the first case, with the mirror large enough to catch all the light radiating through the window, the object behind the mirror would be blocked not only from direct sunlight but also from secondary illumination cast on the ground around the mirror and radiated to it. In that case, of course, the contrast in brightness between the object illuminated by reflection and the body shielded from both direct and secondary illumination would be obvious. In the second case, with the mirror small enough to allow some of the incoming light to illuminate the floor behind it, or to illuminate something white, such as a sheet, on the floor behind it, the object behind the mirror will still be blocked from direct sunlight but will be exposed to secondary radiation from the floor or the sheet. Nonetheless, the illumination on the object receiving the reflected sunlight will be more intense than that on the body receiving the secondary illumination.

[20]In tying reflectivity to the physical structure of the reflecting surface, Alhacen is laying the ground for his subsequent dynamic explanation of reflection by analogy to the physical rebound of a swiftly moving projectile (see paragraphs 3.99-3.102 below). That Alhacen does not conceive of light-radiation as true projectile motion is clear from his insistence that what passes from light sources is formal rather than material, yet common experience shows that certain bodies owe their polish to physical burnishing, which smooths their surfaces. Hence, light seems to act according to the kinetic and dynamic principles that govern the motion of physical bodies.

[21]Here Alhacen is laying out the basic agenda for book 4: to show both empirically and mathematically that reflection from any reflecting surface, no matter its shape, occurs in a plane that is orthogonal to that surface, that the way reflection occurs is governed by the relationship between the normal to the point of reflection and the lines of incidence and reflection, and that this relationship is governed by the equal-angles law.

[22]Throughout the remainder of book 4 and beyond, Alhacen describes convex mirrors as "polished on the outside" (*extra/exterius politum*) rather than as *convexum*. Concave mirrors, on the other hand, he sometimes describes as *concavum* and sometimes as "polished on the inside" (*intus/interius politum*).

[23]As explained earlier in book 3, a "digit" is approximately 3/4" or 1.9 cm; in fact, it is somewhat less, so the plaque is roughly 4.5" by 9", or 11.3 cm. by 22.5 cm. This we know because of his specification of 4 digits for the width of a board used for testing diplopia, that distance, according to Alhacen, being the distance between the pupils of the two eyes; see Smith, *Alhacen's Theory*, 573. Alhacen's choice of measurement here is arbitrary; the crucial thing is that the plaque be twice as long as it is wide. Nevertheless, once the measurements are fixed at 12 by 6, the rest of the construction is contingent on that measure.

[24]Figure 4.3.1, p. 192, which illustrates the final scribing on the plaque, is found only in mss *P1*, *S*, *E*, *O* and *L3* and therefore appears not to be a canonical part of the original text. Among those manuscripts, moreover, the diagram appears in various forms, none of which agrees with that of Risner in his edition of *De aspectibus*.

In order to hew as closely to the manuscript-tradition as possible, I have followed the basic form and lettering of the versions found in mss *E* and *L3*, which are closely related. According to Alhacen's description, side DB of the bronze rectangle is 12 digits long, while side AB is 6. At midpoint E of edge DB a semicircle is drawn with a radius of 6 digits to pass through points B, F and D. Radius FE is drawn perpendicular to DB. Point G is marked on FE at a distance of 1 digit below F. Again from point E, a semicircle of radius GE, i.e., 5 digits, is drawn. Then, points are chosen on arcs FB and FD so that corresponding lines passing through them, such as HE and ME, or EA and its mate on the other side, will form equal arcs and thus equal angles at E. Finally, line OV is drawn 1 digit above and parallel to DB so as to cut lines EH and EA at points L and K, respectively.

[25]Figure 4.3.2, p. 193, represents the plaque with all the excess cut away. The instructions for cutting away the lower portion are open to interpretation because it is unclear whether the radius to which the lines are closest is FE or DE and its correspondent EB on each side of E in figure 4.3.1, p. 192. As I have represented it here—and Risner as well—the cut is made along the last of the three lines, i.e., the one beyond EA and its mate on the other side.

[26]Mss *O* and *E*, as well as Risner, have diagrams representing the instrument that is to be formed according to the initial description here. However, in order to make its overall construction easier to follow, I have illustrated the successive stages in that construction with my own diagrams. Illustrated in figure 4.3.3, p. 193, is the initial scribing of the top of the block, which is 14 digits on each side. The outer circle has a radius of 7 digits, the inner one a radius of 5 digits. Line AE on the top of the block corresponds to line EF on the bronze plaque in figure 4.3.2, p. 193, line BE on the top of the block to line HE on the plaque, and so forth. Hence, angle AEB on the top of the block is equal to angle FEH on the plaque, and so forth.

[27]Figure 4.3.4, p. 194, represents a bird's-eye view of the ring with the remnants of the originally scribed lines on the top surface, which is 2 digits wide, the outer diameter of the ring being 14 digits and its inner diameter being 10 digits.

[28]As specific as it sounds, this procedure is so vague as to be a matter of sheer conjecture. The main problem lies in the meaning of *acuta* as applied to the ruler Alhacen describes. On the one hand, he may have in mind the ruler represented on the left-hand diagram in figure 4.3.5, p. 194, where the lower edge is marked off in equal sections and the ruler actually comes to a point. On the other, he may have in mind the ruler represented in the right-hand diagram, in which *acuta* would best be translated as "sharp-edged." In my reconstruction of the procedure described in this passage, the measuring edge of the ruler is applied to the outer endpoint of one of the lines at the top of the ring and held firmly in place while the ruler is slid along the outer surface until the point touches that surface. Accordingly, the line between the spot where that point touches the ring's outer surface and the endpoint of the line at the top of the ring will be perpendicular to the top edge of the cylinder. Suffice it to say, there is considerable leeway for error in such a procedure; yet, as will become clear later on, the final construction of the instrument based on the scribings described to this point requires extraordinary exactitude.

[29]The procedure here is clear enough in theory, but in practice Alhacen offers no method for finding the midpoint of the semicircle's arc with precision, although

this could be done easily, albeit indirectly, by recourse to Euclid's *Elements*, III.3. It is puzzling that Alhacen did not fall back on the simple schoolboy expedient of drawing intersecting arcs from equidistant points on either side of the point to which the perpendicular is to be dropped, a method based on *Elements*, I.11.

[30]As a standard measure, a grain of barley = approximately 1/3 in or .85 cm., so half a grain of barley = approximately 1/6 in or .42 cm.

[31]Figure 4.3.6, p. 195 shows a cutaway view of the ring with the vertical lines along the inside surface cut by the cavity, whose upper edge lies half a grain of barley below the 2-digit level from the bottom, which is represented by the faint line passing through points A, B, and C of the midmost and the two flanking lines of longitude on the inner surface of the ring.

[32]Although Alhacen does not explain precisely how the bronze plaque is to be inserted, it obviously cannot be done unless the ring is cut in such a way as to allow the plaque to be fit into the cavity. Presumably the two sections would then be glued back together.

[33]This procedure ensures that the axis of each cylindrical hole will lie exactly half a grain of barley above the top surface of the inserted bronze plaque. Figure 4.3.7, p. 195, gives a cutaway view of the ring with the plaque inserted and the holes drilled through the cylinder's wall.

[34]The scribing of lines described to this point is illustrated in figure 4.3.8, p. 196, with the central square formed by the intersection of the outer parallel lines being 4 digits on a side. The space between the outer and inner circles is the same size as the space between the outer and inner edges of the ring. This figure occurs in various forms in mss *P1*, *O*, *L3*, and *E*.

[35]The block, with the square cavity, is represented in figure 4.3.9, p. 196.

[36]Figure 4.3.10, p. 197, represents a bird's-eye view of the ring, with the bronze plaque inserted and the block at the bottom attached. Note that the point of triangular section of the bronze plaque lies directly over the midpoint of the square cavity in the bottom block.

[37]Presumably, the parallel line to which Alhacen refers is line OV in figure 4.3.2, p. 193. Evidently it does not matter which of the lines converging toward E is chosen for the determination of the tube's length as long as that length remains fixed.

[38]The construction of all seven mirrors is illustrated in figure 4.3.11, p. 198. The plane mirror forms a circle whose diameter is 3 digits and whose thickness is indeterminate, but probably fairly small. The concave and convex cylindrical mirrors are formed from a hollow cylinder that is 3 digits high and 6 digits in diameter at the base. The cylinder is cut along chord AB, which is 3 digits long, leaving axis DF of the excised segment less than half a digit long. This follows from the fact that, being half the diameter of the circle, chord AB subtends an arc of 60°. Hence, CD = 3(cosine 30°) = 3(.866)= 2.6, so DF = 3 – 2.6 = .4. Likewise, the spherical concave and convex mirrors are formed from a hollow sphere whose diameter is 6 digits. This sphere is cut along chord AB so that the resulting mirrors will be segments of that sphere having a circular base with a diameter of 3 digits and an axis roughly .4 digits long. The conical mirrors, finally, are formed from a hollow right cone whose

base is a circle with a diameter of 6 digits and whose line of longitude FV is 4.5 digits long, which means that the cone is just over 3.3 digits high along its axis from vertex to base. The resulting mirrors consist of the segment ABV cut along lines AV and BV.

[39]Although Alhacen does not specify the thickness of the panels, it is clear from this instruction that they can be no more than 2 digits thick if they are to fit upright in the square cavity in the block at the base of the ring while touching the point of the triangular section of the bronze plaque inserted into the ring.

[40]Figure 4.3.12, p. 199, which derives from diagrams in mss *P1* and *E*, represents one of the panels described here. Being 4 digits wide and no more than 2 digits thick, it is designed to fit snugly into the square cavity in the block at the base of the ring yet stand upright facing the point of the bronze plaque. That point touches the inserted panel at point X along the midline of longitude AB, which is 6 digits long. Hence, the distance from the base of the panel to point X consists of the 1 digit to which the panel is sunk in the square cavity plus 2 digits minus the half a grain of barley that the bronze plaque lies below the 2-digit mark above the bottom surface of the ring. If a distance of half a grain of barley is marked above point X, the resulting point D will therefore lie precisely 3 digits above point B and will be the midpoint of line of longitude AB. Bisecting the two segments of that line lying above and below point D will therefore yield four sections of 1.5 digits along it.

[41]The insertion of the plane mirror is represented in the top diagram of figure 4.3.13, p. 199, which provides a face-on view of the panel on the left, a side view of the panel on the right, and a view from the bottom of the panel below.

[42]The insertion of the convex cylindrical mirror is represented in the lower left-hand diagram of figure 4.3.13, p. 199, according to the same three views as before.

[43]The insertion of the concave cylindrical mirror is represented in the lower right-hand diagram of figure 4.3.13, p. 199, according to the same three views as before.

[44]The insertion of the convex conical mirror is represented in the upper left-hand diagram of figure 4.3.14, p. 200, according to the same three views as before. Note that the mirror's line of longitude, VF (as represented in figure 4.3.11, p. 198), coincides with the midline along the length of the panel and is therefore in the same plane with it. Notice, too, that since the mirror is formed from a right cone, and since its line of longitude makes a sharply acute angle with the base, the arc on the base of the section inserted into the panel will be visible as represented, assuming that the sides of the cavity into which the mirror is inserted are perpendicular to the plane of the panel's face.

[45]The insertion of the concave conical mirror is represented in the upper right-hand diagram of figure 4.3.14, p. 200, according to the same three views as before. Note that the mirror's line of longitude VF does not coincide with the midline along the length of the panel but is parallel to and below it. Note also that the arc at the base will be visible as represented, assuming that the sides of the cavity into which the mirror is inserted are perpendicular to the plane of the panel's face.

[46]In this case the ruler prescribed by Alhacen would seem to be of the kind represented on the right of figure 4.3.5, p. 194. The insertion of the convex spheri-

cal mirror is represented in the lower left-hand diagram of figure 4.3.14, p. 200, according to the same three views as before. Note that the midpoint of the mirror's surface coincides with the midpoint of the panel's surface and thus lies in the same plane with it.

[47]The insertion of the concave spherical mirror is represented in the lower right-hand diagram of figure 4.3.14, p. 200, according to the same three views as before. Note that the midpoint of the mirror's surface lies below but directly in line with the midpoint of the panel's surface along the mirror's axis.

[48]Although Alhacen's description of the ruler and its use is far from clear in detail, it seems clear enough in a general way. The first step in the procedure is done with the panel outside the ring. Accordingly, as represented in the left-hand diagram of figure 4.3.15, p. 201, the sharply pointed ruler is poised with its measuring edge along midline AB of the panel so that its point lies directly upon the panel's midpoint D and, therefore, directly above the midpoint of the mirror's concave surface. A needle is dropped orthogonally from the ruler's point to the mirror's surface so as to touch its midpoint, and the spot where the needle intersects the point of the ruler is marked. The distance from this mark to the needle's point is thus the length of the mirror's axis—i.e., approximately .4 digits. This distance is then marked off on radius GE of the bronze plaque from point E of the triangle on the plaque to form XE, as represented in the right-hand diagram of figure 4.3.15, which is drawn out of scale in order to make things clearer.

[49]What was set out to be shown, of course, is that the point of the bronze plaque does not reach all the way to the plane of the mirror's centerpoint since it strikes the mirror in front of that plane. As represented in the right-hand diagram of figure 4.3.15, p. 201, the mirror is inserted upright to face the point of the bronze plaque and is then brought up to the plaque until its point touches the mirror at E. The ruler is dropped along midline AB of the panel until it reaches radius GE, which passes through the vertex of the triangle on the bronze plaque. Point Y, where the ruler's point touches GE, is marked. Since the bronze plaque lies half a grain of barley below the mirror's axis DF (as represented by distance DY, which is grossly exaggerated for purposes of clarity), distance YE will be shorter than distance DF; or, as Alhacen puts it, E will be higher than F, the actual midpoint of the mirror.

[50]Having marked point E on the concave mirror, as represented in the right-hand diagram of figure 4.3.15, p. 201, we then bore a hole there so that the vertex of the bronze plaque's triangle can be pushed through the mirror until point X on radius GE of the plaque reaches the midline along the length of the panel—distance XE, which is the height of the mirror's axis, having already been marked on radius GE of the bronze plaque. In thus moving distance XY, the point of the plaque's triangle will reach point E' behind the mirror, EE' thus being equal to XY. Accordingly, line FE' will lie in a plane parallel to the surface of the panel, so it will be perpendicular to line GE and thus to the surface of the plaque. The point of all this will become clear shortly, when Alhacen sets up the apparatus with the panels inserted to demonstrate empirically that light reflects at equal angles from any mirror, no matter its shape.

Alhacen's full description of the construction of the bronze plaque, the wooden ring, the panels, the mirrors, and the insertion of the mirrors is extraordinarily detailed in some ways and extraordinarily vague in others. For instance, although he describes the insertion of the mirrors in exact detail as far as measurements are concerned, he offers no description of precisely how to cut out the cavities in the panels that will accommodate the mirrors. Furthermore, the constructions described by Alhacen require an astonishing exactitude that offers very little, if any, tolerance for error. Particularly salient in this regard is the description of how to accommodate the point of the bronze plaque to the concave spherical mirror. We know that the radius of the mirror is 3 digits,which translates to roughly 2.25 in or 6 cm. We also know that distance FE' in figure 4.3.15 is half a grain of barley, which translates to roughly .17 in or .42 cm. Therefore, chord EH will be approximately .33 in or .84 cm. and, according to the measure of the radius, will subtend an arc of approximately 13°. On this basis, we arrive at a figure for EE' of around .0064 the mirror's diameter, which comes to roughly .014 in or .036 cm., which = .36 mm. Not only is it difficult to believe that Alhacen could have achieved such exactitude given the available technology, but it is doubtful that an adjustment of such negligible magnitude would have made any real difference in the accuracy of the overall experimental setup. This leaves open two possibilities: either the technological resources available to Alhacen were far more sophisticated than expected, or the entire setup, including the ring, the mirrors, and the panels, was only imagined by Alhacen and never actually constructed.

[51]Why Alhacen covers all the holes but the one through which the light passes with parchment (presumably bright white parchment) is clear at this point: i.e., to make the illumination created by the reflected light as clear and bright as possible. The reason for marking the outline of the holes on the parchment will become clear later, when Alhacen shows that the reflected light disperses in the form of a cone rather than proceeding along a perfectly straight shaft with the same diameter as the hole through which it enters the ring. Two things are worth noting at this point. First, the reason the plaque is placed so low in the cylinder is to ensure that the area around it is relatively unaffected by ambient light and thus relatively dark. This, of course, highlights the effect of the reflected light. Second, if enough particles of smoke, dust, or the like are suspended in the air inside the hollow of the ring, the beams of incident and reflected light will actually be visible.

[52]The point of these procedures with the tube is, of course, to narrow the beam of light as much as possible so as to reduce it to something approaching a mathematical ray. In addition, since the iron tube reaches quite close to the mirror (see paragraph 3.22 above), the beam projected by it onto the mirror will have little chance to spread out, as light is naturally apt to do.

[53]Alhacen's point in comparing the three circles—i.e., the circle at the base of the hole, the circle projected onto the mirror, and the circle projected by reflection to the inner surface of the cylinder—is to show that, no matter how much we narrow the beam, light propagates conically, so the base of the cone continually expands the farther the light propagates. Given the small size of the hole through which the light passes, and given how close the mirror is to that hole, determining

the difference in size between the circle of illumination on the mirror's surface and the circle at the inner end of the hole would be extremely difficult. Like the construction of the setup, this procedure demands an astonishing exactitude.

[54]The point here is to show that the expansion of light described above is not due to the axial ray's reflecting at different points on the circle of illumination projected onto the mirror. In other words, the ray imagined to follow the axis of the hole will reflect from only one point on the mirror, the point where the axis of the hole intersects the mirror.

[55]Although the direction of slant is not clear from this description, it seems clear subsequently that the panel is to be slanted backwards so that the reflected light is projected upward, toward the top of the ring, rather than downward, toward its base, because in this latter case the light would be projected onto the bronze plaque rather than to the inside wall of the cylinder. One clear implication of this is that the incident and reflected rays lie in a plane that is perpendicular to the surface of the mirror as well as to the surface of the bronze plaque.

[56]The point of this procedure is to show that, no matter how the mirror is slanted to the left or right, as long as the ray striking it is normal to the plane tangent to its line of longitude—i.e., the plane of the panel's face—the light will reflect back along that normal. Moreover, as Alhacen goes on to show, this fact applies to the concave cylindrical mirror as well as to the convex and concave conical mirrors. Given that the panel is 4 digits wide and that the cavity is also 4 digits wide, and assuming that both measurements are as exact as possible, slanting the panel to the right or left for the sake of this experiment would seem to be impossible, although indeed the results of the experiment are intuitively obvious. Here we have yet another indication that the entire framework of these experiments was imagined rather than actually executed.

[57]At this point the rationale for boring the hole in the concave spherical mirror to accommodate the point of the bronze plaque becomes clear: namely, to ensure that the distance from the centers of the holes to the midpoint of the mirror—i.e., the point of reflection—is precisely the same as the distance of the point on the bronze plaque to the bottoms of the holes. That in turn ensures that the axial rays reaching the midpoint of the mirror will be perfectly parallel to the lines scribed on the surface of the bronze plaque and intersecting at its point.

[58]In other words, if FE′ in figure 4.3.15, p. 201, represents the midline along the longitude of the panel containing the convex spherical mirror whose midpoint is F, then the ruler is placed with its point on F and the panel is moved until the point of the bronze plaque, which is now taken to be at E′ along perpendicular radius G′E′, lies directly in line with the point of the ruler. Thus, when the ruler is lowered until both points touch, the panel will be positioned so that the midpoint of the mirror and the point on the bronze plaque lie on line FE, just as was the case for the concave spherical mirror, and for precisely the same reason.

[59]Thus, Alhacen is extending to reflection what he already established in his analysis of direct radiation: that light invariably acts in the same way, whether it is primary or secondary.

[60]The points made in this paragraph are illustrated in figure 4.3.16, p. 201, where the ellipse represents the common section formed by the plane of the top surface of

the bronze plaque, VHGMO, and the inside surface of the ring. EE″ represents the midline of longitude along the inserted panel. GE represents the bisecting radius of the plaque, G″E″ the midline of the bottom surface of the square cavity dug into block at the base of the ring, and G′E′ the midline of the top surface of the block at the base of the ring. X″Y″ represents the common section of the bottom surface of the cavity in the block and the face of the inserted panel, X′Y′ the common section of the top surface of the block and the face of the inserted panel, and XY the common section of the top surface of the plaque and the surface of the panel inserted into the square cavity in the block. By construction, the plane of the floor of the cavity in the base-block, the plane of the top surface of the base-block, and the plane of the top surface of the bronze plaque are parallel, so the midlines G″E″, G′E′, and GE within those respective planes will be parallel. Moreover, since the panel with its mirror is inserted upright within the cavity in the base-block, all three of those planes will intersect that panel orthogonally along common sections X″Y″, X′Y′, and XY from the bottom up. So the midlines within those planes will be perpendicular to those common sections as well as to midline EE″ of the panel.

[61]In figure 4.3.17, p. 202, VHGMO represents the plane of the top surface of the bronze plaque, GE being the radius of bisection, and HE and ME being corresponding flanking lines on the plaque's top surface. Points A, C, and B represent the centerpoints of the holes at the inner surface of the ring (drawn out of scale for clarity's sake), and AR, CR, and BR represent the axial rays passing through those holes to intersect at R, which lies on midline RE′ along the length of the panel. Accordingly, the plane formed by those axial rays is parallel to the top surface of the bronze plaque, so those axial rays are all perpendicular to RE′, which is perpendicular to the top surface of the bronze plaque. Likewise, lines AM, CG, and BH form segments of the lines of longitude scribed on the inner surface of the ring during its construction, so they are perpendicular to the top surface of the bronze plaque as well as to the plane of the axial rays, so they are all parallel to RE. Moreover, AM, CG, and BH are radii of their respective holes, A, C, and B being the centerpoints, and M, G, and H being the points where those holes are tangent to the top surface of the bronze plaque. Accordingly, RE, AM, CG, and BH are all equal (i.e., half a grain of barley), and, since they lie between equal and parallel lines, so are AR and ME, CR and GE, and BR and HE, all of which are equal to one another, because the lines scribed on the bronze plaque are all radii of semicircle VHGMO.

[62]In other words, since angles GEM and GEH on the bronze plaque represented in figure 4.3.17, p. 202, are equal by construction, then angles CRA and CRB must be equal according to the parallelism and equality of the corresponding lines in the axial plane.

[63]Here, then, Alhacen has established that the plane of reflection, which contains the normal and the incident and reflected rays is orthogonal to the reflecting surface or the plane tangent to that surface at the point of reflection.

[64]Alhacen's point here is illustrated in figure 4.3.18, p. 202. From any random point A, G, or B on the luminous surface represented by AB, a cone of radiation is formed with its base on reflective surface CD and its vertex at the point of radiation. Conversely, at any random point C, H, or D on the reflective surface, a cone of

radiation is formed with its base on the luminous surface and its vertex at the point of reflection. If some point X is taken between the two surfaces, then the light projected from AB through that point to CD will form two cones with shared vertex X and respective bases AB and CH. Likewise, if some intermediate point Y is taken between X and luminous surface AB, then the rays passing through that point from CD to AB will form two cones with shared vertex Y and respective bases CD and EF. Given that reflection takes place at equal angles, moreover, each such cone will generate an equivalent cone after reflection, so that, for instance, cone of radiation AHB will generate equivalent cone A'HB' after reflection.

[65]Accordingly, if EGHF in figure 4.3.19, p. 203, represents one of the circular holes in the ring, AB the luminous source, and CD the plane mirror inside the ring, then the light from AB will be projected to XY on the mirror. If some point is chosen on line GH within the circle at the bottom opening of the hole, then that point will form the vertex of two cones of radiation with their bases on the lighter circles inside AB and XY, so XY forms the spot on the mirror mentioned earlier. Ultimately, the aggregate of rays reaching through the hole from AB to the mirror will form section GXYH, which, in reflecting from the mirror, forms section KXYL. Hence, the light that spreads out from the bottom opening of the hole to the mirror continues to spread out commensurately after reflection, as was empirically determined earlier when the circle of light on the mirror was compared to the circle of light at the hole's opening and to the circle of light projected to the corresponding hole in the ring.

[66]In fact, to this point in the Latin version of *De aspectibus*, Alhacen has not established these two points explicitly, although he has done so implicitly. The first point—i.e., that light-intensity weakens with distance—follows from the fact that luminous sources appear fainter as they get farther from the viewer. The second point—i.e., that concentrated light is more intense than dispersed light—follows from the fact that light radiates in all possible directions from any luminous or illuminated surface, so that the farther it gets from its source, the more it will spread out. Consequently, if an opaque body is placed in the way of that light, the closer that body lies to the source of illumination, the more light will strike its surface and, consequently, the more intensely illuminated it will be. Alhacen makes these two points explicitly in chapter 3 of the first book in the Arabic version of the *De aspectibus*; see esp. I, 3.79; I, 3.82; I, 3.121, in Sabra, *Optics*, vol. 1, 36-37 and 45.

[67]The concentration to which Alhacen adverts here is due to the fact that every point on the luminous surface radiates its form to each point on the reflecting surface, so that the illumination at that point is the sum of the illumination reaching it from every point on the luminous surface.

[68]How reflected light is weakened by distance and dispersal is illustrated in figure 4.3.19, p. 203, where the light shining through opening GH is projected to XY, which is larger than GH, and reflected thence to KL, which is larger than XY. Hence, the total distance the light travels in passing from GH to XY and then to KL is greater than the distance it travels in going from GH to XY, so the light is weakened accordingly. Also, as it goes from GH to KL the area upon which it is projected increases commensurately with the increase in distance, so the same amount

of light is dispersed over a greater area. And, finally, the reflection itself weakens the light. Therefore, by the time it reaches KL, the light is triply weakened. On the other hand, if KL is taken as the luminous source, then the radiation from it that reaches XY on the mirror is concentrated on a smaller area, and when it reflects to GH it is even more concentrated. Hence, if the light is intensified by that concentration enough to counterbalance the weakening due to distance and the reflection itself, the light at GH will be as intense as the light at KL. But if that intensification is not enough to counterbalance the weakening, the light at GH will be weaker than the light at KL, whereas if the intensification overbalances the weakening, the light at GH will be more intense than the light at KL.

[69]Alhacen's point here can be illustrated by reference to figure 4.3.19, p. 203. Let KL represent the luminous surface, and let KXYL represent a segment of a cone of radiation emanating from KL and projected upon area XY of the mirror. Hence, the light at XY will be concentrated, since XY is smaller than KL. Now, if the illumination at XY is more intense than that at KL, then the light at any point on XY will be more intense than the light at any point on KL. Likewise, if the illumination at XY is less intense than that at KL, the light at any point on XY will be less intense than the light at any point on KL. In this case, therefore, even though the light from every point on KL is concentrated at any given point on XY according to the cone of radiation that has its base in the luminous surface and its vertex at that point, the light concentrated at that point will still be less intense than the light at any given point on KL.

[70]Overall, the point of this discussion is that the light shining from any luminous source onto a mirror reaches that mirror in three ways: according to a multitude of cones with their bases on the mirror and their vertices at points on the luminous surface; according to a multitude of cones with their base on the luminous surface and their vertices at points on the mirror; and according to parallel rays connecting points on the luminous surface with corresponding points on the mirror. In the aggregate, all three types of radiation affect the intensity of reflected light through concentration, dispersal, and increasing distance. In fact, Alhacen was well aware that, despite its weakening through distance and reflection itself, parallel light-radiation can be concentrated to great intensity by being focused in concave spherical and parabolic mirrors; see note 94, p. cii above.

[71]It is interesting to note that Alhacen's definition of a minimal quantum of light is contingent on visibility: that is, the smallest effective quantum of light is what is minimally effective in causing sight. This seems to mean that any given luminous surface can be subdivided into smaller and smaller areas until, finally, the subdivision is so small that it can no longer be seen. From where? Alhacen does not specify, but one can suppose that it becomes invisible from a point infinitesimally beyond the surface. Likewise, one can suppose that the standard of visibility is determined by what can be seen by a healthy, normal eye. Furthermore, it seems clear that the size of such spots varies with the intensity of the luminous source, the minimal quantum of effective light on the surface of a brightly luminous source being smaller than that on the surface of a faintly luminous source. By the same token, the size of such spots will vary with the distance between source

and viewer; the farther that distance, the larger the spot. All in all, then, Alhacen's definition of least light is a function more of perceptual psychology than of physics.

[72]That light does not actually radiate from mathematical points follows from the fact that it necessarily inheres in the surfaces of bodies. Being continuous, surfaces are not composed of points, because, by book 1, definition 1 of Euclid's *Elements*, points have no dimension whatever. Hence, no matter how far we subdivide a luminous or illuminated surface, and thus the light on it, such subdivision will never end in points. Reduced to its effective minimum, light occupies a tiny area or spot on the luminous or illuminated surface, that spot being represented mathematically by the point at its center. The light within this spot propagates its form along a shaft bounded by the edges of the spot it occupies on the luminous or illuminated surface so that when this form reaches another body, it will occupy a spot of the same size on the surface of that body. The mathematical line centered within that shaft and connecting the centerpoints of both spots of light will constitute the ray. Hence, the points at ends of the rays are virtual representations of the real spots of light centered on them, and the ray is a virtual representation of the real shaft of radiation centered on it.

[73]For Alhacen's discussion of the swiftness of light-radiation, see 2, 3.60-62, in Smith, *Alhacen's Theory*, 445-47. At this point, Alhacen is transforming the minima of "least light" just discussed into virtual quanta so as to treat light-radiation by analogy to the projection of tiny bodies that interact dynamically with the physical objects against which they are projected. For some discussion of the source of this analogy, see Smith, *Ptolemy's Theory*, 37-38 and 42-43.

[74]Here Alhacen makes clear that light is to be understood as a virtual rather than a real body in motion. Hence, as he stipulates, the activity of light in its propagation, absorption, reflection, and refraction is not due to the actual physical properties of the bodies with which it interacts, although its activity can be likened to that of bodies in motion for the purposes of descriptive "explanation." Thus, although it is not due to actual physical rebound, reflection is like physical rebound in certain fundamental ways and therefore follows its laws in certain fundamental ways.

[75]By now, Alhacen's attempt to describe the activity of light by analogy to projectile motion has forced him into a quandary. On the one hand, he is quick to deny that light-radiation involves actual matter in motion. Yet, on the other, he ties reflection to the physical structure of the surfaces upon which it impinges. Moreover, his account of why light does not reflect from uneven or porous surfaces implies that, when light strikes such surfaces, its rebound is somehow deadened, causing the light to be trapped in those pores. To add even more difficulty to this account, he points out that, even if various parts of the surface are reflective but not arranged uniformly, the light that strikes them will reflect, but in various directions so as to be scattered randomly. In that case, presumably, the light that is seen comes from all parts of the body, not just a privileged spot, as happens in true reflection. This model of deadening and/or scattering, which is apparently meant to explain

mere illumination, is apparently inconsistent with his previous explanation of illumination according to the absorption of luminosity by opaque surfaces, which become sources of luminosity by virtue of that absorption.

[76]The distinction between natural and accidental motion is based on the Aristotelian theory of motion according to which heavy objects, by their very nature as heavy, are intrinsically impelled to move straight toward the center of the universe in free fall. Accidental motion, being extrinsically forced, counters the natural tendency of heavy bodies to drop straight down, causing them to move in other directions, e.g., along the horizontal or straight upward. See esp. Aristotle, *Physics*, 8, 4.254b7-256a3.

[77]In this case, then, the arrow is shot in the same vertical plane as it was when launched along the horizontal, but it is shot along a slanted trajectory either down toward or up toward the mirror, in which case it rebounds back at a slant.

[78]Thus, the resistance posed by the body is a function of the smoothness or continuity of its surface rather than of its physical hardness. This, presumably, is why water can be intensely reflective despite its lack of physical hardness.

[79]Alhacen's analysis of the dynamics of reflection is illustrated by figure 4.3.20, p. 203, in which AB represents the reflecting surface, XY the normal, R the point of reflection, and CR an oblique ray along which light reaches point R. Now, if there were no resistance whatever at R, the light would pass through surface AB unhampered, and it would continue along the rectilinear continuation RD of its original path, so the resulting path below surface AB would have precisely the same disposition as the path above it. On the other hand, if the light is completely resisted at R, it will rebound along RE, which has the same disposition as CR insofar as the angle of reflection XRE is equal to the angle of incidence CRX. The "dynamic" explanation for this follows from Alhacen's reduction of the motion of the light along CR to two components: motion along the normal XY and motion along the orthogonal to that normal, BR. Accordingly, there are two components of resistance at point R, one upward along normal RY, the other along orthogonal AB. But since the light does not pass into the mirror through R, the resistance along AR does not come into play, leaving the light to maintain its original motion in the horizontal direction. On the other hand, there is full resistance along normal RY, so the light is diverted upward with the same amount of motion as before but in the opposite direction. Later, in book 7 of the *De aspectibus*, Alhacen applies the same vectorial analysis to refraction in order to explain why, when it enters an optically dense medium, light is forced toward the normal after passing through point R, now taken as a point of refraction rather than of reflection. For some discussion of this vectorial analysis and its implications, see A. I. Sabra, *Theories of Light from Descartes to Newton* (London: Oldbourne, 1967; reprint, Cambridge University Press, 1981), esp. 72-82 and 93-98; David C. Lindberg, "The Cause of Refraction in Medieval Optics," *British Journal for the History of Science* 4 (1969): 23-38; and A. Mark Smith, *Descartes's Theory of Light and Refraction: A Discourse on Method*, Transactions of the American Philosophical Society 77.3 (Philadelphia: APS Press, 1987).

[80]Presumably, then, even though light loses intensity/speed with distance as well as with the impact of reflection, its overall speed of propagation is so great

that it will not decelerate to the point of stopping no matter how far it goes or how many reflections it undergoes.

[81]My interpretation of this experiment is illustrated in figure 4.3.21, p. 204. The apparatus is set toward the window in such a way that the holes in line with RF and RE are blocked from any illumination passing through the window, presumably to keep such stray illumination from interfering with visual observation through those holes. The colored object is placed in direct sunlight streaming through the window so as to be brightly illuminated. Beams of luminous color will pass from C and D through the two holes in line with CR and DR to reach point of reflection R on the mirror inserted into the apparatus. Those beams will then be reflected from R to their corresponding holes, CR to the hole in line with RF and DR to the hole in line with ER. Since the illumination on the face of colored object CD is secondary, then its radiation to the mirror will be relatively weak, and it will be even weaker after having passed into the hollow of the ring through the narrow holes. Subject to additional weakening by reflection at R, the luminous color passed along RF and RE will be so faint by the time it reaches the inside wall of the ring at the openings of the holes in line with RF and RE that it may be too faint to see, even if it shines on white parchment placed at those openings. Hence, in order to verify that color reflects at equal angles, the experimenter must actually look through the holes to see the reflected color. The problem with this procedure is that it provides only indirect verification, because what is seen is the image behind the mirror of the color at the openings in line with CR and DR. However, if the hole along CR is blocked while the one in line with DR is left open, then the color's image will be seen through the hole in line with RE, but not through the hole in line with FR. And the same will hold *mutatis mutandis* if the hole in line with DR is blocked while the one in line with CR is left open.

[82]Alhacen has already discussed this theory in the context of direct vision, imputing it to the "mathematicians," a group that certainly includes Euclid and Ptolemy. As it is articulated here, Alhacen's understanding of how mirror-images are perceived according to visual rays is unclear. He could be taken to mean that the form is grasped at the object itself, after reflection, or he could be taken to mean that the form is grasped at the mirror. This latter interpretation, however, is inconsistent with his understanding of the theory in 1, å6.58, Smith, *Alhacen's Theory*, 373-374, where he imputes to the visual-ray theorists the idea that, after making contact with a given visible object, the rays transmit the information gathered from that contact back to the eye.

[83]In other words, the form impressed on the mirror is essentially the same as the form that would be impressed directly on the eye. Presumably, Alhacen is referring here to the Democritean theory of "emphasis," in which reflection involves a stamping of the image on the reflecting surface; see, e.g., A. Mark Smith, *Ptolemy and the Foundations of Ancient Mathematical Optics*, Transactions of the American Philosophical Society 89.3 (Philadelphia: APS Press, 1999), 34-35.

[84]I take Alhacen to mean that we should block up all the holes on one side or other of the central hole in the ring that is in line with the radius of bisection on the inserted bronze plaque.

[85]This brief account of how the letter would appear is confusing, in part, because the text, as it stands, makes no distinction among the curved mirrors (i.e., cylindrical, conical, and spherical) according to whether they are convex or concave. In the case of convexity, no matter whether the mirror is cylindrical, conical, or spherical, the image of the letter will appear somewhat smaller than it did in the plane mirror. Likewise, it will be bowed according to the mirror's curvature, and in the convex conical mirror it will appear more bowed toward the top according to the convergence of the lines of longitude as they approach the vertex. As in the case of the plane mirror, however, the letter will appear reversed. In the case of the concave cylindrical and conical mirrors, on the other hand, the image will appear with its proper left to right orientation, whereas in the case of the concave spherical mirror it will appear inverted.

[86]In the first case, when the mirror is held sideways so that its axis lies along the horizontal, the viewer's face will appear frontal but compressed along the vertical. As the mirror is rotated upright toward the vertical, however, his face will still appear frontal, but it will become increasingly elongated until it is maximally distended at the vertical.

[87]It is worth noting that none of the empirical demonstrations Alhacen adduces in this brief chapter militates against the visual-ray theory, at least not as it is applied to the analysis of reflection by Euclid and Ptolemy. Mathematically, in fact, Alhacen's theory of mirror-imaging is based on precisely the same principles as that of Euclid and Ptolemy, neither of whom subscribed to the impression-theory. Alhacen does address the visual-ray explanation of reflection later in paragraph 5.3.

[88]In other words, if Z in figure 4.3.19, p. 203, were a point of luminosity on a visible object, and if KL were a visible object facing mirror CD, then the light from Z would radiate to the mirror in the form of a cone with base XY in the mirror. After reflection, that cone would be extended to base KL so that every ray originating from point Z would be reflected to a given point on KL. By extension, then, if KL were taken as a luminous object, the situation would be reversed, each point on KL radiating to base XY on the mirror and reflecting thence to Z, where all the rays would intersect.

[89]The location Alhacen has in mind here is the point of reflection, not the actual location where the image appears behind or in front of the mirror. What Alhacen establishes here is that the image is always seen through, and therefore in line with, the point of reflection along the line of reflection—a point that follows from his earlier demonstrations at the end of chapter 3 and in chapter 4 that the reflected color or the image of the letter can only be seen through the corresponding hole, which lies along the line of the bronze plaque that forms an angle with the normal equal to the angle formed by the line of incidence with that same normal.

[90]Here, Alhacen takes up the thread left dangling at the end of chapter 4 by explaining why the visual-ray analysis of reflection is flawed. The objection he raises here echoes his general argument against the visual ray theory in 1, 6.58, in Smith, *Alhacen's Theory*, 373-374.

[91]I take this to mean that, in any given instance, there is a perfect point-to-point correspondence between the entire form on the mirror and the luminous object, as

illustrated in figure 4.5.1, p. 204, where the light from points A, B, C, D, and E radiates to corresponding points F, G, H, K, and L so as to reflect to center of sight O. If the points on the entire form on the mirror were distributed disparately, as represented by ray BX, then the resulting reflection from that point would be to Z rather than O. Thus, in any given instance, the entire form of AE will be projected uniformly, so that the form is in perfect correspondence with the object. Only then will it be projected to the appropriate center of sight. Or, to put it another way, any given form at one particular spot on the mirror will be reflected to one, and only one, specific center of sight.

[92]The intent of this passage is far from clear, but there are at least three possible interpretations. First, what might be intended is that the shape of the mirror, in terms of its contour, should be the same as that of the object in cross-section, so that a square object will only appear square in a square mirror. This, of course, is untenable, because the contour of the mirror (e.g., a plane mirror that is circular in outline) has no effect on the shape of the image seen in it. Second, Alhacen might have in mind the surface-shape of the mirror in terms of uniformity. Accordingly, if the mirror's surface is uneven, the image seen in it will also be uneven. Third, and most likely, Alhacen has in mind the distortion due to the curvature of the mirror. Hence, the image seen in a plane mirror will be the proper complement of the object in terms of apparent shape and size. In convex mirrors, on the other hand, the image will bow according to the mirror's curvature, and it generally appears smaller than it should. And, finally, in concave mirrors the image can be distorted in terms of size and orientation.

[93]That is, according to variations in the shape of the mirror, depending on whether it is convex or concave.

[94]Thus, as illustrated in figure 4.5.2, p. 205, BC represents the plane mirror itself and AB a plane continuous with its reflecting surface. R is a point on the mirror to which a ray reaches from some object-point H, and E is the center of sight. Hence, from point R a normal can be erected to the plane of the mirror, and a normal can be dropped to that same plane from E. Being parallel, those two normals form a plane. If, then, line ER is drawn from upper to lower endpoints of those normals, it will form acute angles FER and GRE and will therefore lie in the same plane as those normals.

[95]Again, by reference to figure 4.5.2, p. 205, Alhacen instructs us to continue line FR on the mirror's surface to C, that line being orthogonal to both normals. Then, from right angle GRC acute angle HRG is cut so as to be equal to angle ERG, the former thus constituting the angle of incidence and the latter the angle of reflection. Accordingly, both rays and both normals will lie in the same plane of reflection, and within that plane four cardinal points are situated: center of sight E, object-point H, endpoint F of the normal dropped from E, and point of reflection R.

[96]In other words, if the actual mirror is only a section of a sphere, the eye cannot lie on the mirror itself or on the surface of the sphere of which the mirror is a section.

[97]Alhacen's point here is illustrated in figure 4.5.3, p. 205. Let the circle represent a section of a spherical mirror facing center of sight E, and let ET be tangent to the mirror's surface at T. Then, if ET is rotated about axis EC, which passes through

the mirror's center, T will sweep about the surface of the mirror to form a circular section represented by line TT', according to which the circle is viewed edgewise. Thus, axis EC passes through centerpoint B of that circle, BE forming the axis or pole of that circle's generation. Now, we know from Euclid, III.18, that any line drawn from center C of any great circle within the sphere to any point of tangency, such as T, of that great circle, will be perpendicular to the tangent, so angle ETC is a right angle. Hence, the angle formed by tangent ET with pole BE of circle TT' generated by that tangent will be less than a right angle. If the great circle is then subdivided into quadrants by diameters AC and DC, it is obvious that arc AT cut off by the tangent will be less than a quadrant of that great circle. According to this analysis, then, if the mirror is convex, segment TAT', which is less than half the sphere, will be visible. If it is concave, on the other hand, segment TDT', which is greater than half the sphere, will be visible, assuming that convex segment TAT' is excised to leave the entire section open to view.

[98]Alhacen's demonstration that any point within the visible portion of a spherical mirror can be a point of reflection is illustrated in figure 4.5.4, p. 206. The elliptical section TAT'B backgrounded in light gray in the upper diagram represents the section of a convex spherical mirror that is visible from center of sight E, and EPC, passing through the sphere's centerpoint C, represents the pole or axis of that section, point P being where it intersects the sphere's surface. TPRT' is the arc of a great circle on the spherical surface of the mirror and it passes through point P and some randomly chosen point R. Line DRF is drawn tangent to that arc at R. ARB is the arc of another great circle passing through R, GRK being tangent to that arc at point R. Therefore two tangents DRF and GRK form a plane tangent to the sphere at R, so line XRC is perpendicular to that plane and thus normal to the mirror at point R. Likewise, axis EPC of the visible section of the mirror is normal to the mirror since it passes to C. Accordingly, since EPC and XRC are both normal to the mirror, they lie in the same plane, which is perpendicular to the plane formed by tangents DRF and GRK. Within the plane containing EPC and XRC, as represented in the lower diagram of figure 4.5.4, line of reflection ER strikes the mirror obliquely at angle ERX, so some line of incidence OR can be dropped to R to form angle of incidence ORX equal to ERX. Therefore, R is a point of reflection, and the same will hold for any point on arc TT'. Since an infinite number of planes can be passed along EPC to cut great circles on the mirror, and since every point on each of those great circles can serve as a point of reflection for center of sight E, then every point on visible portion TAT'B of the mirror can serve as a point of reflection for E.

[99]In other words, given viewpoint E, all possible planes of reflection will intersect along normal EPC in figure 4.5.4, p. 206, and those planes will cut great circles on the surface of the sphere.

[100]Alhacen's point here is illustrated in the top diagram in figure 4.5.5, p. 207. Let convex cylindrical mirror T'ABT face center of sight E, let the cylinder be cut by plane ETT', passing through E, and let the resulting common section be circle TT' centered on C. Draw tangents ET' and ET to that circle, and then from points T and T' drop lines of longitude T'A and TB. Neither of the two planes formed by tangent ET' and line of longitude T'A or by tangent ET and line of longitude TB will cut the

cylinder, because if it did—e.g., if plane ETB were to cut the cylinder—then that plane would cut the cylinder along two lines of longitude, TB and some other line of longitude, such as LM. But, in that case, tangent ET would cut the circle at two points, T and L, so it would not be tangent, which is contrary to our original supposition. Therefore, the two planes ET'A and ETB are tangent to the mirror, and the portion of the convex mirror visible to the eye at E is defined by lines of longitude T'A and TB. Although Alhacen specifies at the beginning of this construction that the mirror be convex, it is clear in this case, as in the case of the spherical mirror, that lines of longitude T'A and TB also define the maximum visible section of the concave surface of the cylinder facing viewpoint E, provided that convex segment T'PTBA is excised to expose that entire section to view.

[101]In other words, as illustrated in the top diagram of figure 4.5.5, p. 207, since common section ES of the two tangent planes is perpendicular to the plane of the circle cut on the cylinder, and since axis CQ of the cylinder, as well as lines of longitude T'A and TB, is also perpendicular to that plane, then ES, CQ, T'A, and TB are all parallel to one another.

[102]That reflection can occur to E from any point on the visible portion of the convex cylindrical mirror is illustrated in the middle and bottom diagrams of figure 4.5.5, p. 207. Let E lie in the plane of the cutting circle TT' centered on C, as illustrated in the middle diagram. If the point selected is P, through which the visual axis passes to centerpoint C of the circle, then reflection will occur back along PE, since that line is normal to the mirror. If R is selected to the side of visual axis EP, then a line of longitude RU will pass through that point. Accordingly, plane DFHG tangent to the mirror along that line of longitude will cut both tangent planes between the mirror's surface and center of sight E. The same will hold for any plane tangent along any line of longitude within the visible portion of the mirror. Now, if normal XRC is drawn to plane DFHG, then line ER connecting the center of sight and the point of reflection will form an acute angle, ERX, with that normal, since it lies within right angle DRX. Therefore, within the adjacent right angle XRF, an acute angle XRO can be cut off equal to acute angle ERX. Since all three lines ER, XR, and RO lie in plane ETT' containing the center of sight and the circular section on the mirror, ETT' will constitute the plane of reflection.

On the other hand, if R does not lie in plane ETT', then let it be chosen below that plane, as illustrated in the bottom diagram of figure 4.5.5. If it lies on line of longitude PRU passing through point P, where visual axis EC intersects the mirror's surface in plane ETT', then reflection can occur from any point, such as R, on that line, since we established earlier that ES, which is the common section of the tangent planes, and axis CQ are parallel, and line of longitude PRU is parallel to both. Hence, since EPC lies in the same plane as those three parallels, and since it is normal to the mirror, the plane within which it lies is also normal to the mirror. Within that plane any point on line of longitude PRU will be a point of reflection, for, if we pass a circle of section through that point, extended radius C'RX of that circle will be normal to line of longitude PRU. ER, connecting the center of sight with the selected point of reflection, will therefore form an acute angle with normal XRC', and within the plane of that angle an equal acute angle XRO can be formed with normal XRC′ on the other side. The resulting line of incidence, OR, will therefore lie in the same plane as normal XRC′ and line of reflection ER, and that plane is normal to the mirror. If we choose another point of reflection, such as R′, to the side of PRU, then we can draw a line of longitude R′U through it, and along that line of longitude we can form tangent plane DFHG. We can then form the circle of section through point R′, diameter C′R′X′ of that circle being normal to plane DFHG and thus to the mirror. Hence, line ER′ connecting the center of sight and the point of reflection will form acute angle ER′X′ with normal X′R′C′, and within the same plane an equal acute angle X′R′O′ can be formed with normal X′R′C′. Since the plane formed by line of incidence O′R′, normal X′R′C′, and line of reflection ER′ is normal to the mirror at point R′, that plane will be a plane of reflection.

As with the spherical mirror, so with the cylindrical mirror, the analysis holds for the concave section of the mirror bounded by T′A and TB if we extend normals XRC′ and X′R′C′ to points Z and Z′ on that surface. Since those lines are normal to the concave surface of the cylinder at those points, then lines EZ and EZ′ will form acute angles with the tangents to those points. Within the plane of each of these angles an equal acute angle can be formed with that tangent on the other side of the normal. Since the plane formed by each line of incidence, its respective normal, and its respective line of reflection will be normal to the mirror, it will be a plane of reflection.

[103]That the common section of a cylinder and a plane cutting the cylinder obliquely is a true ellipse was first proved by Serenus in proposition 20 of *On the Section of a Cylinder*.

[104]If the plane of reflection forms an elliptical section on the cylinder's surface, as represented in the top diagram of figure 4.5.6, p. 208, by the ellipse, whose major axis is FG, line FG will be bisected at C, where it intersects axis TU of the cylinder. Pass a plane through point C orthogonal to axis TU to form circle AB. C will therefore be the circle's centerpoint, and its diameter RCZ will constitute the common section of the circle's juncture with the ellipse. Since it coincides with the diameter of circle AB, XRC is normal to the plane tangent to the cylinder's surface at point R along line of longitude DY, and it is also perpendicular to axis TU. Thus, for any center of sight stationed within the plane of the ellipse, reflection will occur to it from point R, provided that R lies within the visible portion of the mirror. On the other hand, reflection cannot occur from any other point, such as L, on the elliptical section, because line LC dropped from it to C will fall obliquely to axis TU and will therefore not be normal to the mirror's surface. Or, to put it another way, line LS normal to the ellipse at point L will pass by axis TU and will therefore not intersect it.

In sum, then, within the plane of reflection formed by the ellipse, the only point from which reflection can occur is that point on the mirror where the ellipse is intersected by a circle, such as AB, whose diameter forms the common section of the circle and the ellipse, this section also forming the minor axis of the ellipse. Being shared by the circle and the ellipse, this is the only line within the ellipse that is perpendicular to axis TU, so it is the only line within the plane of the ellipse that is normal to the mirror's surface. It should be noted that the same analysis applies to the concave surface of the mirror beyond C. Thus, if normal XRC is extended to point Z on the opposite side of the cylinder, point Z will be a point of reflection for any center of sight within the plane of the ellipse. In this case, however, opposite point R can also serve as a point of reflection for that center of sight if more than half the mirror is visible to it. Alhacen makes this point later on in paragraph 5.56 below.

At first glance, the conclusion that no point other than R will serve as a point of reflection in the plane of the ellipse seems to contradict the earlier claim that, for any given center of sight, every point in the visible portion of the cylinder will be a point of reflection. Thus, it seems inconsistent to deny that L can be a point of reflection for the center of sight for which R is a point of reflection. In fact, it can, but not within the plane of the given ellipse. Thus, if E in the bottom diagram of figure 4.5.6 represents a given center of sight, then, as was just shown, R is the only point of reflection for E within the plane of elliptical section RL, because only that section shares diameter XRC of the circle of section ARM passing through R. Accordingly, XRC will be normal to the mirror along line of longitude DR, and line of reflection ER and normal XRC will lie in the plane of elliptical section RL. Now, if we pass circle of section PLU through point L, then its diameter X'L will be normal to the cylinder along line of longitude VL. If line EL is drawn from the center of sight to point L, then that line will form acute angle ELX' with normal X'L. It will also form a plane with that normal, and within this plane an acute angle equal to ELX' can be formed on the other side of the normal. Thus, all three lines will lie in a plane that is normal to the mirror at point L. Moreover, that plane will cut an elliptical section on the cylinder, in this case the section passing through points N and L. Thus, points E and L lie in both of the planes of both elliptical sections, the original one passing through points R and L, and the new one passing through points N and L. And the same holds for any other point on the original section passing through points R and L: another plane of reflection can be passed through it from E according to the diameter of the circle of section at that point, since that diameter is normal to the cylinder at that point.

[105]In other words, more than half the surface of a cylindrical mirror is visible to both eyes if the diameter of the cylinder's base is less than the distance between the eyes. In that case, if both eyes lie within the plane of the ellipse, two points of reflection may be exposed to it, because diameter RC of the circle, which is also the minor axis of the ellipse, is normal to the mirror's surface at both its endpoints.

[106]This, of course, is simply a repetition of the argument made with respect to convex cylindrical mirrors in paragraph 5.16 above.

[107]The situation described to this point is illustrated in figure 4.5.7, p. 209, where the cone with vertex A stands on base-circle GF, which is orthogonal to axis AC passing through centerpoint C of that base-circle. Center of sight E is located below A so that line EA connecting them forms acute angle EAC with axis AC. A plane is passed through E orthogonal to axis AC to cut the cone along circle DB. Tangents EB and ED are drawn to that circle, and through points of tangency B and D lines of longitude AF and AG are drawn. The two planes formed by lines of tangency EB and ED and by lines of longitude AF and AG, respectively, are tangent

to the surface of the cone along the lines of longitude, neither of them being able to cut the cone. Therefore, the lines of longitude AF and AG along which these two planes are tangent to the cone will define the visible portion of the mirror, which constitutes less than half its surface. As in the case of the concave spherical and cylindrical mirrors, so in this one, if the convex segment AGF of the mirror facing the eye were removed, the concave portion defined by lines of longitude AG and AF would be visible to center of sight E, and it would constitute more than half the cone's surface.

[108]In figure 4.5.8, p. 209, center of sight E faces the right cone so that line EA drawn from it to the cone's vertex forms a right angle with axis AC. A plane is passed through the cone orthogonal to axis AC to form circle DKB passing through point H on the axis. H is therefore the circle's centerpoint. The common section of this circle and plane EAHK containing the center of sight and axis AC is straight line KH, which is a radius of circle DKB and is therefore perpendicular to axis AC. Diameter DHB is drawn orthogonally to KH, and through endpoints B and D of that diameter tangents LB and MD are drawn to the circle. From points of tangency B and D lines of longitude BA and DA are drawn to the cone's vertex. Hence, the planes formed by tangent LB and line of longitude AB and by tangent MD and line of longitude AD are tangent to the surface of the cone along those lines of longitude. Since EA and KH lie in the same plane, and since both lines are perpendicular to axis AC, they are parallel to one another. Likewise, tangents MD and LB are parallel to one another as well as to EA and KH, so planes EABL and EADM intersect along EA. Accordingly, the visible portion of the cone defined by planes EABL and EADM constitutes half the cone's surface, since the defining planes are tangent at endpoints D and B of the circular section's diameter (as well as at endpoints F and G of the base circle's diameter). As before, if the visible concave segment of the mirror were removed, the concave segment defined by lines of longitude AF and AG facing the mirror would be visible, and it would constitute half the mirror's concave surface.

[109]The situation just described is illustrated in figure 4.5.9, p. 210. Center of sight E faces the conical mirror from above so that line EA drawn from it to the cone's vertex forms obtuse angle EAC with axis AC. A plane is passed through the cone orthogonal to axis AC to form circle DB. The common section of this circle and plane EAC containing the center of sight and axis AC is diameter KH of the circle. HK is extended beyond the cone's surface, and EA is likewise extended to meet it at point L, EAL and HKL thus forming acute angle ALK and also forming acute angle LAH with axis AC. From L tangents LB and LD are drawn to the circle, and lines of longitude AF and AG are drawn through points B and D of tangency. Thus, the planes formed by tangent LB and line of longitude AF and by tangent LD and line of longitude AG are tangent to the surface of the mirror along those lines of longitude. Those planes also intersect along line EAL, so each contains the center of sight, the vertex of the cone, and the point of intersection L. Hence, the portion of the cone's surface enclosed by the two planes on the side of L is less than half, so the portion seen by E on the other side of L will be more than half. This, of course, means that, if concave segment AFG of the mirror on the side of L were

open to view, it would constitute the maximum section of the concave surface visible to center of sight E, and it would be less than half the mirror's surface.

[110]The point of this discussion is illustrated in figure 4.5.10, p. 210, where center of sight E faces the cone from above so that line of sight EA connecting the center of sight and the vertex of the cone coincides with line of longitude AF on the cone's surface. Accordingly, no matter what point is chosen on that surface, as long as it does not lie on AF, the line from E to it will cut the surface of the cone according to the plane formed by the line of longitude on which that point lies and line of longitude AF. Thus, if points B and D are chosen on the mirror's surface, their respective cutting planes will be ABF and ADF, within which will lie triangles EAB and EAD formed by lines EB and ED from the center of sight to the object-points, the lines of longitude BA and DA extended to vertex A from those object-points, and line EA from the center of sight to the cone's vertex. Accordingly, given any such point, e.g., D, the line extended to it from center of sight E will cut the cone at no other point, such as X, so the line of sight to D will be uninterrupted. It follows, then, that all the points on the mirror's surface, except for those along line of longitude AF, will be exposed to view. On the other hand, if some concave portion of the mirror were open in this situation, for instance, if segment ABD were excised, none of its surface except for line of longitude AF would be visible, because, other than EA, none of the lines drawn from E to any point on its interior surface would cut it. In short, the entire concave surface would be effectively invisible.

[111]Alhacen's point here is illustrated in figure 4.5.11, p. 211, where line of sight EAF enters the cone through vertex A. Accordingly, any line of sight from E to any point on the mirror will cut the mirror on the line of longitude passing through that point. Thus, the plane tangent to the mirror along that line of longitude will pass below the center of sight, and since the plane containing the center of sight, the line of sight, and the line of longitude is orthogonal to that tangent plane, it can be a plane of reflection. Conversely, if any segment of the mirror is removed so as to open the concave surface of the mirror to view, none of that surface will be visible, since none of the lines of sight will intersect that surface.

[112]Although this last sentence is somewhat tortured in Latin, the point of it seems to be that, no matter where the line of sight touches the cone's surface within the visible portion of the mirror, that line of sight lies in a plane that is orthogonal to the plane tangent to the mirror at that point, so not only does that former plane constitute a plane of reflection—or, as Alhacen calls it here, a visual plane (*visualis superficies*)—but the point of tangency is visible. Accordingly, at least by implication, that point of tangency will be a point of reflection not just in theory but in fact.

[113]The claim that reflection will occur from any point on the visible surface of the cone can be easily understood by recourse to figure 4.5.12, p. 211. Let R be a randomly chosen point on the surface of the cone, whose vertex is A and whose axis is AC. If a plane is passed through R and axis AC, it will cut the cone's surface along line of longitude AB. Pass a plane through R orthogonal to axis AC to cut the cone's surface along circle FRG, whose diameter CR is the common section of plane ARC and the plane of the circle. CR intersects line of longitude AB obliquely so as to form acute angle DRB with it. Let the center of sight lie outside the cone within

plane ARC, and draw tangent XY to point R. From point K on axis AC draw KRL normal to the plane formed by tangent XY and line of longitude AB. Accordingly, plane AKRL containing axis AC, line of longitude AB, and normal KRL will be orthogonal to the plane formed by tangent XY and line of longitude AB, so plane AKRL will form a plane within which reflection will occur from R to any center of sight lying within it. For instance, if the center of sight is located at endpoint L of normal KL, then the form of L will reflect back along RL from point R. If the center of sight is located at E or E′ within plane AKRL but outside normal KL, then angles of incidence ORL and O′RL can be formed equal to angles of reflection ERL and E′RL, which means that the forms of O and O′ will reflect from R to E and E′ respectively. No matter where the center of sight is situated within that plane of reflection, then, R will serve as a point of reflection for it. Furthermore, since plane of reflection AKRL is orthogonal to the plane formed by AR and XY, which is tangent to the cone's surface along AR, then, if the lines of incidence and reflection OR and RE are held in place and rotated about axis KL, they will form the same angles of incidence and reflection throughout that rotation. Hence, no matter how the line of sight cuts the cone at R, it will lie in a plane orthogonal to the plane tangent to the line of longitude from point R, so R will be a point of reflection for E no matter where E is situated with respect to line of longitude AB.

[114]If the center of sight lies above the cone on its axis, then clearly the line of sight and the axis will coincide, and every plane passing along that line will be orthogonal to the circle at the cone's base and will intersect it along a diameter. Every such plane will therefore cut the cone along opposite lines of longitude that pass through the endpoints of a diameter on the circle at the base, so any given line of sight extended from the center of sight to the mirror's surface will intersect it at a point on the line of longitude lying in the same plane as the line of sight. Since that plane is orthogonal to the mirror's surface along that line of longitude, it will be a plane of reflection, and every point on that line of longitude will be a point of reflection for the center of sight situated within that plane, because a normal can be extended from the axis through that point. On the other hand, if the center of sight lies anywhere else, an infinite number of planes can be passed through it to cut the mirror along an infinite number of common sections. Of these, only one can be a line of longitude, and reflection will occur from any point on it. If the plane cuts a circle on the mirror's surface, then no reflection can occur in that plane, because every line within it will cut the cone's surface obliquely, so none of them can be normal to that surface. The rest of the planes will form parabolic, hyperbolic, or elliptical sections on the mirror's surface, depending on the obliquity of the cut. Accordingly, if the plane passes through the entire surface of the cone, the cut will be elliptical; if the plane is parallel to one of the lines of longitude, the cut will be parabolic, and if the plane does not cut the entire cone and is not parallel to any of the lines of longitude, the cut will be hyperbolic. That each of these planes *can* be a plane of reflection follows from our analysis of figure 4.5.12, p. 211, where we concluded that R will constitute a point of reflection for center of sight E throughout the rotation of the plane of reflection about normal KRL. During this rotation, that plane cuts the cone along an infinite number of sections, none of which forms a

circle, one of which forms a line of longitude (i.e., AB), and the rest of which form various conic sections depending on the obliquity of cut.

[115]Let the cone with vertex A in figure 4.5.13, p. 212, represent the mirror, and let the plane of reflection cut the mirror at point R along conic section DRE. Extend normal KRL through point R to intersect the cone's axis at K. Circle RB imagined to pass through R will intersect axis AK orthogonally, and AK will pass through its centerpoint C, so, like BCR, all its diameters will intersect the mirror's surface obliquely. If normal KRL is rotated about axis AK, it will form a cone with its vertex at K and its base on circle RB. Therefore, all the lines of longitude on this second cone will be normal to the surface of the cone forming the mirror. Let line KF be one of these lines of longitude, and let it extend through circle RB to G. It will therefore be normal to line of longitude AF on the mirror. Then, through point H, where line of longitude AF intersects conic section DRE formed by the cutting plane, extend line KHM from the vertex of the lower cone. Clearly, since KHM and KFG lie in the same plane with line of longitude AFH on the mirror, KHM must be oblique with respect to that line, since KFG is perpendicular to it. Accordingly, H cannot be a point of reflection in the plane of DRE, nor, for that matter, can any point on section DRE other than R. Thus, within plane of reflection DRE there can be only one point of reflection. Furthermore, since the conic section is tangent to the lower cone along line of longitude KR, that line will form the axis of the section. Hence, if RK intersects edge AB of the cone, the section will be an ellipse with RK as its major axis. If RK is parallel to edge AB of the cone, the section will be a parabola, and if RK does not intersect AB within the cone, the section will be a hyperbola. Which of the three conic sections is formed ultimately depends on the angle RAB formed at the vertex of the cone. In all three cases, K will be the focus of the conic section. This analysis also applies if the mirror is concave so that, instead of viewing the mirror from the left, the center of sight views it from the right. As long as it lies in the plane of conic section DRE, and as long as the major axis of that section passes through R and K, R will be the only point of reflection for that center of sight within that particular plane.

[116]That there can be two, but no more than two, points of reflection in a plane that cuts the imagined cone is illustrated in figure 4.5.14, p. 212. Let the plane be passed through line of longitude AF on the mirror's surface to cut the base-circle of the imagined cone at points R and R' and to form conic section DRR'E on the mirror. By construction we established that lines of longitude KRM and KR'L on the imagined cone are normal to lines of longitude AR and AR' on the mirror's surface. Thus, if those two lines of longitude in the imagined cone lie within the plane of conic section DRR'E, then R and R' can serve as points of reflection for any center of sight lying outside the mirror's surface within that plane. Within that same plane, moreover, lines of reflection from both R and R' can intersect at one and the same center of sight, so, if the center of sight is stationed far enough beyond the cone that both R and R' are exposed to view, reflection can occur from both points to that center of sight. That reflection can occur to that center of sight from no other points on conic section DRR'E is clear from the fact that, if any other point on the conic section is chosen, and if a line from vertex K of the lower cone is extended

through it, that line will not be normal to the section. For instance, if point P is chosen on section DRR'E below base-circle FB, and if line of longitude AP on the mirror is dropped to that point, then, when line of longitude KPQ on the lower cone is extended through it, it will form obtuse angle APQ with line of longitude AP on the mirror, because it passes below point O, where normal KO intersects the same line of longitude. On the other hand, if point H is chosen on section DRR'E above base-circle FB, and if line of longitude AF on the mirror is dropped to that point, then, when line of longitude KHN on the lower cone is extended through it, it will form acute angle AHN with AF, because it passes above point F, where normal KF intersects the same line of longitude. Therefore, no matter how the imagined cone with vertex K is cut, as long as two lines of longitude on that cone lie within the plane of that cut, reflection will occur from the two points on the resulting conic section where it intersects the base-circle of the imagined cone. Since there is an infinite number of pairs of lines of longitude within that lower cone, then there is an infinite number of possible cuts in that cone that will yield a pair of reflection-points on the circle common to it and the cone of the mirror.

It is important to note, however, that cutting base-circle FB of the lower cone at random will not necessarily work, because it can be cut in an infinite number of ways such that none of its lines of longitude lies in the plane of the cut. In that case, of course, the points where the cutting plane intersects the lower cone's base-circle will not serve as points of reflection within that plane. The key to understanding Alhacen's point in this case, as well as in the previous case where there is only one possible point of reflection, is to understand that, no matter how the plane cuts the mirror, the point where it intersects the mirror's axis is the focus of the resulting conic section. Accordingly, if we produce a cone whose vertex is at that point and whose generatrix is normal to the surface of the mirror, the base-circle of that newly generated cone will be intersected by the conic section produced on the mirror's surface by the original planar cut at at least one point, or at most two. In the case of only one point of intersection, the line of longitude on the lower mirror passing through that point will be the conic section's major axis, and that point of intersection will be the sole point of reflection within that plane. If, on the other hand, the base-circle is intersected at two points, then those will be the only two points from which reflection can occur within that plane. By the same token, if the mirror is concave, R and R' will be the only two possible points of reflection for a center of sight stationed to the right of line of longitude AF and within the plane of conic section DRR'E.

[117]This point has already been established in note 98 above.

[118]In figure 4.5.15, p. 213, the circle represents a section of a concave spherical mirror centered on C, E the center of sight, and R the chosen point, which does not lie on diameter EC passing through center of sight E. Draw tangent XRY at point R, and draw normal CR and line of sight ER, both of which form acute angle CRE at R. Whether E lies inside or outside the sphere, ER will fall inside the sphere since it enters through the opening between tangent lines of sight TE and T'E. Now, since normal CR forms right angle CRX with tangent XRY, and since ER lies within that right angle, an angle equal to angle CRE can be cut off from right angle CRY on the other side of CR. Let that angle be CRO. Like line of sight ER, line RO lies inside

the sphere, because angle CRO is acute, and the angle formed by the circle with CR is greater than any acute rectilinear angle (Euclid, *Elements*, III, 16). Therefore, since CRO is an acute rectilinear angle, RO cannot fall outside the circle (i.e., between the circle's outside surface and tangent XRY). According to this analysis, then, in the case of a spherical concave mirror, the center of the sphere, the center of sight, the point of reflection, the object-point, and the endpoint of the diameter extending from the center of sight to the surface of the mirror through its centerpoint will all lie in the plane of reflection, which forms a circular section on the sphere. It is worth noting that, in treating OR as a line of reflection rather than as a line of incidence, Alhacen couches his analysis in terms of visual rays rather than of light-rays. Clearly, however, he does so for the sake of analytic convenience rather than from confusion over whether vision is due to light-rays or visual rays, since he is unequivocal in his rejection of the latter.

[119]Thus, as has already been established in note 100 above, if the cylinder in the top diagram of figure 4.5.5, p. 207, represents a concave cylindrical mirror facing center of sight E, then the maximal portion of it that is visible to E outside the mirror consists of the segment between tangent planes ETBS and ET'AS on the side opposite axis KL from E.

[120]This point has already been established in note 118 above. If the circle in figure 4.5.15, p. 213, is taken to represent the common section of the plane of reflection and a cylinder rather than the common section of the plane of reflection and a sphere, then everything that applies in the analysis of the sphere applies in the analysis of the cylinder.

[121]This point has already been established in note 105 above. Thus, as illustrated in the top diagram of figure 4.5.6, p. 208, there are two possible points of reflection on an elliptical section in a convex cylindrical mirror. Those points, R and Z, lie at the intersection of the minor axis of the ellipse and the circle passing through the endpoints of that axis. In this case, however, only one of those points is visible to a single center of sight, because the visible portion of the mirror is less than half. The visible portion of a concave cylindrical mirror is more than half, however, so both points of reflection can be seen from a single center of sight whether it lies inside or outside the mirror, as long as the opening in the mirror is wide enough to allow both lines of sight to reach those points.

[122]As already established in notes 110 and 111 above, this follows from the fact that in the two cases described all of the lines of sight projected to the mirror will strike its outer surface, so none of them will reach its inside surface, no matter how large an opening in it is provided for the lines of sight to pass through. Thus, as illustrated in figure 4.5.10, p. 210, if line of sight EAF extends along a line of longitude on the mirror, it is clear that no line of sight from E will reach the inner surface of the mirror. This restriction will apply no matter how wide the opening in the mirror is. Likewise, if the line of sight from E to A enters the cone, as in figure 4.5.11 p. 211, then, no matter how wide the opening in the mirror, none of the lines of sight projected from E will reach the inside surface of the mirror.

[123]This point is clear in figure 4.5.7, p. 209, where the inner portion of the cone visible through opening GAF facing center of sight E constitutes more than half the mirror. This situation applies when the line connecting the center of sight and the

cone's vertex forms an acute angle with the axis. When that angle is right, precisely half the inner surface of the mirror will be visible, as illustrated in figure 4.5.8, p. 209, where the tangents defining the visible portion of the mirror pass through the endpoints D and B of the circle's diameter. When the angle formed by line of sight EA and the axis is obtuse, less than half of the mirror's inner surface will be visible, as illustrated in figure 4.5.9, p. 210, where the maximum visible portion of the mirror is defined by concave segment AFG facing the eye.

[124]In other words, when the base forms the only opening in the mirror, and the center of sight faces that opening, the only points that can be reflected to it will be points on the mirror's inner surface within the plane of reflection or points on some object placed inside the mirror.

[125]This point is illustrated in figure 4.5.16, p. 214, where ABOP represents a cross-section of the truncated cone, E a center of sight inside the mirror, and E′ a center of sight outside the base. The forms of D and N reflect to E such that their lines of incidence pass just over edges A and B of the opening in the mirror. Hence, nothing beyond points N and D along horizontal line ND will be visible to E by reflection. Likewise, the forms of F and M reflect to E′ such that their lines of incidence pass just over edges B and A of the opening in the mirror. Hence, nothing beyond points F and M along horizontal line ND will be visible to E′ by reflection. The forms of L and G, on the other hand, reflect to E from just below edges A and B of the opening in the mirror, so nothing between G and L will be visible to E by reflection. Likewise, the forms of K and H reflect to E′ from just below edges A and B of the opening in the mirror, so nothing between K and H will be visible to E′ by reflection. Thus, the field of reflected view for E is bounded on the outside by the largest circle, and on the inside by the next smallest, whereas the field of reflected view for E′ is bounded on the outside by the next largest circle, and on the inside by the smallest. Consequently, although the internal blind spot for E′ is smaller than that for E, its overall field of vision is considerably smaller. Note that Alhacen couches his discussion in terms of the scope of egress for lines of reflection (*pateat exitus lineis reflexionis . . . liberum habebunt linee gressum . . . latior reflexis lineis datur ad egrediendum*), although it is clear that he is really referring to the scope of ingress for lines of incidence. Again, he seems to be using the language of visual rays for analytic convenience rather than out of confusion over whether vision is due to visual rays or light-rays.

[126]Note the sharp distinction between "form" and "image," the latter being applied to what is seen in a mirror, which is an illusion, whereas the form itself is a representation.

BOOK FIVE OF ALHACEN'S *DE ASPECTIBUS*

Topical Synopsis

arc on the invisible portion of the mirror below the limit-point defined in proposition 12, all possible image-locations will lie outside the sphere defining the mirror. **[2.120-124]** PROPOSITION 15: There is a point in the invisible portion of the mirror at which no image-location is possible. **[2.125-132]** PROPOSITION 16: The form of any given object-point will reflect to any given center of sight from only one point on a spherical convex mirror. **[2.133-136]** PROPOSITION 17: If two object-points are taken on the same normal, the image-location of the object-point nearer the center of curvature lies farther from the center of curvature than the image-location of the object-point farther from the center of curvature, and the point of reflection for the nearer object-point lies farther from the center of sight than the point of reflection for the farther object-point. **[2.137-140]** PROPOSITION 18: Given an object-point and a center of sight facing a convex spherical mirror and lying the same distance from the center of curvature, to find the point on the mirror's surface at which the form of the one reflects to the other. **[2.141-157]** PROPOSITION 19, LEMMA 1: If a diameter is taken in a circle, and if a point is taken on that circle's circumference, a line can be drawn from that point to the extension of the diameter beyond the circle such that its extension from the point where it intersects the circle to the point where it intersects the diameter is equal to a given line. **[2.158-166]** PROPOSITION 20, LEMMA 2: From any given point on a circle outside its diameter, a line can be drawn through the diameter to the opposite arc of the circle so that the segment on it between the diameter and that arc is equal to a given line. **[2.167-173]** PROPOSITION 21, LEMMA 3: Given a right triangle, and given a point on one of the sides forming the right angle, a line can be drawn from that point to the other side forming the right angle and intersecting the third side facing the right angle in such a way that the segment of the line between the point of intersection and the side on which the point does not lie is to the segment of the side opposite the right angle from the intersection-point to the side containing the given point as some given line is to another given line. **[2.174-185]** PROPOSITION 22, LEMMA 4: Given two points, and given a circle, to find the point on that circle such that the line tangent to the circle at that point bisects the angle formed by the lines drawn from the aforementioned two points to that tangent-point. [**2.186-192]** PROPOSITION 23, LEMMA 5: Given a circle, given a diameter in it, and

[2.339-350] PROPOSITION 36: If the object-point and the center of sight lie on the same normal but on opposite sides of the center of curvature, the form of the one can reflect to the other from any point on a circle on the mirror's surface, but if the two points lie at different distances from the center of curvature, there is a limit to how much greater the one distance can be than the other if reflection is to occur. **[2.351-373]** PROPOSITION 37: If the two points lie on different normals at equal distances from the center of curvature, the form of the one can reflect to the other from one, two, three, or four points on the mirror, depending on where the points are located with respect to the mirror's surface. **[3.374-383]** PROPOSITION 38: If the two points lie on different normals at different distances from the center of curvature, reflection can occur from only one point on the arc opposite the arc directly facing the two points. **[3.384-388]** PROPOSITION 39: Given two different normals, and given some point other than the bisection-point on the arc opposite the arc directly facing those normals, that point can serve as a reflection-point for an infinite number of point-pairs on the given normals, provided that those points are not equidistant from the center of curvature. **[3.389-391]** PROPOSITION 40: Given two points on different normals at unequal distances from the center of curvature, reflection can occur from one to the other from only one point on the arc opposite the arc directly facing those points. **[2.392-396]** PROPOSITION 41: Given a center of sight and a point of reflection on the arc facing that center of sight, reflection can occur at that point to the center of sight from an infinite number of object-points. **[2.397-405]** PROPOSITION 42: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, and if reflection occurs from some point on the arc facing those points, the reflected angle will be greater than or less than the angle adjacent to the angle formed by the normals facing the arc from which reflection occurs. **[3.406-409]** PROPOSITION 43: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, and if the form of the one reflects to the other, then the reflected angle cannot be equal to the adjacent angle. **[3.410-424]** PROPOSITION 44: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, and if there are two reflections, both reflected angles cannot be less than the adjacent angle. **[3.424-438]**

PROPOSITION 45: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, it is possible for the form of the one to reflect to the other from two points on the facing arc. **[2.439-461]** PROPOSITION 46: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, the form of the one can reflect to the other from only two points on the facing arc such that the reflected angle is greater than the adjacent angle. **[3.462-464]** PROPOSITION 47: Given an object-point and a center of sight lying on different normals at unequal distances from the center of curvature, to find the point or points of reflection at which the reflected angle is greater than the adjacent angle. **[2.465-470]** PROPOSITION 48: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, and if they lie outside the great circle on the mirror formed by the plane of reflection, the form of the one can reflect to the other from only one point on the mirror. **[2.471-486]** PROPOSITION 49: Depending on whether and how the line connecting the center of sight and the object-point cuts the great circle formed by the plane of reflection on the mirror, there can be as few as one and as many as four reflections. **[2.487]** For every point of reflection in concave spherical mirrors there will be an image. **[2.488-489]** It is extremely difficult to see all the images formed in concave spherical mirrors because the head blocks many of them. **[2.490]** Even though each image is seen separately by each of the two eyes, the object appears single in concave spherical mirrors.

[2.491] Preliminary observations. **[2.492-499]** PROPOSITION 50: If the plane of reflection cuts an elliptical section on the surface of the cylinder composing the mirror, and if the center of sight and a point of reflection are determined within that section, then, depending on where the object-point is located on some given normal, the image-location can lie behind the mirror, between the mirror and the center of sight, at the center of sight itself, or beyond the center of sight. **[2.500-503]** PROPOSITION 51: If the object-point and the center of sight lie on the axis of the cylinder from which the mirror is composed, the form of the one will reflect to the other from any point on a circle on the mirror's surface, and the resulting images will lie on a circle outside the surface of the mirror. **[2.504-519]** PROPOSITION 52: Given an object-point and a center of sight facing a concave cylindrical

[BOOK FIVE]

This book is divided into two parts. The first part constitutes the book's prologue; the second [focuses] on images.

[CHAPTER 1]

The first part

[1.1] It is clear from the fourth book that the forms of visible objects are reflected from polished bodies, and sight apprehends them in polished bodies according to reflection. And it has been shown how objects are apprehended through the reflection of [their] forms. Moreover, sight perceives the visible object at a determinate and principal location of reflection when there is no change in the spatial disposition of the visible object with respect to the eye. The form perceived in a polished body is called an image. And in this book we shall explain image-locations in polished bodies, and we shall discuss how to gain a scientific understanding of these locations, as well as how to establish that science by demonstration, and [we shall also discuss] how [those] image-locations are rationally determined.

[CHAPTER 2]

The second part, in which images are discussed

[2.1] The image-location of any point is the point where the line of reflection intersects the normal imagined [to extend] from a point on a visible object to the line tangent to the common section of the surface of the mirror and the plane of reflection, or [to the common section] of the plane that coincides with [the plane of the] mirror and the plane of reflection. We shall demonstrate [this point as follows].[1]

[2.2] Take a plane mirror, set it up horizontally, and stand a smooth, straight wooden rod upright upon the mirror. Let the mirror be large enough that the whole rod can be seen [in it], for, unless the whole rod is visible, error can arise. Let a black spot be marked on the rod. To the eye a rod the same size as this one will appear behind the mirror and directly in line with the actual rod and orthogonal to the mirror, and in the rod that is seen [in the mirror], the black spot marked [on the actual rod] will appear to lie the same distance behind the mirror's surface as [it lies] on the rod above that surface. Furthermore, if the [actual] rod is inclined to the mirror, the rod seen [in the mirror] will appear inclined with the same slant, and the spot marked on the rod seen [in the mirror] will appear to lie the same distance from the mirror's surface as the spot marked [on the actual rod]. Moreover, if some [other] rod is dropped orthogonally to the mirror from the spot marked [on the actual rod], this rod will also appear [behind and] orthogonal to the mirror from the point on the apparent rod, and [it will appear] right in line with the actual rod. The same thing happens for multiple points marked on the rod. The very same thing will happen if the mirror is raised or lowered [by tilting].

[2.3] From this, then, it is evident that the image of a visible point appears on the normal extended from the visible point to the surface of the mirror, and that in this mirror what is perpendicular to the mirror's surface is perpendicular to the common section of the mirror's surface and the [plane of] reflection.

[2.4] The same thing can be shown in the case of a right cone placed upright with its base lying flat upon the plane mirror, for [in that case] another cone will appear [behind the mirror] directly in line with it and sharing the same base, the vertices of these cones lying the same distance from the [surface of the] mirror. And it is obvious that, if a straight line is extended from vertex to vertex, it will be perpendicular to the [shared] base, and thus to the [surface of the] mirror, since that surface and the base [of the cones] coincide, so the [image of] the cone's vertex will appear on the normal dropped from that vertex to the mirror. Likewise, from some point on the cone draw a line to the corresponding point on the cone seen [behind the mirror]. That line will be orthogonal to both the [cone's] base and the mirror's surface, so the image of any point on the cone lies on the perpendicular imagined [to extend] from that point to the mirror's surface.[2]

[2.5] But whatever point on a body may face a plane mirror, a cone can be imagined with its vertex at that point, and this cone is orthogonal to its base as well as to the surface of the mirror or its continuation. And another cone opposite this one can be imagined sharing the same base and orthogonal to the mirror, and the perpendicular [extended] from vertex to vertex [of the two cones] will be orthogonal to the mirror, so the image of any point facing the mirror lies on the normal [extended] from the point to the surface of the mirror or to its extension. But it is obvious that in [all] mirrors the perception of forms will be attained only along lines of reflection, so the image of the point seen [in the mirror] lies on a line of reflection, and every such line is straight, so the image of any point [seen in a mirror] lies at the intersection-point of the normal [dropped] from that point to the surface of the mirror and the line of reflection. And it falls on plane mirrors, and the common section of the mirror's surface and the plane of reflection is unique [in coinciding] with the line tangent to the point of reflection, so it is evident that in plane mirrors the appropriate image-location is the point where the normal [dropped] from the visible point to the line tangent to the common section of the mirror's surface and the plane of reflection intersects the line of reflection.[3]

[2.6] In the case of spherical mirrors that are polished on the outer surface [i.e., convex] what we have claimed will be evident. Find [such] a mirror with a surface large enough that the [entire] form of the thin rod [just used with the plane mirror] can be seen when the rod is stood upright on it. The form of the rod [seen in the mirror] will appear directly in line with the rod [itself], and on the form of the rod [the form of] the point marked on it will appear [to lie] the same [relative] distance behind the mirror's surface as it lies [above the mirror] on the actual rod.[4] Moreover, if the rod is narrower at one end than at the other, its form will appear conical in this kind of mirror, but this involves a visual illusion that we will explain later.[5]

[2.7] Now, along with this rod form a cone that is orthogonal to a circular base that is rounded off perfectly, and let it also be applied to this mirror. A cone continuous with this one will appear [behind the mirror and] standing on the same base, but [it will appear] smaller than the [actual] cone. That the image should appear conical is clear from the fact that all the lines [extended] from the vertex of the cone seen [in the mirror] to the circle at the base appear equal, and if the cone is moved a bit on the mirror's surface away from where all of it is visible, i.e., so that some of it disappears from view, then, as long as the point of reflection on the mirror is exposed to sight, the image of the cone will be visible.[6] And if the eye is drawn away from the mirror or approaches it, provided it follows the [original] line of reflection extending between it and the point of reflection, the image of the cone will be visible, but the movement [of the eye] toward or away will be along this line in such a way that the point of reflection is marked. Extend a line from the location of the eye to the mark according to which the movement occurs.

[2.8] Now, since the image of the cone is orthogonal to the cone's base, and since that base forms one of [an infinite number of possible] circles on the sphere, the line [extended] from the vertex of the [actual] cone to the vertex of its image will be orthogonal to that circle, and it will pass through its center. It will also be orthogonal to the sphere and will pass through the sphere's center, and it will be orthogonal to the plane tangent to the sphere at the point where this line passes through [its surface]. Likewise, it will be orthogonal to the line tangent to the [great] circle on the sphere that passes through that point, and this tangent forms the common section of the plane of reflection and the plane tangent to the sphere at that point, and this line is tangent to the circle on the sphere that forms the common section of the sphere's surface and the plane of reflection. Thus, the line extended from the vertex of the cone to the vertex of its image is perpendicular to the line tangent to the common section of the plane of reflection and the mirror's surface, and that section is a circle.

[2.9] Therefore, the image of the vertex will be seen on this perpendicular, and it is evident that the image of the vertex lies on the line of reflection, so the image of the vertex will be perceived at the intersection of the line of reflection and the normal dropped from the [actual] vertex to the sphere or to the tangent to the circle forming the common section of the sphere's surface and the plane of reflection. Furthermore, from any given point facing this [sort of] mirror a cone can be imagined orthogonal to the mirror's surface or to its extension, and the vertex of this cone is the given point. The line from that point to the image of that point will lie in the plane of reflection as well as on the normal to the mirror's surface or its extension in the way discussed before, for the point seen and its image always lie in the plane of reflection, and so does the line extended from the point seen to its image.

[2.10] In the case of cylindrical mirrors [polished] on the outer surface, the things we claimed about the rod and the cone [in the previous example] are not observed, because in these mirrors what is straight does not appear straight, and there occurs a visual error whose cause we will explain later.[7] Still, for [any] single point on a visible object [seen in such a mirror] the image-location that has been described can be observed.

[2.11] For instance, having set up the apparatus [described] in the previous book, insert the panel with the [convex] cylindrical mirror attached in it so that the midline of the portion of the mirror [contained by the panel] lies in the plane of the panel. And do not let this panel pass [below] the bronze plaque, but let it stand orthogonally to it so that the [bottom endpoint of] the line along the height of the panel stands on the line that bisects the triangle of the bronze plaque. When the panel is set up on this plaque, fill [the space below] with wax and level the wax so that it lies in the same plane as the [top surface of the] plaque, and this is [done] so that the panel can be placed perfectly perpendicular to the plaque.[8]

[2.12] Then find a pointed ruler, sharpen its edge, and lay this sharpened edge along the midline of the surface of the [cylindrical] ring.[9] Let it reach along this line [to the mirror], and mark the spot where it touches the [mirror in the] panel. Then lay a needle along this line, and on its end let a small white object be attached, and do not let the needle reach all the way to the [mirror in the] panel.

[2.13] Let the eye[s] be placed in the plane of the ruler, and let one of them be closed. The image of the [small white] object [at the end of the needle] will be seen along the line extended from the point marked [on the mirror] to the point of the needle, and that line is of course perpendicular to the surface of the panel, which is tangent to the cylinder [from which the mirror is formed] along its line of longitude; and that line is perpendicular to the cylinder's line of longitude, which lies in the plane of the panel and forms the common section of the cylinder's surface and the plane of reflection. Also the line of [the cylinder's] longitude and the line perpendicular [to it] lie in the plane of reflection.

[2.14] Now, if the location of the eye is shifted, and if it moves along [the outer edge of the upper] surface of the [cylindrical] ring, then, just as before, the [small white] object [at the end of the needle], the image of the object, and the [point of the] needle will appear [to lie] on the same line. Moreover, that line is perpendicular to the midline along the length of the cylinder, and this perpendicular lies in the plane of reflection, for the [top] surface of the [cylindrical] ring intersects the cylinder along a circle that is parallel to the base of the cylinder, and the eye lies in the plane [of this circle]. And later on we shall demonstrate that, when the eye and the visible object lie in a plane parallel to the base of the cylinder, that plane constitutes the plane of reflection.[10] In this situation, however, the common section of the cylinder's surface and the plane of reflection is a circle, and the normal upon which the image and the [small white] object are seen falls orthogonally to a line tangent to the circle.[11]

[2.15] When all this is done, remove the needle and place the sharp-edged ruler along the midline [of the cylindrical ring's upper surface] so that it reaches the midline along the longitude of the panel, and attach the sharp-edged ruler firmly to the [cylindrical] ring's [upper] surface with wax. Then remove the panel with the mirror in it and take a pointed ruler and lay its sharpened edge upon the midline along the length of the panel, and along the sharpened edge draw a line upon the mirror with ink. Then take a moderate-sized wax triangle, one of whose sides is equal to the height of the panel containing the mirror, let it be moderately thick, and let the surface of this triangle be as flat as possible. Then attach the wax triangle firmly to the panel containing the cylinder below the panel's base, and place the side of the triangle that is equal to the height of the panel upon the side of the panel's base. When that is done, the length of this triangle at the base of the [panel containing the] cylinder will be equal to the [height of the] panel, and in order to make the triangle's surface as flat as the surface of the panel, place the triangle between a panel and a flat surface and compress it until it is quite flat. Place a sharp-edged ruler on the surface of this triangle, and cut off the bottom edge of this triangle along the edge of the ruler, and this edge will form a straight line. This line will form the base for the panel containing the mirror.

[2.16] Then place the panel on the surface of the [bronze] plaque in the instrument, and place the edge of its base, which lies along the length formed by the side of the wax triangle, on the line of longitude of the bronze [plaque], as was done before. The plane of the panel containing the mirror will be perpendicular to the bronze plaque, and this plane cuts the bronze plaque along the line of longitude of the bronze plaque, and this plane is tangent to the surface of the mirror along the line on the mirror's surface. And this plane is the surface of the panel containing the mirror, and the angle [along the edge] of the pointed ruler will be attached along the midline of the [top] surface of the ring, and the mirror's surface will be tilted downward to that surface on the side of the triangle's vertex, since one side of the panel has been raised according to the breadth of the triangle, while the other side [of the triangle] beyond the vertex of the triangle is the plane of the bronze [plaque], and the line that lies along the midline of the mirror will be inclined.

[2.17] And when the side of the wax triangle is [positioned] along the line of longitude of the bronze [plaque], the panel that contains the mirror will move along the line of longitude of the bronze plaque, and the [base] side of the [wax] triangle will move with it, if it is [positioned] on the [bronze plaque's] line of longitude. And let it move back and forth until the point of the sharp-edged ruler meets some point on the [mid]line on the surface of the mirror and the mirror fits snugly against the sharp-edged ruler, and erase the line drawn in ink on the mirror. Mark the point on the surface of the mirror that is directly under the end of the sharp-edged ruler. And remove the sharp-edged ruler, and apply the needle, and let the needle lie along the midline of the [top] surface of the ring, and let it be attached firmly with wax. And the imaginary line from the point of the needle to the point marked on the surface of the mirror will be perpendicular to the plane of the panel which is tangent to the mirror's surface at the marked point, and it will lie perpendicular to any line extended through that point in the plane tangent to the mirror. Thus, it will be perpendicular to the straight line tangent to the common section of the top surface of the ring and the [mirror's] surface.

[2.18] Now, place the eye in the plane of the ring at its apex [on the midline along its top surface], and it will look into the mirror until it perceives the form of the small [white] object on the needle, and then it will perceive that object, the point marked on the mirror, and the image of that object [all in a line]. And the line passing through the small [white] object and the point marked on the [mirror's] surface is perpendicular to the plane tangent to the mirror's surface at the point marked on it. And this surface on the ring is among the planes of reflection, and the small [white] object and the centers of sight lie in that plane, and the point of reflection is in that plane, and we will prove this later on. And in this case the image of the small [white] object will lie upon a straight line extended from the small object [itself] directly to the plane tangent to the mirror's surface, and since this line is perpendicular to the straight line tangent to the common section of the surface of the mirror and the plane of reflection, which is the [top] surface of the ring, and since the plane of reflection is among the oblique planes that cut the cylinder between the lines of longitude on the cylinder and the circles parallel to its bases, then, because the panel and the mirror in it are at a slant, the common section of this plane [of reflection] and the surface of the mirror is among the cylindrical sections [i.e., an ellipse].[12] And we will explain the image-location in this same way if the orientation of the panel containing the mirror is changed and slanted on its surface at a smaller or greater obliquity.

[2.19] It is therefore evident from these things that the image is perceived where the normal extended from the visible point to the surface of the mirror intersects the line of reflection, and this is the predicted location. If lines are drawn from the visible point to the mirror's surface, the one that is perpendicular is shorter than any of the others, for any of the others first intersects the common section of the plane tangent to the mirror's surface to which the perpendicular falls and the plane of reflection that reaches the mirror, and any line extending in this plane from the visible point to this common section is longer than the perpendicular, since it subtends a greater angle, so what was proposed [is demonstrated].[13]

[2.20] The same procedure can be applied in the case of the conical mirror [polished] on the outer surface, and the same thing will be shown whether the images of objects are seen in [planes formed by] conic sections, or whether they are seen in [planes formed by] cuts made along lines of longitude.[14]

[2.21] In concave spherical mirrors some images are perceived behind the mirror, some on the mirror's surface, and some in front of the mirror, and some of these images are perceived according to reality, whereas some are perceived in ways that do not accord with reality.[15]

[2.22] That all of those images seen according to reality appear at the point where the normal [dropped from the object] intersects the line of reflection will be demonstrated as follows. Fashion a cone that is orthogonal to its base [i.e., a right cone], and let the diameter of the base be smaller than the radius of the sphere [from which the concave mirror is formed], and let the line along the length of the cone be longer than that radius. Now, [from the vertex] toward the base, cut off a length [along the cone's line of longitude] equal to that radius, inscribe a circular section [parallel to the cone's base at that point] and cut the cone according to this circle. Then, in the middle of the mirror draw a circle the same size as the base of the cone that is left [after the cut], fit the cone to this circle, and attach it firmly with wax.

[2.23] Place the center of sight where the image of the cone can be perceived, and direct light upon it so that the perception may be clearer.[16] You will not see a cone conjoined with this one, but you will perceive [the image of] this [cone] extending behind the mirror, so a certain cone directly in line [with it] will appear with its base behind the mirror along with the wax section of its cone.[17] And if a line of longitude is marked with ink on this cone, this line will appear to extend along the surface of the cone that appears [in the mirror], and since the vertex of the cone lies at the center of the sphere, the line extending from the vertex along the length of the cone will be perpendicular to the tangent to any circle on the sphere that passes through the endpoint of the line.

[2.24] Hence, any line of longitude on the cone that appears [in the mirror] is perpendicular to the line tangent to the common section of the plane of reflection and the surface of the sphere, that line being the common section, and it is a [great] circle. And any point on the cone [lying] on this perpendicular is seen, and every [such] perpendicular lies in the plane of reflection, because the visible point and its image lie on the normal as well as in this plane of reflection. But every image is perceived along the line of reflection, so the image of any point on the cone will lie at the point of intersection of the normal and the line of reflection.

[2.25] Furthemore, points whose images are perceived in front of the mirror—i.e., ones that lie between the center of sight and the mirror—[are perceived this way] since the line extended from any of them to the center of the mirror will cut the side of the line lying between the visible object and the mirror. In order to observe this [fact], remove the cone from the center of the mirror and replace it at the side. The vertex will be the center of the mirror, and let the distance between the center of sight and the mirror be greater than the sphere's radius. Then take a thin white rod, and stand it on the mirror so that the center of the mirror lies directly between the top of the rod and the center of sight, and direct your line of sight to the point on the mirror from which the line extended to the vertex of the cone lies between the top of the rod and the center of sight. Then, look into the mirror until neither the top of the rod nor the rod itself appears, and the form of the top of the rod will appear above the mirror and nearer to the eye than the vertex of the cone. And the vertex of the cone, the top of the rod, and the image of the top of the rod will lie on the same straight line, and this line is perpendicular to the line tangent to the common section of the mirror's surface and the plane of reflection. For the plane of reflection passes through the center [of sight] and the visible point, and the line passing through these two points lies in the plane of reflection, and the common section is a circle. And this line will form a diameter in this circle, because the center of that circle is the center of the sphere, so this line will be perpendicular to the line tangent to the circle at the endpoint of this line, and this line passes through the visible point as well as through its image. And so, any point [whose image is seen] in front of the mirror is perceived on the same line as the center [of the sphere] and the image, and any point is seen on the line of reflection, so [it is seen] at the point of intersection of the normal and the line of reflection.

[2.26] And the things that are perceived in these mirrors as they exist in reality are those whose images appear behind the mirror or in front of the mirror, and beyond these [types of images] there are none that sight may perceive as they exist in reality in this [sort of] mirror, for these latter types of objects do not permit their true images to appear. Images that appear on the surface of this [type of] mirror belong to the last category, and we will explain this when the discussion turns to visual errors. Therefore, any point that is perceived as it actually exists in this [sort of] mirror appears at the intersection of the normal and the line of reflection, and this normal passes from the visible point to the center of the sphere and falls orthogonally on the tangent to the common section [of the mirror and the plane of reflection].[18]

[2.27] In the case of concave cylindrical mirrors the image varies, for its location will sometimes be on the surface of the mirror, sometimes behind it, and sometimes in front of it. And in all these cases [the image] is sometimes perceived as it actually exists, sometimes not.

[2.28] If you want to observe the image-location in these cases, do what you did [before] in the case of cylindrical mirrors [polished on the] outer surface. Insert the panel with the concave cylindrical mirror on it into the ring, as it was inserted earlier, and likewise [position] the needle and the small [white] object at the end of the needle. Place your eye directly in front of the center of the circle [passing through the middle of the mirror] and at the middle of the [top] surface of the ring, raise the eye a little bit above the ring's [top] surface, and let it look until it sees the image of the object and perceives the form of the object, the object itself, and the point marked on the mirror along the same line that is perpendicular to the mirror's surface—and [determine] this according to sense-induction.[19] The image will lie behind the mirror, and the reflection will occur from a point on the line [of longitude] that lies in the middle of the mirror.

[2.29] Then place the center of sight in the plane of the [top surface of the] ring, but outside the midline, until it sees the image of the small [white] object [on the needle]. You will see it in front of the mirror, and you will see the object, its image, and the point marked on the mirror along one straight line perpendicular to the straight line tangent to the circle that is parallel to the base of the mirror and [lying] upon the point marked on the mirror's surface. And the plane of this circle forms the plane of reflection in this case, and it is the top surface of the ring, and the point of reflection is a point on that circle.[20]

[2.30] Afterwards, with your other hand place another needle with a small object attached to its end, and lay it on the [top] surface [of the ring] and the axis in such a way that the object and the point marked [on the mirror] lie on the same line according to sense-induction. Let the eye lie in the plane of the [top surface of the] ring between its endpoint and the middle [of the ring's top surface]. It will see the image of the object, and it will see this image as well as its [generating] object along with the point marked on the surface of the mirror on the same straight line.[21]

[2.31] If, however, the straight line [of longitude in the middle of the panel] is slanted according to the small triangle we fashioned—and the center of sight should be on the midline of the ring—you will see the image in front of the mirror, but on the same straight line as the body and the point marked [on the mirror's surface]. And this reflection will occur from one of the cylindric sections [i.e., ellipses], because the mirror is inclined, and we know that an image is perceived only along a line of reflection. Therefore, it is clear that the image-location lies where the normal intersects the aforementioned line of reflection, since the [image] is perceived properly, and even though the image may not be perceived clearly, these images will still belong to the category of those that are properly seen.

[2.32] In the same way you can observe [the image] in a concave conical mirror at the intersection of the normal with the lines of reflection. It is therefore evident that in all mirrors images are perceived at the aforementioned location, which is likewise referred to as the "image."

[2.33] We will now explain why visible objects are perceived through reflection where the image is located and why the image lies on the normal [dropped] from the visible object to the surface of the mirror. When the visual faculty grasps a form by reflection, it grasps it immediately without an accurate determination, and it grasps its distance through estimation. It may perceive this distance accurately through a close and careful scrutiny, or it may not. And we have explained this [process] in the second book, where it was claimed that the visual faculty grasps distance through a deduction based on the size of the object and the particular angle under which the size is perceived. The perception of a visible object whose location is unknown clearly occurs in this way. Also, objects [whose locations are] known are perceived in this way, for they are compared to known objects and known sizes or distances.[22] When sight perceives some object through reflection, it perceives the distance of the image solely by estimation; then, after a close scrutiny, it grasps the distance and verifies [it] deductively on the basis of the size of the visible object and the angle of the cone according to which the form is reflected to the eye.

[2.34] Thus, if the visible object is among things familiar [to the perceiver], the visual faculty grasps its distance according to an already-known distance [occupied by such a body] subtending an angle equal to this one and [lying at] a distance similar to this one. Likewise, when the visible object is unknown, its magnitude will be compared to another magnitude belonging to familiar objects, and the distance of this image is grasped according to a deduction based on the size of the angle that the image subtends at the center of sight at the time of reflection. Moreover, the place where the form of the visible object is perceived through reflection [is such that] the form reaching the eye directly from it at a [given visual] angle will arrive according to the very same cone according to which the form is reflected to the eye, and the same cone will encompass the entire form as it does at the image-location. Hence, when it grasps the visible object through reflection, the visual faculty grasps it where the image lies, for, because the form perceived through reflection at the image-location is identical to the form as perceived directly, it is contained by that [same] cone, and this is why it is perceived at the image-location.[23]

[2.35] We will now explain why the image will be perceived on the normal. We know that a point that is perceptible to sight exists not intellectually but sensibly, and its form is sensible. I say therefore that in the case of plane mirrors, since the image does not appear on the surface of the mirror but behind it, it is more appropriate and reasonable for it to appear upon rather than outside the normal. For, since it has been defined according to a spot on the normal, the image's distance from the point of reflection on the mirror—which is a segment of the line of reflection extended from the location of the image to the point of reflection—is equal to the distance of the point that is seen from the point of reflection. Since the surface of the mirror is orthogonal to the normal, then the line extended [on that surface] from the point of reflection to the normal forms a side common to the two triangles, and it will bisect the normal. Hence, two sides of one of the triangles will be equal to two sides of the other, and an angle [of one will be equal] to an angle [of the other], for both are right, so the base [of the one will be equal] to the base of the other.[24]

[2.36] Thus, if the image appears on the normal, it will lie the same distance from the point [of reflection] and the eye as the object from which it originates, and the image will occupy an equivalent location with respect to the point of reflection as the point that is seen does with respect to the same point [of reflection], and the location [of both the image-point and object-point] will be equivalent with respect to the center of sight, so in this situation both the point seen and its image will appear according to reality. If, however, the image lay beyond the normal, then, since it must lie on the line of reflection, it will lie either beyond or in front of the normal with respect to the eye. If the image lies beyond, then it will lie farther from the point of reflection and from the eye than the point seen, so it will subtend a smaller angle in the eye than the point [itself]. Moreover, it occupies a smaller area on the eye, so, when it should be equal, it will appear smaller than it[s generating object]. On the other hand, if it lies in front of the normal, it will appear larger [than its generating object-point] because it lies nearer.[25]

[2.37] In the case of a convex spherical mirror, the image appears on the normal, for the image that appears is either of the center of the eye or of some other point.[26] If the image is of the center [of the eye], then I say that it is more appropriate for the image of the center of the eye to appear upon the normal dropped from the [center of the] eye to the center of the sphere than [to appear] anywhere else, for if the form propagates directly along this normal to the center of the sphere, then it will always maintain a uniform disposition with respect to the eye, and so the form that faces any point on the sphere and moves orthogonally toward the center will maintain a uniform disposition with respect to the eye. And the disposition of the form will be the same on one normal as it is on another, for the center of the sphere has the same situation with respect to every point on the sphere, and all normals of this sort are identically situated.

[2.38] On the other hand, if the image propagates to some point on the sphere outside the normal, its disposition with respect to the eye will change, for it will have a different situation outside the normal than it does on the normal itself, and the normal moves outside the mirror rather than inside it. But if it appears outside the normal, it will not maintain a [uniform] disposition, yet it was [agreed earlier that it is] more fitting for the image to maintain [a uniform] disposition than for it to change [its situation] if the visual faculty is to perceive the visible object with certainty. Accordingly, the image of the [eye's] center appears on the normal, yet we cannot [on that account] determine a specific point for that image on the normal, because there is nothing to be found in any point on the normal to give it precedence over any other point in determining where the image should appear on it. But we know that at whatever point on this normal the image [of the eye's center] appears, it invariably appears continuous with the [whole] eye appearing [in the mirror], and it always maintains [the same] location and situation with respect to the entire form [of the eye] that is seen.

[2.39] The image of any point on the eye other than the center reaches [the mirror] obliquely, so it does not maintain a uniform disposition with respect to the object-point that is seen [i.e., the center of sight], and the normal dropped from the point seen to the mirror['s surface] falls to the center of the sphere, and it is upon that normal that the image maintains a uniform situation. There is thus no point where the perceived image will maintain a uniform disposition except upon that normal, and since it must be perceived along the line of reflection, it will be perceived at the intersection of this line with that normal.[27] We have thus explained the reason for this phenomenon, but the [essential] character of natural objects is related to the [essential] character of their [underlying] principles, and the [underlying] principles of natural objects are hidden.[28]

[2.40] The same way of proving this will obtain in the case of a concave spherical mirror, as well as in the case of a concave or convex conical [mirror]. In general, for any kind of mirror, image-location will be on the normal, for there is nowhere outside the normal where the form will maintain a uniform and equivalent disposition.

[2.41] With these points clarified, it remains for us to rationally define the image-location for every kind of mirror.

[2.42] We say that, for each and every point that is perceived by sight in a plane mirror, when it lies outside the normal that falls from the center of sight to the surface of the plane mirror, the line along which the form of that point is reflected to the center of sight will intersect the normal dropped to the mirror's surface from that point. Moreover, this intersection-point, which constitutes the image-location [of the object-point], lies behind the mirror['s surface], and this [image] will lie the same distance from the mirror's surface as the [object-]point viewed [in the mirror]. And the visual faculty perceives the image of the point viewed [in the mirror] only in that location, and only one image of any point perceived by sight in such a mirror will be seen.

[2.43] Moreover, for any point that sight perceives in a convex [spherical] mirror, when its form radiates outside the normal dropped from the center of sight to the center of the mirror, the line along which the image is reflected to the eye will intersect the line extended from that point to the center of the mirror, that line being the normal dropped from that point, and it is orthogonal to the line tangent to the common section of the plane of reflection and the surface of the mirror. Moreover, where the point of intersection (which constitutes the image-location) lies with respect to the mirror's surface will depend upon where the center of sight lies with respect to the mirror's surface. The intersection-point may lie beyond the mirror, on the mirror's surface, or behind the mirror['s surface]. But the visual faculty perceives all [these] images behind the mirror, even though their locations may vary, and it does not perceive the location of any image on the mirror's surface through deduction. Also, no matter what point is perceived by sight in this sort of mirror, it yields only one image.[29]

[2.44] In the case of a convex cylindrical mirror (and the same holds for a convex conical mirror), if any given point that is perceived by sight lies outside the normal dropped from the center of sight orthogonal to the plane tangent to the mirror's surface, the line along which the form is reflected to the center of sight intersects the normal dropped directly from that point to the line tangent to the common section of the plane of reflection and the surface of the mirror. And some of the images in these mirrors lie beyond the mirror's surface, some on the surface itself, and some inside it.[30] But the visual faculty grasps all of the images in these mirrors behind the mirror's surface, and whatever point the visual faculty may perceive in these mirrors produces only one image.

[2.45] In the case of a concave spherical mirror, some of the lines along which the forms of visible points are reflected intersect the normals extended straight from those points to the lines tangent to the common sections of the surface of the mirror and the plane of reflection, and some are parallel to these normals. For those that intersect the normals, the point of intersection, which constitutes the image-location, sometimes lies behind the mirror outside the point of intersection—and this perception is unclear.[31] Moreover, of the visible objects that sight grasps in this [sort of mirror], some yield one image only, some two, some three, and some four, but there can be no more [than four].

[2.46] In a concave conical or a concave cylindrical mirror, some of the lines along which forms are reflected intersect the normals extended straight from the visible points to the lines tangent to the common sections [of the mirror's surface and the plane of reflection], and some are parallel to the normals. In the case of those that intersect the normals, the intersection for some of the points lies behind the mirror, and the intersection for others lies in front of the mirror. Some of those that lie in front of the mirror will lie between the mirror and the eye, some at the center of sight [itself], and some beyond the center of sight. The perception by sight of visible objects in this [sort of mirror] sometimes occurs at the [proper] image-location, which is the point of intersection, and sometimes outside the point of intersection. And of those objects that are perceived [in this sort of mirror], one yields a single image only, another two images, another three, and another four, but there is no way that a thing can yield more than four. And we will demonstrate all these points theorematically.

[Plane Mirrors]

[2.47] **[PROPOSITION 1]** Let A [figure 5.2.1, p. 220] be the object-point, B the center of sight, and DGH a plane mirror. Let G be the point of reflection and DGH the common section of the plane of reflection and the mirror's surface. From point G draw EG perpendicular to that common section. From point A draw AH perpendicular to the mirror's surface, and continue it behind the mirror. Let AG be the line along which the form [of point A] reaches the mirror and BG the line along which it is reflected to the center of sight. Accordingly, BG, EG, and AG lie in the plane of reflection, and since [normal] EG is parallel to [normal] AH, while BG is oblique to EG, BG will intersect AH. Let it then intersect at point Z. I say that ZH = HA.

[2.48] Since angle BGD = angle AGH,[32] [since angle AGH = angle HGZ, which = vertical angle BGD], since [right] angle AHG = [right] angle ZHG, and since side HG is common, then [by Euclid, I.26] triangle [AHG] = triangle [ZHG]. Hence, ZH = AH.

[2.49] Now, if we wish to determine where the point of reflection lies according to normal [AHZ], let a segment along the normal be cut off below the mirror equal to the segment on it [from object-point A] to the mirror, i.e., so that ZH = AH. Then extend line BGZ from the center of sight to point Z. I say that G is the point of reflection.

[2.50] Since AH and HG = HG and HZ [leaving AH = HZ and HG common], and since [right] angle [AHG] = [right] angle [ZHG], then [by Euclid, I.4] triangle [AHG] = triangle [ZHG]. Therefore, angle ZGH = angle HGA. But [vertical angle] ZGH = [vertical] angle DGB. It follows, therefore, that angle BGE [in right angle EGD] = angle EGA [in right angle EGH], and so G is the point of reflection. And what was proposed [has] thus [been demonstrated].

[2.51] **[PROPOSITION 2]** Furthermore, let A [figure 5.2.2, p. 220] be the center of sight, let AG be the normal to the plane mirror, and let D [be the point] on the eye's surface [where it] intersects this normal. I say that there is no point other than D on this normal that may reflect from this mirror to the eye.

[2.52] For if [some such] point may be taken on this normal beyond the eye, let it be H. Its form will not reach the mirror along the normal because of the interference of the opaque body [of the eyeball], and so its form is not reflected along the normal.

[2.53] But if it is claimed that [its form] can be reflected from some other point on the mirror, let that [point] be B. Its form will propagate to point B along line HB, and it is reflected along line BA. Let angle HBA be bisected by line TB [so that HBT is the angle of incidence and TBA the angle of reflection]. Therefore, TB will be normal to the mirror's surface. But TG is normal to that same [surface], so two perpendiculars will have been dropped from the same point [T] to the mirror's surface, which is impossible.

[2.54] The demonstration that the form of point D cannot be reflected from any point other than G on the mirror will be the same, so that form is reflected only along the normal. If, however, it is claimed that the form of any point selected between G and D on this normal is conveyed to the center of sight by reflection, the disproof follows from the fact that the body [of the selected point] will be either opaque or transparent.

[2.55] If it is opaque, then the point's form will proceed to the mirror along the normal and will return along the same line to the center of sight, but because of the [point's] opacity that form cannot pass through to reach the center of sight.

[2.56] On the other hand, if that point is transparent, its form will mingle with it and fuse with it upon its return from the mirror along the normal, and it will not be reflected to the center of sight.

[2.57] Moreover, by the preceding method, it can be proven that the form of any point selected on this normal between G and D cannot be reflected to the center of sight from any point on the mirror other [than G]. By the same token, the form of [any] point selected between A and D is reflected to the center of sight neither along the normal nor along any other line, for the points lying between the center of the eye and its surface are absolutely transparent, so their form is neither conveyed back nor reflected in such a way as to be sensed. And since no point other than D selected on the surface of the eye faces the mirror at a right angle, any such point will be seen upon the normal dropped from it to the mirror, and its image [will lie] the same distance behind the mirror's surface as the point itself [lies above it]. And since D appears continuous with the other points on the surface of the eye, and since its image appears continuous with the other images [of those points], the image of D will appear [to lie] the same distance from the mirror's surface as D lies from it.[33]

[2.58] Hence, it is evident that the image of any point seen in the mirror will appear on the normal, and the distance of the image from the mirror's surface and [the distance] of the visible body [from the mirror's surface] is the same.

[2.59] **[PROPOSITION 3]** Furthermore, the form of a point viewed in a plane mirror is reflected to the same center of sight from only one point. Let A [figure 5.2.3, p. 220] be the center of sight, B the point viewed, and ZH the mirror. If, then, it is claimed that the form of B is reflected to the center of sight from two points on the mirror, let one of them be point D, the other E. Draw line AB from the point viewed to the center of sight; this line will either be perpendicular to the mirror or not.

[2.60] If it is not perpendicular, we know that the line lies in the plane of reflection [which is] orthogonal to the surface of the mirror, and it will lie in only one such plane. For if [it lies] in two, it will be common to two orthogonal planes, and if a point is chosen on it and a line is extended from it in either plane to the common section of that plane and the surface of the mirror, this line will certainly be orthogonal to the mirror. Likewise, from that [same] point extend a line in the other plane to the common section of that plane and the surface of the mirror, and this line will be orthogonal to the mirror, so from the same point two perpendiculars will have been dropped [to the same plane].[34]

[2.61] Thus, since BA lies in only one orthogonal plane, and since the three points A, B, and E lie in the same orthogonal plane, AE and EB will also lie in the orthogonal plane that contains AB, and so will EB and DB. Therefore, EA and EB [will lie] in the same plane as DA and DB. But angle AEH = angle DEB [by supposition], and angle HEA > angle ADE, because it is an exterior [angle of triangle AED], so [angle] BED > [angle] ADE. But [angle] BDZ = [angle] ADE [by supposition], while [exterior angle] BDZ [of triangle BED] > [interior angle] BED, so [angle] ADE [which = angle BDZ, by supposition] > [angle] BED, but it was [just] claimed that it is smaller. It follows, therefore, that reflection may occur from only one point.

[2.62] On the other hand, if AB is perpendicular to the mirror, then it has already been claimed [in proposition 2] that there is only one point on the line dropped orthogonally from the center of sight to the mirror whose form is reflected from the mirror to the center of sight. And it has already been demonstrated that the image of that point reflects from only one point [on the mirror], so what was proposed [has been demonstrated].

[2.63] **[PROPOSITION 4]** Furthermore, when any point is viewed by both eyes, one and the same image appears to both, and it does so in the aforementioned [image]-location. Now, it is obvious that the form of the point does not reflect to both eyes from the same point on the mirror. For if the line of reflection proceeding to one eye were to form the same angle with the normal erected to the mirror's surface [at the point of reflection] as the line along which the form reaches the mirror, then another line could not be chosen in the same plane to form with the normal an angle equal to this one.[35] Hence, from this point no [form incident along the same] line will be reflected to the other eye. Reflection must therefore occur from different points on the mirror.

[2.64] Let those points be T and Z [figure 5.2.4, p. 221]. Let QE be the plane mirror, A the point viewed, B and G the two centers of sight, and AD the normal [dropped from object-point A to the mirror]. It is evident, then, that BT, AT, [segment DT of] ET, and AD lie in the same plane orthogonal to the surface of the mirror. Likewise, AD, AZ, and GZ lie in the same orthogonal plane, and DT is the common section of plane ADTB [and the mirror's surface], while DZ is the common section of plane ADZG [and the mirror's surface]. If BT and GZ lie in the same orthogonal plane, TDZ will form a single line, and normal AD will lie between the two aforementioned perpendiculars [dropped] to the mirror's surface from the two centers of sight [i.e., the perpendiculars dropped from B and G to the plane of the mirror containing line QDE], or it will lie outside them [figure 5.2.4a, p. 221].

[2.65] Whichever the case, line BT of reflection will cut from normal AD a segment behind the mirror equal to segment AD [above it, as demonstrated in proposition 1 above]. Likewise, GZ will cut from the same normal a segment behind the mirror equal to that segment [AD above the mirror]. Those two lines of reflection will therefore intersect the normal at the same point [H] behind the mirror. Hence, the image of point A will be perceived by both eyes at the same point on the normal, so there will only be one image, and it will be the same [for both eyes], and it will lie at the same place as it would if it were viewed by only one eye.

[2.66] If, however, points T and Z do not lie in the same plane orthogonal to the mirror [figure 5.2.4b, p. 221], precisely the same proof will apply, since each line of reflection cuts a segment on the normal [below the mirror] equal to the segment above it, and the intersection of the lines of reflection with the normal will be at the same point, so what has been proposed [is demonstrated].

[2.67] On the other hand, if point A lies on the normal dropped to the surface of the mirror from only one center of sight, it is perceived by that same center of sight behind the mirror at a point on the normal that lies as far from the mirror's surface [below it] as A lies from it, so the form of A appears continuous with the forms of other points that appear in proximate locations. Also, the image of A is perceived by the other eye at the same point on the normal, and so only one image of point A is seen by both eyes, and [it appears] at the same point on the same normal, which is what was proposed.[36]

[Convex Spherical Mirrors]

[2.68] **[PROPOSITION 5]** What we claim will be clear in the case of spherical convex mirrors. Let A [figure 5.2.5, p. 222] be the point seen, B the center of sight, and G the point of reflection. It is clear that BG and AG lie in a plane [of reflection] orthogonal to the plane tangent to the sphere at point G. Let ZGQ be the [great] circle that forms the common section of the plane of reflection and the surface of the sphere [from which the mirror is formed]. Let PGE be the line tangent to this circle at the point of reflection. Let HG be the normal to this line [at the point of reflection]. It is evident that HG should reach the center of the circle. But if not, then, since the line extended from the center of the sphere to point G is also perpendicular to line PGE, two lines perpendicular to one [and the same] line will have been drawn from the same point on the same side [of that line].[37]

[2.69] Now, let N be the center of the sphere, and let a line, i.e., AN, be extended from the point viewed to the center of the sphere, that line being normal to the plane tangent to the sphere at the point on the sphere through which it passes. And since it is manifest that [line of reflection] BG intersects the sphere, because it lies between HG and GP, which form a right angle, it will intersect line AN. And since normal HG lies in the plane of reflection, the center of the sphere will lie in the same plane, and so [normal] AN [will lie] in the same plane as [normal] HG.

[2.70] Accordingly, let D be the intersection of [line of reflection] BG with [normal] AN. It is clear that D will be the image-location, and this analysis must be understood [to obtain] when the line extended from the point seen to the center of sight is not perpendicular to the mirror.

[2.71] **[PROPOSITION 6]** Now, let line PGE [figure 5.2.6, p. 222] intersect line AN. Let the point of intersection be E, and this point is designated as the endpoint of tangency. I say that in this case the line extending from the center of the sphere to the image-location is longer than the line extending from the image-location to the point of reflection: that is, DN > DG.

[2.72] For angle [of reflection] BGH = angle [of incidence] HGA [by previous supposition], but angle BGH = [vertical] angle NGD. Therefore, angle [of incidence] HGA = that same [angle NGD], and EG is perpendicular to HGN, so angle AGE = angle EGD [because angle AGE = angle BGP = vertical angle EGD]. Therefore, AG:DG = AE:ED [by Euclid, VI.3].

[2.73] Let a line [AH] be drawn from point A parallel to DG, and let it intersect line HN at point H. Accordingly, angle NGD = [alternate] angle GHA. But angle NGD = angle AGH [by previous conclusions]. Therefore, angle GHA = that same angle [AGH], so the two sides AG and HA [of triangle AGH] are equal. Therefore, AH:DG = AG:DG. But AH:DG = AN:DN [by Euclid, VI.4, because triangles AHN and DGN are similar], so AN:DN = AG:DG. Therefore, AN:AG = DN:DG [by Euclid, V.16]. But AN > AG, because it subtends more than a right angle in triangle ANG [by Euclid, I.19 and 32]. Accordingly, DN > DG, which is what was proposed.

[2.74] **[PROPOSITION 7]** I say, further, that the line dropped orthogonally to the sphere from endpoint of tangency E, i.e., the segment [EF] of line EN [between E and the surface of the sphere], is shorter than the radius [of the sphere].

[2.75] Let F [figure 5.2.7, p. 223] be the point where [normal] AN [passing through endpoint of tangency E] intersects the sphere's surface. I say that EF < NF.

[2.76] For, as has [already] been claimed [in the previous proposition], AG:DG = AE:ED, but AN:DN = AG:GD [by previous conclusions]. Therefore, AN:DN = AE:ED. Therefore, AN:AE = DN:DE [by Euclid, V.16]. But AN > AE, so DN > DE, and so DN > EF, from which it follows that NF > EF, which is what was proposed.[38]

[2.77] **[PROPOSITION 8]** Now, let G [figure 5.2.8, p. 223] be the center of sight, D the center of the sphere, and DZG the normal [dropped] from the center of sight to the sphere. I say that the form of no point other than the point on the surface of the eye [through which normal GD passes] is reflected along this normal.

[2.78] For the forms of the points selected beyond the center of the eye are not reflected for the reason given above [in proposition 2]. And the same holds for the points lying between the surface of the eye and the mirror. I say, as well, that no point on this normal is reflected from any other point on the mirror.

[2.79] For if it is claimed that [reflection does occur] from another point, let that point be A. Line GA will be the line of reflection, and from that point [on the normal, which is assumed to reflect its form to G, i.e., X] we will imagine a line to A, that line [AX] being the one along which the form proceeds [to the mirror]. These two lines form an angle at A, and diameter DA will necessarily divide that angle, since it is normal to point A, and the normal bisects the angle produced by the form's line of incidence and its line of reflection. And so diameter DA will intersect normal GD between the selected point and G [i.e., at point E]. And so the two straight lines [GD and DE, which consists of DA and its rectilinear continuation AE] will intersect at two points [D and E] and will form a plane.

[2.80] It therefore follows [from the impossibility of such a double intersection] that the form of only the one point that lies on the surface of the eye may be reflected from the mirror along the normal, and it must appear at its original image-location according to its continuity with other [surrounding] points.[39]

[2.81] **[PROPOSITION 9]** Furthermore, let GA and GB [figure 5.2.9, p. 224] be lines drawn tangent to the sphere from the center of sight, and mark off the [great] circle upon which the plane formed by these lines cuts the sphere. AB will be the visible portion of this circle. I say, therefore, that some of the image-locations perceived according to reflection from this portion lie inside the mirror, some on the mirror's surface, and some outside the mirror. Each one of these cases must be accounted for.

[CASE 1]

[2.82] From point G [figure 5.2.9] draw a line that cuts the circle, and let the segment of it that forms the chord on the [intersected] arc of the circle be equal to the circle's radius. Let that [cutting] line be GHK, and let HK be the [segment on it forming the] chord that is equal to the radius. Then, from point H draw normal DHM. I say that the [image-]location of a form reflected from point H will lie inside the sphere.

[2.83] From point H draw a line OH that forms with MH an angle [OHM] equal to angle MHG. Points [incident to the mirror] on this line, and on no other, will be reflected from point H to the center of sight [G]. Accordingly, choose some point O on it, and draw line OD [i.e., the normal dropped from the object-point] from that point to the center of the sphere. OD will be perpendicular to the plane tangent to the sphere at the point where OD passes through it. But, by construction, angle OHM = angle MHG, so, by the same token, it is equal to [angle MHG's] alternate angle KHD. But [angle] KHD = [angle] KDH, because they are subtended by equal sides [of equilateral triangle KDH, i.e., radii DH and DK].

[2.84] Accordingly, angle OHM = angle KDM, so lines KD and OH are parallel. Therefore, if they are extended indefinitely, they will never intersect.[40] But line OD will intersect the line [HK] joining KD and OH, and so, no matter what point is chosen on line OH, the line extended from that point to point D [i.e., the normal] will intersect the line of reflection inside the sphere, and that line will be perpendicular to the sphere, as represented by OD [so it will be the normal on which the image must lie]. Hence, the image of any point on line OH will appear inside the sphere [where the line of reflection GH intersects normal OD].

[CASE 2]

[2.85] Now, the arc on the circle lying between point H and the point where the normal dropped from the center of sight passes through [the circle] is HZ. I say that, from whatever point on this arc reflection occurs, the image-location will lie inside the sphere.

[2.86] [Here is] the proof. Let I [figure 5.2.9a, p. 224] be the point selected, draw line GIS from the center of sight to intersect the circle at that point, and draw normal DIT from that point. Make line PI form an angle [PIT] with [normal] IT that is equal to angle TIG. It is clear that only points on line PI are reflected from point I to the center of sight. It is also clear that line IS > line KH [by Euclid, III.8], so it is longer than [line] SD [which = line KH, by construction]. Therefore, angle SDI > angle SID [by Euclid, I.19] , so it is greater than angle GIT [which is vertical to angle SID], and therefore greater than angle TIP.

[2.87] Hence, line PI and [line] SD will never intersect [in the direction of P and S],[41] and the line extended to point D from any point on line PI will intersect line SI inside the sphere, SI being the line of reflection [continued inside the circle]. And any line drawn [to D] from any point on line PI will be perpendicular to the sphere, just as PD is. Moreover, since the image-location lies at the intersection of the normal [dropped] from the point viewed and the line of reflection, the image of any point on line PI will lie inside the sphere. It is therefore evident that the appropriate location of all images [produced by reflection from points] within arc HZ will be inside the sphere, which is what was proposed.

[CASE 3]

[2.88] Moroever, if some point [of reflection] is chosen on arc HB, I say that the image will sometimes lie inside the sphere, sometimes on the surface of the sphere, and sometimes outside the sphere.

[2.89] Choose some point on this arc, let it be N [figure 5.2.9b, p. 224], and from point G draw line GNQ to intersect the circle. Then draw normal DNF, and draw line EN to form with normal [DNF] an angle [ENF] equal to angle FNG. Since line NQ < line KH [by Euclid, III.8], it is also shorter than line QD [since QD = KH, by construction], and so angle QDN < angle DNQ [by Euclid, I.19], and therefore smaller than angle GNF [which is vertical to angle DNQ], and therefore also smaller than angle ENF [which = angle GNF, by construction]. Hence, line EN and [line] DQ will intersect. So let the intersection be at point E. It is obvious that line EQD is normal to the sphere, and it intersects line of reflection GNQ at point Q, which is a point on the sphere. Therefore, if reflection occurs at point N, the image of point E will appear at point Q, and this point lies on the surface of the sphere.

[2.90] If, however, a point, such as R, is chosen on line EN beyond E, the normal dropped from that point to the center of the sphere, e.g., RD, will intersect line of reflection GNQ beyond point Q [e.g., at point L]. This lies outside the sphere, so the image of any point chosen on line EN beyond E will lie outside the surface of the mirror.

[2.91] But if some point [such as C] is chosen on line EN in front of point E, the normal dropped from it to the mirror [along CD] will intersect line GNQ inside the sphere, because [it intersects it] at a point that lies between N and Q. Accordingly, the image of any point chosen on line EN between E and N will appear inside the sphere.

[2.92] The very same proof will hold for any other point chosen on arc BH. And so, for any point on arc BH, one image alone lies on the surface of the mirror, others lie inside the mirror, and others lie outside it. Moreover, what has been demonstrated for arc ZB can be shown in exactly the same way for arc ZA, and the very same demonstration will apply for any [great] circle on the sphere and the portion [of it] that is selected to face the center of sight and be bisected by normal GD.

[2.93] Accordingly, if the center of sight remains immobile with normal GD fixed in place, and if line GHK is rotated uniformly about normal [GD as axis], it will cut off a circular portion of the sphere with its rotation, and the image of any point [reflected from any point] within this portion will appear inside the sphere.[42]

[2.94] On the other hand, if line GB tangent [to the sphere] is rotated uniformly about normal [GD as axis], it will cut off a portion of the sphere larger than the previous one, and from any point on the portion that constitutes the excess of the latter over the former, [one] image reflected [to the eye] will find its location on the surface of the sphere, and of the remaining images, some [will lie] inside the sphere, and some outside it.[43]

[2.95] From these facts, we know that in this [sort of] mirror any image appears on a diameter of the sphere [that forms the normal, and that image may appear] inside the sphere, outside it, or on its surface. Moreover, any diameter upon which an image may appear, whether on the surface of the sphere or outside that surface, lies below the point on the sphere where the tangent extended from the center of sight to the sphere touches the sphere, that point being at the very edge of the visible portion [of the mirror]. We [also] know [from this analysis] that every line of reflection cuts the sphere in two points, the point of reflection and some other one.[44]

[2.96] **[PROPOSITION 10]** It remains for us to determine image-locations more specifically. I say that, if a diameter is chosen [on a convex spherical mirror], and if a line is extended to it from the center of sight to cut the sphere so that the segment lying between the point of intersection on the sphere and the point on the diameter to which it reaches is equal to the segment of the diameter lying between that latter point and the center [of the sphere], that point does not constitute the location of any image.

[2.97] For instance, let AG [figure 5.2.10, p. 225] be a [great] circle on the sphere, H the center of sight, and ED a diameter, or normal, in the sphere. Let HZ be a line intersecting the sphere at point F and meeting ED at point Z, and let ZF = ZD. I say that Z does not constitute the location of any image [on normal DE].

[2.98] For it is evident that there is no image-location at any point other than on ED, because the image of any point lies on the diameter extended from it to the center of the sphere. That the image-location of any point on ED does not lie at Z will be demonstrated as follows.

[2.99] From point D draw normal DFN to point [of reflection] F, and at point F form angle QFN equal to angle NFH. It is therefore evident that angle QFN = angle ZFD [which = vertical angle NFH]. But [angle] ZFD = angle ZDF, because they are subtended by equal sides [of isosceles triangle ZFD, sides ZF and ZD being equal, by construction]. Therefore, [angle] QFN = angle ZDN, so [by Euclid, I.27] line [of incidence] FQ is parallel to line ED [which forms the normal].

[2.100] Hence, if they are extended indefinitely, they will never meet. Accordingly, the form of no point on [normal] ED will pass to point F along [line of incidence] QF, and there can be no image-location for any point at point Z unless its form passes to F along line QF.[45] The same disproof will hold for any diameter that is chosen, so what was proposed [is demonstrated].

[2.101] I say, as well, that no point on line ZD can be the location for any image.

[2.102] For let point P be taken [on line DZ], and draw line [of reflection] HP to intersect the sphere at point [of reflection] B. Then draw normal DBM, and make angle [of incidence] TBM = angle [of reflection] MBH. It is clear that [angle] TBM = [angle] PBD [which = vertical angle MBH], and it is clear that angle DPH > angle PZF, because it is exterior [to triangle PZH]. Therefore, the two remaining angles [PDB + DBP] of triangle DPB < the two remaining angles [ZDF + DFZ] of triangle DZF [since angle PZH is common to both triangles PZH and DZF]. But angle PDB > angle ZDF. It therefore follows that angle DBP < angle DFZ. But angle DFZ = angle ZDF [since triangle DZF is isosceles by construction], so angle DBP < angle ZDF. Hence, it is even smaller than angle PDB [since we have already established that angle PDB > angle ZDF], so angle [of incidence] TBM [which = angle DBP] < angle PDB. Therefore, lines TB and ED will never intersect [on the side of T and E], and so no image [created by reflection] from point B is produced at point P. The same holds for the image of any other point [on line of incidence TB], and likewise for any point on line ZD. It follows, therefore, that the whole of ZD is void of image-locations.

[2.103] **[PROPOSITION 11]** Furthermore, if a diameter, other than the one imagined [to extend] from the center of sight to the center of the sphere, is taken between the lines drawn tangent to the sphere from the center of sight, and if the point on it we [just] described as the limit of image-locations is determined, I say that the image-locations of points on that diameter lie only at points on it between the surface of the sphere and the aforementioned limit.

[2.104] For example, let BZ and BE [figure 5.2.11, p. 225] be the tangents, B the center of sight, A the center of the sphere, BHA the visual axis [which is normal to the mirror from the center of sight], and DA the selected diameter with G [the point at which DA intersects the sphere and T] the limit [of image-locations].[46] I say that images of points [whose forms are reflected to B] occur only at points between G and T on DA.

[2.105] That they will not occur at point G or [any point] outside the sphere's surface is clear from what was said above [in proposition 9], i.e., that the diameter on which the image-location will occur at the surface of the mirror or outside it lies below the point of tangency [E]; and since diameter DA lies between the lines of tangency, there will be no image-location on it either at or outside the [mirror's] surface. On the other hand, it will be demonstrated that the image may fall at any point taken between G and T.

[2.106] Choose [such] a point, let it be Q, and draw line [of reflection] BQ to cut the sphere at point C. Now, draw normal ACL, make angle [of incidence] DCL = angle [of reflection] LCB, draw line BT to cut the sphere at point F, and draw normal AF. Accordingly, triangle ACB contains triangle AFB, so angle AFB > angle ACB. It follows, then, that angle AFT [adjacent to larger angle AFB] < [angle] ACQ [adjacent to smaller angle ACB]. But angle AFT = angle FAT, because they are subtended by equal sides [of isosceles triangle FAT]. Therefore [angle] ACQ > angle CAQ [because angle CAQ < angle FAT < angle ACQ], so [angle] LCB [which = vertical angle ACQ] > [angle] CAQ, from which it follows that [angle] DCL > [angle] CAQ [since angle DCL = angle LCB, by construction]. CD and AQ will therefore intersect. Let D be the [point of] intersection. Thus, the form of point D should reflect at point C along line CB, and its image-location is Q. And the same proof holds for any [other] point selected between G and T.

[2.107] **[PROPOSITION 12]** It remains for us to determine image-locations in the invisible section of the sphere.

[2.108] Accordingly, let AC and AG [figure 5.2.12, p. 225] be the tangents [defining] the visible portion [of the sphere], A the center of sight, B the center of the sphere, ADBZ the visual axis [which is normal to the mirror from the center of sight], and ZCG the [great] circle on the sphere lying in the plane of the tangent lines. From the center [of the sphere] draw diameter BG to point [G] of tangency. It is evident that angle ZBG > a right angle, for, since angle BGA in triangle BAG is right, angle GBA < a right angle, so [adjacent angle] ZBG will be greater [than a right angle]. Accordingly, let HBG be a right angle. HB will thus be parallel to line of tangency AG. Hence, when extended [indefinitely] they will never intersect, whereas any diameter between H and G will intersect line AG.

[2.109] Draw line AMO from point A to intersect the sphere so that chord MO = radius OB, and let diameter BO intersect line AG at point T. I say that there is an image-location at any point on TO, that there is an image-location at no other point on diameter TB, and that O and T are the limit-points for image-locations [on that diameter].[47]

[2.110] Take [such a] point, let it be K, and draw [line of reflection] ANK to cut the sphere at point N. Then draw normal BNY, and form angle [of incidence FNY] with FN that is equal to angle [of reflection] YNA. It is clear that [line of incidence] FN will not lie between B and T, because [in that case] it must cut either the sphere or tangent AP in two points.[48] Therefore, the form of point F will propagate along FN to point N and will be reflected to A along line AN, and its image will appear at point K. And the same proof holds for any other point chosen [as an image-location between T and O].

[2.111] **[PROPOSITION 13]** I say, further, that no matter what diameter may be chosen within arc OG, it will contain image-locations inside the mirror, as well as one [image] on the surface of the mirror, and other [images] outside the mirror.

[2.112] Accordingly, take point L [in figure 5.2.13, p. 226], and draw diameter BL until it intersects [line of tangency] AP at point E. Then draw line AL to intersect the sphere at point R. It is obvious [by Euclid, III.8] that RL < LB, since it is shorter than MO, which is equal to the [sphere's] radius [by construction from the previous proposition]. Accordingly, if from A a line is drawn to diameter LB such that the segment [of it] lying between the circle and the diameter is equal to the segment of the diameter [extending] from the point where that line meets the diameter to the center [of the circle], it will fall between L and B. For if it were to fall between L and E, then RL >

LB, and every line falling between the center [of the sphere] and that equal segment [on LB] would be longer than the segment on the diameter where it terminates, according to the proof [in proposition 10] devoted to explaining the limit of image[-locations].[49]

[2.113] So let I be the point where the line[-segment between arc DG and normal BL] equal [to the segment on BL] will fall. I say that there is an image-location at any point on EI [other than E or I]. And the same demonstration will apply [in this case] as applied for TO [in the previous theorem].

[2.114] Accordingly, the image-locations on diameter EB are divided among those that lie inside the mirror [i.e., between L and I], those that lie outside the mirror [i.e., between E and L], and the single one, i.e., at point L, that lies on the mirror's surface. And in this way you can demonstrate [the same thing] for any diameter that passes through arc OG.

[2.115] **[PROPOSITION 14]** Moreover, if some diameter is taken within arc OH, [any] image-location on that diameter will lie outside the mirror.

[2.116] Take diameter BQ [figure 5.2.14, p. 227], and let it intersect the tangent [AG] at point P. Then draw line ANQ to cut the sphere at point N. It has already been established [by construction in propositions 12 and 13] that MO = [radius] OB, but NQ > MO, so NQ > [radius] QB. Moreover, the line drawn at the circumference [of the sphere] to diameter PB and equal to the segment of BP lying between it and the center [of the sphere] will not fall between Q and B. For if it were to fall [there], then, according to the previous demonstration, NQ < QB [which contradicts the conclusion drawn above].

[2.117] It follows, therefore, that the line equal [to the previously defined line] should fall between P and Q. That it may not fall at point P is clear from the fact that angle PGB is a right angle [and so tangent AGP cannot be a line of reflection]. Therefore, PB > PG [because it subtends a larger angle in triangle BGP]. So it will fall in front of P [i.e., between P and the mirror].

[2.118] Let the point at which it falls be S [so that the line-segment on AS between S and the point at which AS cuts the circle is equal to SB]. S will therefore constitute the limit for image-locations, and any point between P and S will constitute an image-location, and the proof for this is the same as above.

[2.119] From these theorems it is evident that all the images [produced] on diameters within arc HO lie outside [the mirror]; on diameter FB, one lies on the surface, at O; and all the rest lie beyond, i.e., in TO; however, of all the images on [any] diameter within arc OG, some lie inside [the mirror], some outside it, and one on its surface.

[2.120] **[PROPOSITION 15]** Furthermore, no diameter can be selected in arc HZ [figure 5.2.15, p. 227] that contains an image-location, for no diameter selected there intersects tangent AP [since HB is parallel to AP].

[2.121] Now, from some point on such a diameter [outside the sphere], draw a line [tangent] to the sphere. It will of course touch [the sphere] within segment GZC, not within segment GDC, unless it cuts the sphere. Accordingly, no form of any point on such a diameter will reach the portion [of the mirror] visible to the eye.

[2.122] Moreover, what has been claimed for arc GH can be demonstrated in the same way for the portion of arc CZ that corresponds to it. So if an arc equal to arc HZ is taken on the other side of Z, there will be no image-location on any diameter within that arc.

[2.123] The same method is used in proving this for any circle [on the sphere], so if line HB is rotated [about axis BZ] while angle HBZ is kept constant [throughout its rotation], it will describe with its motion a portion of the sphere within which there is no diameter with an image-location. If, however, HB is held in place [so that angle HBZ remains constant], and if OB is rotated [about axis BZ], it will describe a portion [of the sphere] within which all of the images [on any diameter lie] outside [the surface, by proposition 14], but on diameter TB one [image lies] on the surface, the rest outside [by proposition 12]. Finally, if arc OG is rotated, it will form a portion [of the sphere] within which some of the images lie on the surface, some outside the sphere, and some inside [by proposition 13].

[2.124] Yet the visual faculty does not perceive which images lie on the surface of the sphere or which [lie] outside, nor does it determine anything in the process of perceiving them except that they lie behind the visible portion [of the sphere].[50] At this point, then, the image-locations in these [sorts of] mirrors have been determined.

[2.125] **[PROPOSITION 16]** Furthermore, in this [sort of] mirror, the form of a visible point can be reflected to the eye from only one point on the mirror.

[2.126] Let the [visible] point be B [figure 5.2.16, p. 228], let A be the center of sight, and let A not stand on the normal dropped to the center [of the sphere through the point of reflection]. I say that [the form of] B is reflected to A from only one point on the mirror, and it yields only one image to the eye in this [kind of] mirror.

[2.127] It is clear that its form can be reflected from some point. Let that point be G, and draw [line of incidence] BG and [line of reflection] AG. Let N be the center of the sphere, draw diameter BN [i.e., the normal dropped from object-point B] to intersect the sphere's surface at point L, and let the limits of the [visible] portion [of the mirror] facing the center of sight be D and E [that portion being subtended by tangents AE and AD]. Let line AG intersect normal [BN] at point Q, which is the image-location.

[2.128] It is obvious that A, N, and B lie in the same plane orthogonal to the sphere. And since all planes that are orthogonal to the sphere and that contain BN intersect along BN, and since only one plane containing line BN can be extended through point A, it is clear that A and B lie in only one plane that is orthogonal to the sphere, not in several [such planes]. Moreover, since the visible point [B] and [center of sight] A must lie in the same plane that is orthogonal [to the sphere] at the point of reflection, it is clear that the reflection of [the form of] point B to the eye will occur only on the [great] circle that lies in plane ANB within the sphere. Accordingly, let that circle be DGE. I say, again, that reflection will occur from no point other than G on this circle.

[2.129] If it is claimed that [such reflection occurs] from point L [on normal BN], then, since BN is orthogonal [to the sphere] while AL is not orthogonal, and since the form that is incident along the perpendicular is necessarily reflected back along the perpendicular, it is obvious that [the form of] B does not reflect to A from point L. Nor, by the same token, [does it reflect] from another point on arc LE. For, to whatever point on that arc a line is dropped from point B, it will form an obtuse angle with the tangent at that point on the side of E, whereas the line dropped from point A to that point will form an acute angle with that tangent on the side of L. Therefore, if reflection were to occur from that point, an acute angle would equal an obtuse angle.

[2.130] Likewise, reflection cannot occur from any point on [arc] GL. Take some [such] point, let it be Z, and draw line AZO to intersect normal [BN] at point O. Then draw the line tangent to the circle at point Z, a line that necessarily falls between [i.e., intersects] BG and BL, and let it be MZ. Let FG be the line tangent to the circle at point G. It is clear from the above that BN:NQ = BF:FQ [from proposition 7 above, F being the endpoint of tangency]. So too, BN:NO = BM:MO [from proposition 7 above, Z being the point of reflection and M the endpoint of tangency]. But BN:NQ > BN:NO [by Euclid, V.8]. Therefore BF:FQ > BM:MO, which is clearly impossible, since BF < BM, while FQ > MO. It therefore follows that reflection may not occur from point Z.

[2.131] On the other hand, it will be proven as follows that reflection may not occur from any point on arc GD. Take some [such] point, and let it be T. Draw line BT, and [draw] line ATH to intersect BN at point H. Then draw CT tangent to the circle at point T. Accordingly, BN:NH = BC:CH [by proposition 7 above, T being the point of reflection and C the endpoint of tangency], and BN:NQ = BF:FQ [as established above]. But BN:NH > BN:NQ [by Euclid, V.8]. Hence, BC:CH > BF:FQ, which is clearly false, since BF > BC, while CH > FQ. It therefore follows that reflection of [the form of] point B may occur from no point on arc GD, so [the form of] any given point is reflected to the eye from only one point on the sphere. Hence, there will be only one line of reflection for each visible point, and so [there is] only one image of [any] single point.

[2.132] Moreover, if point B lies on the visual axis [which is the normal dropped from the eye to centerpoint N of the sphere], it is obvious that it is reflected from only one point, because [it is reflected] only along the normal, and its image will be unique and will lie in the appropriate image-location because of its continuity with the other points [as established in proposition 8].

[2.133] **[PROPOSITION 17]** Furthermore, if two points are taken on a given diameter on the same side of the [sphere's] center, the image-location of the point nearer the [sphere's] center will lie farther from the center of the sphere than the image-location of the point farther from the sphere's center. Also, the point of reflection for the point nearer the [sphere's] center will lie farther from the center of sight than the [point of reflection] for the point farther from the center of the sphere.

[2.134] I say, for example, that the image-location for point C [figure 5.2.17, p. 229] lies farther from the center [N] than does the image-location for point B, and that the point of reflection for point C lies farther from point A [the center of sight] than the point of reflection for point B, i.e., point G. I say [further] that [the form of] point C is reflected [to A] from a point on arc GL only.

[2.135] For it is evident [from the previous theorem] that it will not be reflected from any point on arc LE, nor from point L, nor from point G, since [the form of] B is reflected from that point. But if it is claimed that [it is reflected] from some point on arc GD, let that point be T, and let CT be the line along which the form [of C] radiates to the mirror. Then draw normal NT, which will necessarily bisect angle CTA, and draw normal NGK. Angle NTA > [angle] NGA [by Euclid, I.21]. It therefore follows that angle PTA [adjacent to angle NTA] < angle KGA [adjacent to angle NGA], so angle CTP < angle BGK [because, by supposition, angle CTP = angle PTA, and angle BGK = angle KGA, and we have just concluded that angle PTA < angle KGA]. But [by Euclid, I.32] angle CTP = angle TNC + angle TCN, because it is an exterior [angle of triangle TCN]. In addition, angle BGK [which is an exterior angle of triangle BGN] = angle GNB + angle GBN. Accordingly, the [sum of the] two angles TNC and TCN < the [sum of the] two angles GBN and GNB , which is impossible, since angle TNC contains [angle] GNB as a part, while angle TCN > [angle] GBN.

[2.136] Hence, it follows that point C may only be reflected from [some point among the] points lying between G and L. All lines drawn from point A to normal BN through such points fall on points [between L and Q] that lie farther than point Q from the center of the sphere, and they fall to points on the sphere [between L and G] that are farther away from the center of sight [A] than point G, and so what was proposed [has been proven].

[2.137] **[PROPOSITION 18]** Now, given a [convex spherical] mirror and given a visible point, to find the point of reflection.

[2.138] Let B [figure 5.2.18, p. 229] be the visible point, [let] A [be] the center of sight, and draw two lines from these points to the center of the mirror. If those lines are equal, it will be easy to find [the point of reflection]. For a [great] circle on the sphere will be defined in the plane within which those two lines lie, and we know that reflection occurs from only one point on that circle. Accordingly, the angle those two lines form at the [sphere's] center will be bisected.

[2.139] Draw the line [GN] that bisects the angle [and extend it] beyond the sphere. It will of course be perpendicular to the line tangent to this circle at the point [G] through which it passes. And if two lines [AG and BG] are drawn to that point, one from the center of sight and the other from the visible point, they will form two triangles with that normal and the two original lines, two sides of these triangles [i.e., BN, NG and AN, NG] being equal [respectively] and angle [BNG of the one, being equal] to angle [ANG of the other].[51] Hence, the point on the circle through which that normal passes constitutes the point of reflection, which is what was set out [to be demonstrated].

[2.140] On the other hand, if the line extending from the visible point to the center of the sphere is not the same length as the line extending from the center of sight to that same centerpoint, we must set forth some preliminary things, one of which is as follows.

[2.141] **[PROPOSITION 19, LEMMA 1]** If a diameter is taken in a circle, and if a point is taken on its circumference, a line can be drawn from that point to the extension of the diameter beyond [the circle] such that its extension from the point where it intersects the circle to the point where it intersects the diameter is equal to a given line.

[2.142] For instance, let QE [figure 5.2.19, p. 230] be the given line, GB the diameter of circle ABG, and A the given point [on its circumference]. I say that I can draw a line from point A such that [the segment extending] from the point where it intersects the circle to diameter GB is equal to line QE, which will be proven as follows. Draw the two lines AB and AG, which are either equal or unequal.

[CASE 1]

[2.143] Let them be equal, and add a line to QE—let EZ be this additional line—such that the rectangle that will be formed by the whole line augmented by the additional line and the additional line [i.e., (QE + EZ),EZ = QZ,EZ] = AG^2.[52] Accordingly, since QZ,EZ = AG^2, QZ > AG. For if EZ were equal to, or longer than AG, it is impossible that QZ,EZ = AG^2 [because the whole, QZ, is greater than its part, EZ, whose square = AG^2]. On the other hand, if it is shorter, it is clear that QZ > AG.

[2.144] Accordingly, extend AG [figure 5.2.19a, p. 230] until it is equal [to QZ], and let the result be AGT. Then, placing a compass-point at A, draw a circle according to radius AGT, and [since] this circle will intersect diameter BG, let it intersect [it] at point D. Then draw line AD, which will necessarily intersect circle [AGB], for if it were tangent at point A, it would be parallel to BG and would never intersect it. Accordingly, let it intersect [the circle] at point H, and draw line GH.

[2.145] It is clear that, since ABGH constitutes a quadrilateral within the circle, the two opposite angles, i.e., ABG and AHG, sum up to two right angles [by Euclid, III.22]. But [angle] AGB = [angle] ABG, since they are subtended by equal sides, according to construction. Therefore, angle AHG = angle DGA [because, as previously concluded, angle AHG = two right angles – angle ABG, and angle DGA = two right angles – angle AGB, which = angle ABG], and angle HAG is common to the whole triangle ADG and triangle AHG that forms part of it. It therefore follows that angle HDG = angle HGA and that [the one] triangle [DGA] is similar to [the other] triangle [AHG], so [by Euclid, VI.4] DA:AG = AG:AH. Hence, DA,HA = AG^2 [by Euclid, VI.17]. But DA = TA [since they are both radii of circle TD], so DA = QZ [since TA = QZ, by construction]. And [therefore] AH = EZ, and DH = QE, which is the given line, and so what was proposed [has been demonstrated].[53]

[CASE 2]

[2.146] Now, if AB and AG are not equal [figure 5.2.19c, p. 231], draw a line from point G that is parallel to AB, let it be GN, and take some line ZT, and form an angle at point Z with line ZF [i.e., angle TZF] equal to angle AGD. Then, from point T draw a line parallel to ZF, let it be TM, and from angle TZF cut off an angle with line ZM [i.e., angle MZT] equal to angle NGD, for this line [MZ] will necessarily intersect TM, since it lies between parallels [TM and ZF]. Let the point of intersection be M. It follows, therefore, that angle MZF = angle AGN [since, by construction, angle TZF = angle AGD, and angle TZM = angle NGD, so angle MZF (which = angle TZF – angle TZM) = angle AGN (which = angle AGD – angle NGD)].

[2.147] Now, from point T draw a line TO parallel to line ZM, and this line will necessarily intersect FZ. Let the point of intersection be K. Take some line I that is to line ZT as [line] BG is to [line] EQ, [which is the] given line [i.e., I:ZT = BG:EQ, by Euclid, VI.12]. Then, at point M produce a conic section [i.e., a hyperbola] in the way Apollonius describes in the fourth proposition of book 2 of his *Conics*, and let this conic section be UCM, which may not intersect lines KO and KF [that form its asymptotes].[54] In this section draw a line equal to line I, i.e., MC, extend it to lines KT and KF, and let O and L be the intersection-points [where MC extended cuts asymptotes KT and KF]. Thus, as the same [book of Apollonius' *Conics*] will demonstrate, OM = CL.[55]

[2.148] From point T draw line TF parallel to [line] CM, and at point A form an angle [GAN] with line AND that is equal to angle ZFT. It is evident that this line [AND] will intersect GD, since angle AGN = angle FZM, and angle GAN = angle ZFT. Hence, line AD will either be tangent to the circle or will intersect it, because, if it is not tangent, and if arc AB > arc AG, it will cut arc AB, whereas if AB < [AG], it will cut arc AG.

[SUBCASE 2A]

[2.149] Accordingly, let it be tangent at point A. Since angle GAN [in triangle GAN] = angle ZFT [in triangle ZFY], and since angle AGN = angle FZY, the third angle [ANG] = the third angle [ZYF], so [by Euclid, VI.4] triangle AGN will be similar to triangle ZFY. Likewise [in triangles AGD and FZT], since [angle] AGD = angle FZT [while angle ZFT = angle GAD, and the remaining angles ZTF and ADG are equal], then triangle AGD will be similar to triangle FZT. Therefore AN:AG = FY:FZ, and AG:GD = FZ:ZT, so AN:GD = FY:ZT [by Euclid, V.22].

[2.150] But since TM is parallel to FL [by construction], and since FT is parallel to ML [by construction], FT = ML, so it will be equal to CO, since MO = LC [by Apollonius, II.8]. But MO = YT, since it is parallel to it, and YM is parallel to TO. It therefore follows that FY = CM. But CM = I [by construction]. Thus, FY = I. But I:ZT = BG:EQ [by construction]. Therefore, AN:GD = BG:EQ [because we established earlier that AN:GD = FY:ZT].

[2.151] However, angle GAN = angle GBA, as Euclid demonstrates in the third [book of the *Elements*, prop. 32]. But angle NGD = [alternate] angle ABG, since NG is parallel to AB [by construction]. Therefore, angle NGD [in triangle NDG] = angle NAG [in triangle AGD], and angle NDG is common [to both triangles], so the third [angle, i.e., DNG] = the third [angle, i.e., AGD], and so triangle NDG is similar to triangle ADG. Hence, AD:GD = GD:ND, so AD,DN = DG^2 [by Euclid, VI.17].

[2.152] But AD^2 = BD,DG, as Euclid demonstrates [in III.36], and AD^2 = AD,DN + AD,NA [by Euclid, II.2]. And BD,DG = DG^2 + BG,GD, as Euclid demonstrates [in II.3]. Therefore, when equal terms are subtracted [i.e., GD^2], it follows that AD,AN = BG,DG.[56] Hence [by Euclid, VI.16], the second [AN] is to the fourth [DG] as the third [BG] is to the first [AD], so AN:DG = BG:AD. But it has already been established that AN:GD = BG:EQ. Therefore, EQ = AD, which is what was proposed.

[SUBCASE 2B]

[2.153] If, however, AD is not tangent to the circle but cuts it, and if AG > AB [figure 5.2.19d, p. 232], then it will cut [arc] AG. Let it cut at point H, and draw line AG.

[2.154] It is obvious that the two angles AHG and ABG sum up to two right angles [by Euclid, III.22]. But angle NGD = [alternate] angle ABG [because NG is parallel to AB, by construction]. Thus, angle AHG + angle NGD = two right angles. Accordingly, angle NGD = angle NHG [since angle AHG + angle NHG = two right angles], and angle NDG is common [to triangles NGD and HGD], so the third angle [DNG] = the third angle [DGH], and so triangle HGD is similar to triangle NDG. Therefore, HD:DG = DG:DN, so HD,DN = GD^2 [by Euclid, VI.17].

[2.155] But AD,DH = BD,DG, as Euclid demonstrates,[57] and AD,DH = DH,DN + DH,AN [by Euclid, II.1]. Moreover, BD,DG = BG,GD + GD^2 [by Euclid, II.3]. Therefore, when equal terms are subtracted (i.e., GD^2 and DH,DN), it follows that DH,AN = BG,DG,[58] so [by Euclid, VI.16] the second term is to the fourth (i.e., AN:GD) as the third is to the first (i.e., BG:DH). But it has already been proven that AN:DG = BG:EQ. Thus, EQ = DH, and so what was proposed [has been demonstrated].

[SUBCASE 2C]

[2.156] On the other hand, if AG < AB (and let [AD] cut [the circle] on arc AB), then let H [figure 5.2.19e, p. 232] be the point of intersection, and draw line HG. It is evident that angle NGD = angle ABG [by construction]. But [by Euclid, III.21] angles ABG and AHG are equal, since they are subtended by the same arc [AG]. Therefore, angle NGD = angle AHG, and angle NDG is common [to triangles NGD and HGD], so the third angle [GND] = the third angle [HGD], and the triangles [NGD and HGD] are similar. Accordingly HD:GD = GD:DN, so HD,DN = GD^2 [by Euclid, VI.17].

[2.157] But HD,DA = BD,DG [since, by Euclid, III.36, both are equal to the square on the tangent drawn from D], and HD,DA = DN,HD + AN,HD [by Euclid, II.1]. Moreover BD,DG = DG^2 + BG,DG [by Euclid, II.3]. Hence, with equal terms subtracted [i.e., GD^2 and HD,DN], HD,NA = BG,DG. Accordingly, AN:DG = BG:HD [by Euclid, VI.16]. But it has already been established that AN:DG = BG:EQ. Therefore, EQ = HD, which is what was proposed, because from point A we have drawn a line to cut the circle, and [the segment] from the point of intersection [H] to the diameter [BG] is equal to the given line.

[2.158] **[PROPOSITION 20, LEMMA 2]** Now, from a given point on a circle outside its diameter, a line can be drawn through the diameter to the circle so that the segment on it between the diameter and the circle is equal to a given line.

[2.159] For instance, let ABG [figure 5.2.20, p. 233] be the given circle, BG its diameter, A the given point, and HZ the given line. I say that a line can be drawn from point A to pass through diameter BG such that the segment [ED] from the diameter to the circle is equal to line HZ.

[2.160] The proof [is as follows]. Draw lines AB and AG, and at point H form with line HM an angle [MHZ] equal to angle AGB, and at the same point form with line HL an angle [LHZ] equal to angle ABG. Then, from point Z draw a line ZN parallel to line HM, and it will intersect HL; and from point Z draw line ZT parallel to HL, and let it intersect HM at point T. At point T construct conic section [i.e., hyperbola] TP, which Apollonius will describe in his book on conics [II.4], and that [conic] section will never touch either of the lines ZN and HL [i.e., the asymptotes] between which it lies. Between these same lines construct conic section [i.e., hyperbola] CU facing the first one.

[2.161] Thus, if the shortest of all the lines extending from point T to [conic] section CU is equal to diameter BG, a circle produced according to this shortest line [as radius] when the point of a compass is placed at point T will be tangent to [conic] section CU. On the other hand, if the shortest of all the lines extending from point T to [conic] section CU is shorter than diameter BG, the circle produced in the aforementioned way according to [radius] BG will intersect [conic] section CU at two points.[59]

[2.162] Accordingly, let CT be the shortest [line], and [let it be] equal to diameter BG, and it will intersect ZQ and HF when it is extended to the [conic] section lying between them.[60] From point Z draw a line parallel to this one, and it will intersect HM and HL just as its parallel [CT] does. Let it be MZL, let it intersect [those lines] at points M and L, let Q be the point where CT intersects ZN, and on diameter GB form an angle equal to angle HLZ, and let it be DGB. Then draw the two lines AD and BD.

[2.163] It is evident that, since angle GAB is a right angle [by Euclid, III.31], the other two angles in triangle AGB sum up to a right angle, so angle LHM is right [since it consists of the two angles MHZ and ZHL that were constructed equal to angles AGB and ABG, which sum up to a right angle], and it is equal to angle GDB [which is a right angle by Euclid, III.31]. Also [by construction], angle HLM [in triangle HLM] = angle DGB [in triangle DGB]. Hence, the third [angle HML is equal] to the third [angle DBG], and the [first] triangle [HML] is similar to the [second] triangle [GDB], so GB:BD = LM:MH.

[2.164] But, since angle ADB = angle BGA, because they are subtended by the same arc [AB], and since angle BGA = angle MHZ, by construction, then angle ADB = angle MHZ. In addition, we already know that angle GBD = angle HMZ [by the similarity of triangles HML and DGB]. Hence, the third [angle in triangle DEB, i.e., angle DEB, is equal] to the third angle [in triangle MHZ, i.e., angle MZH], so triangle DEB is similar to triangle MHZ. Let E be the point where line AD intersects diameter BG. Accordingly, BD:DE = MH:HZ. Hence, BG:DE = LM:HZ.[61]

[2.165] However, Apollonius demonstrates [in *Conics*, II.16] that, when two [hyperbolic] conic sections lie opposite one another between two [asymptotic] lines, and when a line is drawn from one section to the other, the segment of that line lying between one of the sections and one of the [asymptotic] lines is equal to the opposite segment lying between the other section and the other [asymptotic] line, so QC = TF. But TQ = MZ, since it is parallel to it [by construction] and lies between two parallels [HM and ZN, which are parallel by construction]. Therefore MZ = FC [since FC = TQ], and ZL = TF [since they are parallel and lie between TZ and HL, which are parallel by construction]. Hence, ML = TC, so GB:ED = TC:HZ, and since TC = BG, ED = HZ, which is what was proposed.

[2.166] However, if the line extending from T to [conic] section CU is the shortest line and is shorter than diameter BG, then extend it beyond the [conic] section until it is equal. Then, according to its length [as radius], produce a circle that will intersect the [conic] section at two points, and from those points the lines extending to T will be equal to BG. Then, from point Z draw a line parallel to both, and in that case two lines equal to the given lines will be drawn from point A in the prescribed manner, and this will be proven in precisely the same way.[62]

[2.167] **[PROPOSITION 21, LEMMA 3]** Moreover, given a right triangle, and given some point on one of the sides forming the right angle, a line can be drawn from that point to the other side forming the right [angle] and intersecting the third side facing the right angle in such a way that the segment of this line that lies between the point of intersection and the side on which the given point does not lie is to the segment of the side opposite the right angle from the [point of] intersection to the side containing the given point as a given line is to a given line.

[2.168] For instance, ABG [figure 5.2.21, p. 235] is the given triangle with ABG the right angle, and the given point D, which is on side GB, is either in or outside the triangle. I say that from point D a line can be drawn to cut side AG and intersect side AB in such a way that the segment [TQ] on it between sides AB and AG is to the segment [TG] of the side AG extending from that line to point G in the same proportion as E is to Z, those being the given lines [i.e., TQ:TG = E:Z].

[CASE 1]

[2.169] The proof [is as follows]. Let point D lie on triangle ABG itself [as represented in figure 5.2.21], and draw from it a line DM that is parallel to AB. Produce a circle upon the three points G, M, and D [by Euclid, IV.5], and draw line AD. Since it is evident that angle GMD = [alternate] angle GAB, it will be greater than angle GAD. With line MN cut from [angle] GMD an angle equal [to GAD], and let it be DMN, and [by Euclid, VI.12] draw line H so that AD is to it as E is to Z [i.e., AD:H = E:Z]. Then, from point N, which lies on the circle, draw a line to diameter GM[63] that is equal to line H, according to the previous account [in proposition 19, lemma 1], let it be [on line] NL, and let point C be where it intersects the circle [so that LC = H].[64] Draw line GC, and from point D draw a line to point C, a line which, when extended, necessarily intersects the other line [i.e., AB], since it falls between two parallels [AB and DM, which are parallel by construction] and forms with one of them an acute angle [i.e., MDT]. Let them intersect, then, and let Q be the point of intersection.

[2.170] It is evident that angle GMD = angle GCD, because they are subtended by the same arc [GND], and angle GMD = angle GAB [by construction]. It therefore follows that angle GCQ = angle GAQ [because they are adjacent to equal angles, i.e., GCD and GAB]. Let T be the point where DQ intersects AG, and [so] angle GTC [in triangle TCG] = [vertical] angle ATQ [in triangle ATQ, while angle GCT (i.e., GCQ) = angle TAQ (i.e., GAQ), as established above]. Hence, the third [angle TGC] = the third [angle TQA], so triangle ATQ is similar to triangle TCG. Thus, QT:TG = AT:TC.

[2.171] But angle NMD = angle TAD [i.e., GAD, by construction], and [it is also equal to] angle NCD [because they are both subtended by arc ND], so [angle] NCD [= vertical angle TCL in triangle TCL, and it also] equals [angle] TAD [in triangle TAD]. In addition, angle CTL is common to both triangles, so the third [angle TLC] = the third [angle

TDA], and [so] the [one] triangle is similar to the [other] triangle, i.e., [triangle] TLC to triangle TAD. Hence, TA:CT = AD:LC [but TA:CT = QT:TG, as established above], so AD:LC = QT:TG. But LC = line H, and AD:H = E:Z. Therefore, QT:TG = E:Z, which is what was proposed.

[CASE 2]

[2.172] If, on the other hand, D is taken on that [same] side [that includes the right angle but extends] beyond the triangle [figures 5.2.21a and 5.2.21b, p. 235], then [in figure 5.2.21a] draw [a line] from point D that is parallel to AB, and let it be DM, and extend AG until it intersects DM at point M. Then produce a circle passing through the three points G, D, and M, and draw line AD. Angle GAD [which is exterior to triangle DAM] will of course be greater than [interior] angle GMD. Form an angle equal to it [i.e., equal to GAD], and let it be NMD, and from point N, which is a point on the circle, draw a line equal to line H such that AD is to that line as E is to Z, and let that line be [on line] NCL, which is extended to diameter MG [so that CL = H and, therefore, that AD:CL = E:Z]. Let the point of intersection be L.

[2.173] Thus, since angle NMD and angle NCD sum up to two right angles [by Euclid, III.22], and since angle NMD = angle TAD, the two triangles TCL and TAD will be similar.[65] In addition, since the two angles GCD and GMD are equal [by Euclid, III.21], the two triangles GCT and TAQ will be similar [so that DT:LT = TQ:TG = AD:CL], and so AD:CL (which = H) = QT:TG, and so E:Z = QT:TG, which is what was proposed.[66]

[2.174] **[PROPOSITION 22, LEMMA 4]** Now, given two points, i.e., E and D, and given a circle, to find a point on it such that the line tangent to the circle at that point bisects the angle formed by the lines drawn from the aforementioned points to that point.

[2.175] For example, from point E [figure 5.2.22, p. 236] draw line EG to the center of the given circle, extend it to the circumference, and let [this extended line] be ES. Then, draw line GD, and let line MI be cut at point C so that IC:CM = EG:GD [by Euclid, VI.10]. Then bisect MI at point N, and draw perpendicular NO. On point M form with line MO an angle [OMN] that is half of angle DGS. It is clear that it will be smaller than a right angle, whereas angle ONM [will be] a right angle [by construction]. Thus, MO will intersect NO. Let it intersect at point O, and from point C draw a line CKF to the triangle [NMO] such that KF:FM = EG:GS [by proposition 21, lemma 3, case 2]. Then, with line AG, which is extended to the circle, form an angle at point G that is equal to angle MFK, and let it be angle AGE. Finally, draw the two lines AG and DG. I say that A is the point we are seeking.

[2.176] Draw line EA. Accordingly, since [angle] MFK = angle AGE [by construction], and since KF:FM = EG:GA [by construction], because GA = GS [since G is the circle's center, by construction], then triangle AGE will be similar to triangle MFK [by Euclid, VI.4]. Hence, angle FMK = angle EAG, and angle AEG = angle MKF.

[2.177] From point A, then, draw a line that forms with line AE an angle [EAZ] equal to angle NMK, and let it be line AZ, which will necessarily intersect line GE, because KF:FM = EG:GA, and angle GAZ = angle FMC [since angle EAG = angle FMK, and angle EAZ = angle KMN, by construction, so EAG + EAZ = GAZ = FMK + KMC = FMC]. Therefore, just as line MO will intersect [line] FK at point F, AZ will intersect GE. Let the intersection occur at point Z, and draw AZ to point Q so that AZ:ZQ = MC:CI [by Euclid, VI.12], and draw line EQ.

[2.178] Then, from point A draw a [line] parallel to EQ, and let it be AT. Angle AQE = [alternate] angle QAT [because QA intersects parallels AT and EQ], and since the two angles ZEA and EAT sum up to less than two right angles [because angle ZEA + adjacent angle AEG = 2 right angles, and EAT < AEG], AT will necessarily intersect EZ. Let T be the point of intersection. It is clear that angle AEG = angle MKF [because, as we concluded above, triangles AGE and MFK are similar]. When line EL is drawn from point E perpendicular to AZ, angle AEL = angle MKN, since [by construction] angle EAL [in triangle EAL] = angle KMN [in triangle KMN], and angle ALE = angle MNK, since both are right angles [so the remaining angles AEL and MKN are equal]. It follows, therefore, that angle LEZ = angle NKC [since both are adjacent to equal angles, i.e., LEG (= AEL + AEG) and NKF (= MKF + MKN), respectively], and right angle ELZ = [right] angle KNC [so triangles ELZ and NKC are similar]. It follows that angle EZL = angle KCN. Therefore, [exterior] angle EZQ [of triangle EZL] = [exterior] angle KCI [of triangle NKC].

[2.179] So it is evident that triangle EAG is similar to triangle FMK, triangle EAL is similar to triangle KMN, triangle ELZ is similar to triangle KNC, and triangle EAZ is similar to triangle KMC. Accordingly, AZ:ZE = MC:CK, and QZ:ZA = IC:CM [by construction], and [so, by Euclid, V.22] QZ:ZE = IC:CK, so triangle QZE is similar to triangle ICK, while triangle QLE is similar to triangle IKN. [Accordingly] NM:NI = AL:LQ [since triangles KNM and AEL are similar], and so AL = LQ [since NM = NI, by construction], EQ = EA, angle EQZ [in triangle EQZ] = angle LAT [in triangle ZAT, by construction], and angle EZQ = [vertical] angle AZT. Hence, the third [angle, i.e., ZEQ] = the third [angle, i.e., ATZ], and triangle EZQ is similar to triangle ZAT, so QZ:ZA = EZ:ZT = EQ:AT = AE:AT [since AE and EQ are equal by previous conclusions]. But QZ:ZA = EG:GD [because EG:GD

= IC:CM, by construction, and QZ:ZA = IC:CM, by construction]. Therefore AE:AT = EG:GD.

[2.180] Now, at point A form an angle equal to angle GAE, and let it be UAG. It is clear that angle GAL is half of angle UAT,[67] but it is [also] half of angle DGU [since, by construction, OMN is half DGU, and GAL = OMN], so angle UAT = angle DGU. But angles TAU and TUA sum up to less than two right angles, since AT and UT intersect, so the two angles TUA and DGU sum up to less than two right angles. Therefore, AU will intersect DG.

[2.181] I say that it will intersect at point D, because [by Euclid, VI.4] it will form a triangle with lines UG and GD that is similar to triangle AUT, for they will have angle AUG in common, and angle TAU = angle UGD [by previous demonstration]. Therefore AU:AT is as UG is to the line [X] that AU cuts from GD [i.e., AU:AT = UG:X], and EA:AU = EG:GU [by Euclid, VI.3], since angle UAG = angle GAE [by construction].

[2.182] Therefore, since EA:AT = EG:GD [by previous conclusions], EA:AT is compounded from EA:AU and AU:AT [i.e., EA:AT = (EA:AU):(AU:AT)].[68] EG:GD will be compounded from these same ratios [i.e., EA:AU and AU:AT, so EG:GD = (EA:AU):(AU:AT), but AU:AT = UG:X, by previous conclusions] so it will be compounded from EG:GU and GU:X [i.e., EG:GD = (EG:GU):(GU:X)]. But it is compounded from EG:GU and GU:GD [i.e., EG:GD = (EG:GU):(GU:GD)]. Therefore the line that AU cuts off from GD is line GD [i.e., GD = X]. Therefore AU cuts GD at point D.

[2.183] Accordingly, from point A extend tangent AH. GAH will therefore be a right angle. But [angle] GAL is half of angle DGU [by previous conclusions]. Hence, angle LAH is half of angle DGE, since these two [i.e., DGU and DGE] sum up to two right angles [so their halves, GAL and LAH, must sum up to a right angle—i.e., GAH, which is right by construction]. But since angle TAU = angle DGU, angle TAD [adjacent to TAU] = [angle] DGE [adjacent to DGU]. Therefore, angle LAH is half of angle TAD [because LAH is half of DGE, by previous conclusions], and angle EAL is half of angle EAT [by previous conclusions]. Thus, angle EAH [which = EAL + LAH] is half of angle EAD, so AH bisects angle EAD, which is what was proposed.

[2.184] On the other hand, if the angle at point A [i.e., UAG] = angle GAE, and if AU does not fall on line ES either outside or inside the circle, then let it be parallel [figure 5.2.22a, p. 236]. Accordingly, [alternate] angle UAG = [alternate] angle AGE. But the same angle [i.e., UAG] = angle GAE [by supposition], so angle GAE = angle AGE. Thus EG = AE. Likewise, angle TAD = angle ATG, since they are alternate. But it has already been established that angle TAD = angle DGT [by previous conclusions]. Therefore, angle ATG = angle DGT, and by the same token the two angles ADG and DGT are equal. Therefore, the two angles ADG and TAD are equal [as are angles ADG and ATG, so triangle AMD is similar to triangle GMT, and both triangles are isosceles because of the equality of the angles at points A and D and at points G, and T].

[2.185] From these conclusions it will therefore follow that the line AU cuts from DG is equal to line AT [i.e., GM = MT, and AM = MD, so AM + MT = GM + MD]. And it has already been established that EG = AE. Hence, EG is to the line AU cuts from DG [i.e., X] as AE is to AT [i.e., EG:X = AE:AT]. But it has already been established that AE:AT = EG:GD. Therefore the line that AU cuts off from DG is GD, and since [angle] TAD = angle DGT, [angle] LAH will be half of angle TAD, as was claimed above, and [angle] EAL [will be] half of [angle] EAT [by previous conclusions]. Accordingly, [angle] EAH will be half of angle EAD, which is what was proposed.

[2.186] **[PROPOSITION 23, LEMMA 5]** Moreover, given a circle with G as its center [figure 5.2.23, p. 237], given GB as a diameter within it, and given point E outside the circle, a line can be drawn from point E to diameter GB that cuts the circle in such a way that the segment of that line [extending] from the circle to the diameter [i.e., DZ] is equal to the segment of the diameter between that line and the [circle's] center [i.e., ZG].

[2.187] For instance, from point E draw EC perpendicular to the diameter, and draw line EG. Take a line QT equal to EC, and on QT form a segment of a circle such that any angle within that segment [e.g., QPT] is equal to angle EGB [by Euclid, III.33], and then complete the circle. From the midpoint [L] of QT extend a perpendicular [FL] in both directions to the circle. This will of course be a diameter of this circle. Then, from point Q draw a line to this diameter to intersect it at point F, and extend it to point P on the circle such that FP is half of GB [by proposition 20, lemma 2], and draw line PT and line TF. From point P draw line PU parallel to the diameter. Let it intersect TF at point U, and from point U draw UO parallel to TQ. From point T draw TN orthogonal to PQ, from point T draw TS parallel to PQ, and from point U draw UH orthogonal to PQ. Then, from angle BGE cut off an angle BGD equal to angle QPU, and draw line EDZ. I say that DZ = ZG.

[2.188] Now, from point D draw DI orthogonal to BG, and from point D draw tangent DK. It is clear that, since diameter FL is perpendicular to QT as well as to OU, and since PU is parallel to that diameter, angle OUP will be a right angle. And since OU is bisected by the diameter [FL] along the orthogonal, FO = FU, so angle FOU = angle FUO.[69] However, since the [remaining] two angles POU and OPU [of triangle POU] sum up to a right angle, angle FUP = angle FPU [because triangle OFU = triangle PFU, since

OU and OP are both bisected by FL, and OP is bisected by FU], so FP = FU, and so it equals FO. And therefore PO = BG [because FP was constructed to be half of BG, and it forms half of PO], and it is also equal to GD [since GD and BG are both radii], so EC:GD = TQ:PO [since TQ = EC, by construction].

[2.189] But since right angle KDG = angle GID [which is right by construction], while angle IGD is common, triangle IGD will be similar to triangle KGD [by Euclid, VI.4], and GD:DI = GK:KD. Yet angle KGD = angle OPU [by construction], whereas right angle KDG = [right angle] OUP, so triangle KDG is similar to triangle OUP [by Euclid, VI.4], and KG:KD = OP:OU. Hence, DG:DI = OP:OU. Accordingly, EC:DI = QT:OU.[70]

[2.190] But QT:OU = TF:FU, since triangle TFQ is similar to triangle OFU [and QT and OU and TF and FU are corresponding sides]. However, angle UTS = angle HFU, since it is alternate to it [because PQ and ST are parallel, by construction], and right angle UST = [right] angle FHU [both being right by construction. Consequently] triangle UST will be similar to triangle HUF, so TU:UF = SU:UH, and so [by Euclid, V.18] TF:UF = SH:UH. But TN = SH, since it is parallel to it and since both lie between two parallels [PQ and ST]. Therefore, TF:UF = TN:UH, so QT:OU [which = TF:UF, by previous conclusions] = TN:UH, and EC:DI [which = QT:OU, by previous conclusions] = TN:UH.

[2.191] But since right angle GID = [right] angle PHU, and since angle IGD = angle HPU [by construction], triangle IGD will be similar to triangle HPU [by Euclid, VI.4], and ID:GD = HU:UP, so EC:GD = TN:UP.[71] Yet, since angle CGE = angle NPT [by construction], and since right angle GCE = [right angle] PNT, [then triangles CGE and NPT are similar, so] GE:EC = PT:NT. Hence, GE:GD = PT:UP.[72]

[2.192] But angle DGE = angle UPT [because angle QPT was constructed equal to angle EGB, and angle DGB = angle HPU, so the remainders DGE and UPT are equal]. Therefore, triangle DGE is similar to triangle UPT. Accordingly, angle GDE = angle PUT. It therefore follows that angle GDZ [of triangle GDZ] = angle PUF [of triangle PUF], and angle DGZ = angle UPF, so the third [angle DZG] = the third [angle PFU, so the triangles are similar], and DZ:ZG = UF:FP. But UF = FP [by previous conclusions]. Therefore, DZ = ZG, which is what was proposed.

[2.193] **[PROPOSITION 24, LEMMA 6]** Furthermore, given a right triangle ABG [figure 5.2.24, p. 238] with right angle ABG, and given point D on either BG or AB, to draw a line from point D to side AG intersecting it at point Q [on one side] and intersecting the other side [at point T] in the other direction such that the sum [of the lengths from D to the respective sides] is to GQ as E is to Z [i.e., (TD + DQ = TQ):GQ = E:Z].

[2.194] For instance, from point D draw DM parallel to AB, and produce a circle passing through the three points D, M, and G. MG will be a diameter [since angle MDG is right, by construction]. Draw line AD, and [by Euclid, VI.12] let H be a line in proportion to which AD is as E is to Z [i.e., AD:H = E:Z]. Since angle DMG = angle BAG, cut from it [an angle] equal to angle DAG, and let it be CMD. Draw MC until it meets the circle at point C, and from that point draw a line to diameter MG and [extend it from point of intersection L] to the circle so that LN = line H [by proposition 20, lemma 2]. Then draw line NG, and [draw] line DN to intersect AG at point Q.

[2.195] Accordingly, since angle DMC = angle DNC, because they are subtended by the same arc, and since angle QNL = angle DAQ [because QNL = DMC, which = DAG, by construction], and since angle NQL = [vertical] angle DQA, then [by Euclid, VI.4] triangle NQL is similar to triangle DQA. Therefore, AQ:QN = AD:NL.

[2.196] But, since angle DMG = angle DNG [because they are subtended by the same arc DCG, then angle] QNG = angle TAQ [because TAQ = DMG, which = DNG]. Let T be the point where DN intersects AB, and [since] angle TQA = [vertical] angle NQG, [then, by Euclid, VI.4] triangle TQA will be similar to triangle NQG, and AQ:QN = TQ:QG. Therefore, TQ:QG = AD:LN [since AQ:QN = AD:LN, by previous conclusions]. But NL = H, and AD:H = E:Z [by construction]. Therefore, TQ:QG = E:Z, which is what was proposed.

[2.197] Furthermore [as demonstrated in proposition 20, lemma 2], it is possible for two lines like CN to be drawn, and in that case two lines can be drawn from D equal to TQ such that each of them is to the segment it cuts off from AG as E is to Z, and the proof will be identical.[73]

[2.198] **[PROPOSITION 25]** With these things established, and given a [convex] spherical mirror, it will be [shown how] to find a point of reflection on it.

[2.199] For instance, let A [figure 5.2.25, p. 239] be a center of sight, B a visible point, and G the center of the sphere [forming the mirror], and draw lines AG and BG. Take the plane within which these two lines lie, and take the [great] circle [that forms the] common [section] of this plane and the mirror. Accordingly, the point of reflection will be found on this circle.

[2.200] Take some other line MK, and divide it at point F such that FM:FK = BG:GA [by Euclid, VI.10]. Bisect MK at point O, and from point O draw perpendicular CO, and from point K draw line KC to CO to form an angle [OCK] with CO that is equal to half of angle BGA. Then, from point F draw line FP to CK, and let it intersect CO at point S so that SP:PK = BG:radius GD [by proposition 24, lemma 6]. From angle BGA cut off an angle equal to angle SPK, i.e., [angle] DGB, and draw lines SK and BD.

[2.201] Accordingly, BG:GD = SP:PK [by construction], and so triangle SPK will be similar to triangle BGD [by Euclid, VI.6, because angle SPK = angle DGB, by construction], and [therefore] angle SKP = angle BDG. But, according to what we established earlier [in proposition 24, lemma 6], we may draw from point F to CK another line like SP [i.e., S'P'] such that it is to the segment it will cut off from CK as SP is to PK [i.e., S'P':P'K = SP:PK], and on that basis a line other than SK [i.e., S'K] will be drawn from point K to OS to form another angle [CKS'] with CK [i.e., other than the original CKS] that is either greater than, or less than angle CKS. If the larger of these angles is not greater than a right angle, no point of reflection will be found [as will be demonstrated below]. Accordingly, let angle CKS be greater than a right angle, and the point [of reflection] is found as follows.

[2.202] Angle BDG will be greater than a right angle [since BDG = CKS, by previous conclusions, and CKS is greater than a right angle, by stipulation]. Draw tangent NDY, and since angle PKO is smaller than a right angle, cut off angle QDG equal to it from angle BDG. Thus, since angle SPK = angle QGD [i.e., DGB, by construction], triangle FPK will be similar to triangle QGD [by Euclid, VI.4, so angle DQG = angle KFP, and so] angle DQB [adjacent to DQG] = angle KFS [adjacent to KFP], and triangle DQB will be similar to triangle KFS [since angle SKP = angle BDG, and angle QDG = angle PKF, leaving remaining angles QBD and FKS equal].

[2.203] Now, extend DQ [beyond Q], and from point B draw perpendicular BZ to it. Accordingly, angle BQZ = angle SFO [since triangles BQD and SKF are similar and BZ and SO are dropped orthogonally to corresponding sides from the corresponding vertices]. Also, right angle BZQ = [right] angle SOF, and so triangle BQZ is similar to triangle SFO [by Euclid, VI.4].

[2.204] Extend DZ to point I, and let ZI = ZD. It is therefore evident that ZQ:QB and QB:QD are as OF:FS and FS:FK, and from this ZD:QD = OK:FK [by Euclid, V.22], so [by Euclid, V.18] ID:QD = MK:FK [since ID and MK are, by construction, twice ZD and FK, respectively], and so [again, by Euclid, V.17] IQ:QD = MF:FK, and IQ:QD = BG:GA [because MF:FK = BG:GA, by construction].

[2.205] Now, draw line BI and [draw line] DL parallel to it. Triangle LDQ will be similar to triangle BQI [by Euclid, VI.4, since angle QBI = alternate angle DLQ, angle BIQ = alternate angle LDQ, and angle BQI = vertical angle DQL], and [so] IQ:QD = IB:DL. And since IZ = ZD [by construction], and since BZ is perpendicular [to ID], BD = BI, so BD:DL = BG:GA [because IQ:QD = BG:GA = IB:DL = BD:DL, by previous conclusions].

[2.206] Now, from point D draw line DH to form an angle [HDL] with line LD that is equal to angle BGA. Then, since HL and DL intersect, the two angles LHD and LDH will sum up to less than two right angles, and so the two angles AGH [which = LDH, by construction] and DHG that are equal to these sum up to less than two right angles, so HD will intersect GA. I say that it will intersect at point A.

[2.207] It is evident that right [angle] GDN = the [sum of the] two angles OCK and OKC [in triangle OKC where KOC is a right angle, by construction, and angle NDG is also right, by construction], and angle OKC = angle GDQ [by construction]. It follows that angle QDN = angle OCK [because angles QDN and QDG sum up to a right angle, as do OCK and OKC, and angle QDG = angle OKC in similar triangles QDG and FPK], so angle QDN = half of angle BGA as well as half of angle HDL [since angle QDN = angle OCK = half of angle BGA, by construction, and angle HDL = angle BGA, by construction]. But angle QDB is half of angle BDL, because BQ:QL = BD:DL, since triangle DLQ is similar to triangle BQI [by previous conclusions], and BD = BI [by previous conclusions]. It therefore follows that angle NDB is half of angle HDB, and so [angle] BDN = [angle] NDH. It follows, moreover, that [when radius GD is extended beyond the circle to E, angle] BDE = angle HDG [because NDE and NDG are both right angles, and NDH = NDB, so NDE – NDB = BDE = NDG – NDH = HDG]. But angle HDG = vertical angle EDA, so [angle] BDE = [angle] EDA, and so D is the point of reflection. I say this is so if HD intersects AG at point A, which will be demonstrated as follows.

[2.208] Draw line HT parallel to BD. It is clear that angle BDE = angle HDG [by previous conclusions]. But [angle] BDE = [alternate] angle HTD, so [triangle HDT is isosceles, and so] HT = HD. But [given that HT is parallel to base BD of triangle BDG, triangles BDG and HTG are similar, so], BD:HT = BG:GH, as Euclid demonstrates [in VI.2]. Thus, BD:DH [which = HT] = BG:GH. But, when it is extended, HD will intersect GA, and it will form a triangle [HGA] similar to triangle HDL [by Euclid, VI.4], since it has angle LHD in common [with triangle HDL], and since angle HDL = angle HGA [by construction]. Therefore, HD is to DL as HG is to the line [X] that HD cuts from GA [i.e., HD:DL = HG:X]. But BD:DL is compounded from BD:DH and DH:DL [i.e., BD:DL = (BD:DH):(DH:DL)]. It is therefore compounded from BG:GH and GH:[X, i.e.,] the line HD cuts off from GA [i.e., BD:DL = (BG:GH):(GH:X)].[74] But BD:DL = BG:GA [by construction]. Therefore, BG:GA is compounded from BG:GH and GH:[X, i.e.,] the line HD cuts from GA [i.e., BG:GA = (BG:GH):(GH:X)]. But it is compounded from BG:GH and GH:GA [so GA = X]. Therefore, GA is the line HD cuts off from GA, and so it will intersect it at point A, which is what was proposed.

[2.209] If, however, angle CKS is not greater than a right angle, I say that reflection will not occur to the center of sight from any point on the mirror.

[2.210] For if it is claimed that [such reflection] can [occur], let D be the point of reflection, and draw line AD to point H on diameter BG. Make angle LDH equal to angle AGB, draw tangent NDY, and make angle QDN equal to half of angle AGB.

[2.211] It is evident [from previous conclusions] that triangle HDL is similar to triangle HGA, so DH:DL = HG:GA. But BD:DH = BG:GH, which will be evident from the fact that HT is parallel to BD [by construction]. So BD:DL = BG:GA. However, since angle BDE = [alternate] angle HDG, angle BDN will be half of angle BDH [by previous conclusions]. But [angle] NDQ is half of angle HDL [by previous conclusions]. Therefore, [angle] BDQ is half of angle BDL [by previous conclusions], so BQ:QL = BD:DL [by Euclid, VI.3].

[2.212] From point B draw BI parallel to DL, and let DQ intersect it at point I. Bisect DI at point Z, and draw BZ. Triangle BQI will be similar to triangle QDL [by Euclid, VI.4]. Therefore, BQ:QL = BI:DL, and so BI = BD [by previous conclusions]. But IQ:QD = MF:FK [by previous conclusions], so ID:QD = MK:FK [by Euclid, V.18], DZ:QD = OK:FK [by Euclid, V.17, since OK = half MK, and IZ = half ID], and ZQ:QD = OF:FK [by Euclid, V.17].

[2.213] It is evident that BZ is perpendicular [to IQ]. Extend it until it intersects DG at point X, which is possible, because angle DZX is a right angle, while ZDX is less than a right angle. And it is obvious that BG:GD = SP:PK [by construction]. Therefore, since it is claimed that angle CKS is not greater than a right angle, I say that an angle greater than a right angle will be formed at point K by a line that intersects CO at the point from which a line passing to CK through point F is to the segment of CK [cut off by it] as BG:GD [i.e., SP:PK = BG:GD].[75]

[2.214] For example, it is clear that, since angle QDN = angle KCO [by previous conclusions], angle QDG = angle CKO. Accordingly, at point K form an angle equal to BDQ [i.e., SKO], and suppose that the line forming this angle intersects CO at point S, and draw SFP. It is evident that, since right angle BZD = [right] angle SOK, triangle BZD will be similar to [triangle] SOK, and BZ:BD = OS:SK. But QZ:QD = OF:FK [by previous conclusions]. Hence, angle ZBQ = angle OSF, and angle QBD = angle FSK, so triangle BGD is similar to triangle SPK. Therefore, SP:PK = BG:GD, which is what was proposed.[76]

[2.215] Furthermore, it is impossible for two angles to be erected on MO such that both of them are larger than a right angle. For if both such angles were larger than a right angle, then, since an angle equal to angle SKM may be formed upon the same center, another angle different from this one will be formed upon the same center which will form on KM another line like SK. And so reflection will occur from point D as well as from some other point on this circle, which is impossible, since it has already been demonstrated [in proposition 16 above] that for any one center of sight there is [only] one point of reflection, and it has now been shown how to find it.[77]

[2.216] Given two eyes, even though there are two points of reflection, there will nonetheless be a single image according to sense-deduction, and there will be a single image-location.[78] We will demonstrate this on the assumption that the two lines extending from the centers of the eyes to the center of the circle are equal.

[2.217] Now, if the location of the visible point is the same with respect to both eyes, so that the lines [extending] from the visible point to the centers of the eyes are equal, the proof will be simple, because the visual axes cut an arc of reflection on the circle, and they form equal angles with the line extending from the visible point to the center of the sphere [i.e., the normal], and the arcs lying between this line and the visual axes are equal. And if the points [of reflection] are selected according to the previous proof [in proposition 25 above], the arcs on the circle lying between these two points and the point on the circle that lies on the normal extended from the visible point [to the circle's center] will be equal, which will be easily demonstrated by repeating the preceding demonstration.

[2.218] And this is the case whether the points of reflection lie in the same plane of reflection, or in different ones; those arcs will still be equal, the lines extending from the centers of the eyes to the points of reflection will be equal, and the lines [extending] from the visible point to the same points will be equal. In addition, the lines extending from the centers of the eyes to the points of reflection will necessarily intersect one another, and the proof is obvious that the intersection will be at the same point on the normal dropped from the visible point, and at this point one and the same image will appear to both eyes, which is what was proposed.[79]

[2.219] Furthermore, the arrangement of images is the same as the arrangement of the visible points [producing them]. For if a line [forming a cross-section] is taken on a visible object, and if two lines are drawn from its endpoints to the center of the sphere, it will form a triangle within which the images of all the points on that [cross-sectional] line will be included. And if there is a point on that line that is identically disposed [with respect to both eyes], the image of a point farther from it will lie on a normal farther from its normal, and one nearer [will appear] on a nearer [normal]. And so, in images [any] portion maintains its [relative] position as it actually exists among the visible points [within that portion].

[2.220] Moreover, given a line on which there is [such] a uniformly disposed point, any point on that line will be uniformly disposed with respect to both eyes, according to the previously discussed way, and it will have a single image because of the equality of angles formed by that line with the visual axes. In addition, if a line is chosen that bisects the angle formed by the two lines from the centers of the eye to the visible point, the location of any point on that line, no matter how far it is extended, will be the same for both eyes, just as was the case for the other line, and the same method of demonstration applies.

[2.221] Aside from these two lines, no other one can be found that maintains the same location, so, when a visible point is perceived on the normal, its image will fall at different points on that normal, but [the two resulting images will lie] an imperceptible distance from each other. Also, the image of any point, no matter how many eyes may view it, always maintains a uniform relative position, so the image appears unified, as has been claimed in the case of direct vision. For, even though they may fall at different places [on the normal], still, because of their insensible distance [from one another], the images do not appear disparate unless their relative positions are disparate. In a similar vein, when the distance of a point from one eye is slightly greater than it is from the other, the image-locations will [only] be imperceptibly separated, so they appear fused, and one [image] is melded from them, in which case the image-locations may be partially rather than wholly distinct.[80]

[Convex Cylindrical Mirrors]

[2.222] In convex cylindrical mirrors, the common section of the plane of reflection and the mirror's surface is sometimes a straight line, sometimes a circle, and sometimes a cylindric section [i.e., an ellipse].[81]

[2.223] When the common section is a straight line, the image-location will lie on the normal extending from the visible point to the surface of the mirror, and that image-location lies just as far [below the mirror] from the common section [i.e., the line of longitude] as the visible point lies [above it]. [This is subject to] the same proof as applies to the plane mirror [in proposition 2 above].

[2.224] However, when the common section is a circle, the image-location will sometimes lie inside the circle, sometimes outside it, and sometimes on the circle itself. This phenomenon has the same explanation as applies in the convex spherical mirror [in propositions 11-15 above].

[2.225] But if the common section is a cylindric section, I say that some of the image-locations lie inside the mirror, some on the mirror's surface, and some outside the mirror. These claims will be explained individually.

[2.226] **[PROPOSITION 26]** Let ABG [figure 5.2.26, p. 241] be a cylindric section, B the point of reflection, E the visible point, and D the center of sight. From point B draw a normal to the plane tangent to the mirror at point B, let it be TBQ, and from point E draw perpendicular EKQ to the plane tangent to the mirror at point K [to form the normal dropped from the object-point]. Let the line tangent to the mirror at point B be CU; let the line tangent to the mirror at point K be KM. I say that the two normals TB and EQ will intersect.

[2.227] Draw lines EB and DB, and draw line KB. It is obvious that KM will fall within figure EKB, and line BC [will fall] within the same figure. Therefore, BC will intersect EK. Let it intersect at point C. It is evident that angle TBK > right angle [CBT], and angle EKB is likewise greater than right angle [EKM], so TB and EK will intersect [since angles BKQ and KBQ, adjacent to angles EKB and TBK are acute]. Let Q be the point of intersection. Likewise, [angle] DBK is greater than a right angle. Therefore, DB and EK will intersect. Let H be the point of intersection. Accordingly, H is the image-location. I say, as well, that EQ:QH = EC:CH, and also that QH > HB.

[2.228] Draw HF parallel to EB. It is clear that angle EBC = angle DBU [because angle of incidence EBT = angle of reflection DBT by construction]. It is therefore equal to angle CBH [which is vertical to angle DBU]. It follows that [angle] EBT = angle HBQ, since TBC is a right angle, and QBC is a right angle [and angle EBT = TBC – EBC, whereas angle HBQ = CBQ – CBH, and EBC = CBH, so the remainders EBT and HBQ are equal]. Thus, since CB bisects angle EBH, EC:CH = EB:BH [by Euclid, VI.3].

[2.229] But angle EBT = [alternate] angle HFB [since HF and EB are parallel, by construction], so HF and HB are equal [insofar as they form equal angles with FB]. But EB:HF = EQ:QH [because, by Euclid, VI.4, triangles EBQ and HQF are similar, since angle QHF = alternate angle QEB, and angle HFQ = alternate angle EBQ]. Accordingly [since EC:CH = EB:BH = EB:HF = EQ:QH, by previous conclusions], EC:CH = EQ:QH, which is what was proposed.[82] On this basis, moreover, because EQ:QH = EB:BH, and because EQ > EB, QH > HB, which is [also] what was proposed.

[2.230] From this it is evident that, if a perpendicular is dropped to a plane tangent to segment GB [of the mirror], it will intersect TB. By the same token, any [perpendicular] dropped to segment AB [of the mirror] will intersect TB. And these conclusions are evident when the visible point does not lie on the visual axis. For [when it does] it is clear from earlier discussions [in proposition 8 above] that the form of one single point reaches the mirror orthogonally and is reflected back along the same line, and this point [whose form reaches the mirror along the] normal lies on the surface of the eye, for a point taken outside the eye cannot be reflected along this normal because it cannot reach the mirror along this normal according to the aforementioned reasoning [i.e., that the body of the eye gets in the way].

By the same token, it could not be reflected from a point on the mirror other than from a point on the normal, because [in that case] two normals would happen to intersect and form a triangle with two right angles, just as was shown above [in proposition 8].

[2.231] **[PROPOSITION 27]** Furthermore, take [some] cylindric section [figure 5.2.27, p. 242], select point A on it, draw tangent [E]AT to the section [at point A], and within the [section on the] mirror take DA perpendicular to [E]AT.

[2.232] It is obvious that AD divides the [cylindric] section into two portions, in either of which there is a single point to which the tangent will be parallel to AD. Accordingly, let G [be a point in one of the two portions of the section] to which the tangent intersects AD at point H, and to this tangent draw perpendicular QG, which will necessarily intersect HD, as was shown in the preceding figure [i.e., 5.2.26]. Let D be the point of intersection, draw line GA to P, and draw line QA. Accordingly, angle QAH is equal to, greater than, or smaller than angle HAP.

[2.233] Let it be equal. The form of point Q will therefore reach A and will be reflected to P, which is the center of sight, and the image-location will be a point on the cylindric section, i.e., G [because G is where line of reflection PA intersects normal QD].

[2.334] If, however, some point, such as point F, is taken above point Q, angle FAH < angle HAP. Make it equal to [angle] NAH. [Line of reflection] NA will intersect [normal] GQ inside the cylinder. Let it [do so] at point K. It is evident, then, that the image of point F will lie at point K, and the images of all points beyond Q [will lie] inside the cylinder.

[2.235] But if some point, such as C, is taken between Q and T, angle CAH > angle HAP. Make it equal to HAM. It is clear that [line of reflection] MA will fall on [normal] GQ, but outside the [cylindric] section. Let it [do so] at point O. Thus, the image of point C will lie at point O, and the images of all points lying between T and Q will lie outside the [cylindric] section between T and G.

[2.236] Moreover, if angle QAH < angle HAP, then cut [an angle] from HAP equal to it, and let it be HAN. It is evident that the image of Q will lie at point K, and the images of all points above [Q] will lie within the [cylindric] section. But if point C is taken below [Q] so that angle CAH = angle HAP, the image of C will lie on the [cylindric] section, [the images of] all [points] between C and Q [will lie] inside [the cylinder], and [the images of] all [points] between C and T [will lie] outside [the cylinder].

[2.237] If, however, angle QAH > angle HAP, make [angle] HAM [figure 5.2.27a, p. 242] equal to it. It is clear that MA will intersect the [cylindric] section [at some point beyond A on arc AG]. Let it intersect at point B, draw the tangent to point B, and let it intersect DH at point L. Now, angle DLB will be acute, angle HLB will be obtuse, and, when it intersects HG, LB will form an acute [angle] with it. Draw SB perpendicular to LB at point B. It will intersect HG, it will form an acute angle with it, and its vertical angle will likewise be acute. Let HG intersect QA. Let U be the point of intersection, and it forms an acute angle with QA at point U [i.e., angle HUA is acute], so SB and QU intersect. Let the intersection be at point Z. It is therefore evident that the form of point Z will reach the mirror along ZA, and it is reflected along AM [since QAH = HAM, by construction], and B [will be] the image-location. Moreover, the images of points above Z on line ZS will lie inside the [cylindric] section [because both the new ZAH and MAH will be more acute], whereas [the images] of points below Z [on line ZS] will lie outside the [cylindric] section [because both the new ZAH and the new MAH will be more obtuse], which was what was proposed.[83]

[2.238] **[PROPOSITION 28]** Moreover, reflection occurs to a center of sight from only one point on a cylindrical mirror, as, for example, [the form of] point B [figure 5.2.28, p. 243] is reflected to [point] A from point G. I say that it does not reflect to the same point from any point on the mirror other than from point G.

[2.239] For if the entire axis [CED] of the mirror lies in the plane of reflection ABG, the common section of the mirror's surface and the plane of reflection will be a line of longitude [FGN] on the mirror. And since the center of sight [A], the visible point [B], the point of reflection [G], and the point [E] on the axis where the normal [to the point of reflection] falls [all] lie in the [same] plane of reflection, only one plane can be assumed within which that line of longitude, or the axis, and points A and B lie, so reflection can only occur to A from some point on the line of longitude [FN]. But it has already been demonstrated [in proposition 3] that reflection cannot occur to [point] A from any point other than G on the line of longitude, so in this case reflection occurs to A from only one point on the mirror.[84]

[2.240] But if the plane [of reflection] A'B'G is parallel to the base of the cylinder, the common section [of this plane with the cylinder] will be a circle [GH] parallel to the base. And it has already been demonstrated [in proposition 16] that reflection to [point] A' cannot occur from any other point on that circle. But if reflection were to occur from some other point on the mirror [outside circle GH], the normal dropped from that point would fall orthogonally to the axis [CED], and it would intersect line A'B' at some point [K]. From that point draw a line [KLI] to the axis in the plane parallel to the base of the cylinder. It will of course be orthogonal to the axis, and so two perpendiculars [KE and KI] will form with the axis a triangle [KEI] two of whose angles [KEI and KIE] are right angles, which is impossible. It is therefore evident that in this case B does not reflect to A except from point G.[85]

[2.241] If, however, plane ABG cuts the mirror according to a cylindric section, I say that reflection occurs from point G only.

[2.242] From [the center of sight at] point A [figure 5.2.28a, p. 243] produce a plane parallel to the base of the cylinder, let it be EZI, and likewise from point [of reflection] G produce a plane parallel to the base of the mirror [i.e., GSP] and in that plane draw line TG from the [mirror's] axis [TQ] to point G. This line will therefore be perpendicular to the plane tangent to the mirror at point G. Let it intersect [line] AB at point K, and from point G draw GZ, the line of longitude on the mirror, and let TQ be the axis. Then, from [object-]point B draw BH perpendicular to plane EZI, and draw lines AZ and HZ. In that same plane [i.e., EZI] draw line ZQ from Z to the axis. It will be perpendicular to the axis, since the axis is perpendicular to this plane [EZI within which it lies], and it will be perpendicular to the plane tangent to the mirror at point Z. Let it also intersect line AH at point L. I say that the form of point H is reflected to A from point Z.

[2.243] From point A draw [line] AM parallel to line KG, and it will intersect BG. Let M be the point of intersection. It is evident that [line] GZ is parallel to line BH, since [by construction] both of them are perpendicular to parallel planes [i.e., EZI and GSP], so line BGM lies in the same plane [BGMH] as these lines [GZ and BH]. Accordingly, the three points M, Z, and H lie in this plane. But AM in turn is parallel to KG, and LZ is parallel to KG, because GZ is parallel to TQ and lies between parallel planes [i.e., EZI and GSP]. Therefore, LZ is parallel to AM, so they lie in the same plane [AHZM], and line AH lies in it too. Hence, the three points M, Z, and H lie in this [same] plane. But it has already been shown that they lie in plane M[G]BH. So these two planes [AHZM and MGBH] intersect along a common line [MZH]. Therefore, HZM is a straight line.

[2.244] Since G is the point of reflection, it is evident that angle [of reflection] AGK = angle [of incidence] KGB, and so angle AGK = angle AMG [which is alternate to angle KGB between parallels AM and KG]. But it is equal to MAG, because it is alternate to it [between the same two parallels]. Hence, [within isosceles triangle MAG] AG and MG are equal. But since GZ is orthogonal to any line within plane AZH, [then, by the Pythagorean Theorem] $MG^2 = MZ^2 + GZ^2$ [in right triangle MZG]. By the same token, $AG^2 = AZ^2 + GZ^2$ [in right triangle AZG]. Hence, AZ = MZ [because AG = MG, by previous conclusions], so angle AMZ = angle ZAM. But [since AM and LZ are parallel, by previous conclusions] angle AMZ is also equal to [alternate] angle LZH, and angle ZAM = angle LZA, since they are alternate angles. Therefore, angle AZL = angle LZH, so, when it reaches point Z, the form of point H is reflected to point A.

[2.245] Hence, if it is claimed that the form of B can be reflected to A from some point [D] other than G, that other point will lie either on line of longitude GZ or on some other [line of longitude]. If it lies on GZ, draw from it a perpendicular [DL′], which will necessarily intersect line AK and will be parallel to line AM [because, by supposition, it must lie in a plane of reflection that includes points A and B, and it must be normal to the mirror]. In addition, the line extending from point B to that point [D] will necessarily intersect line AM [because line AM is in the plane of reflection that includes A and B], and [so] that point and point M will lie in the same plane.

[2.246] Furthermore, that line [BD] will either fall on point M or [will fall] on some other point. If [it falls] on point M, then two [different] straight lines will have been drawn from point B to point M [i.e., BGM and BDM]. On the other hand, if [it falls] to another point on line AM [i.e., N], draw a line from that point to point Z, and it is demonstrated that this line forms a straight line with HZ, as was demonstrated for line ZM. And so two straight lines will have been drawn from point H to pass through point Z and fall at different points on AM, which is impossible.

[2.247] It is therefore evident that [the form of] B can be reflected to A from no point other than G on line GZ. If it is claimed that [it can be so reflected] from a point taken outside this line, draw the line of longitude on the mirror that [lies] on this point, and from the point on circle EZI where this line [of longitude] falls, it is demonstrated that [the form of] H is reflected to A according to the previous proof. But it has already been demonstrated that [the form of] H is reflected to A from point Z, so [the foregoing conclusion] is impossible [by proposition 16 above]. It therefore follows that [the form of] B is reflected to A from only one point on the mirror, which is what was proposed.

[2.248] **[PROPOSITION 29]** Furthermore, given that [the form of] point B is reflected to A, it will be possible to find the point of reflection, and this will be demonstrated by reversing the [previous] proof.

[2.249] From point A [figure 5.2.28a, p. 243] produce a plane parallel to the cylinder’s base, and this plane will cut the cylinder along circle EZI. Draw line BH orthogonal to this plane from point B, and find point Z within this plane from which the reflection of [the form of point] H occurs to A [by proposition 25 above]. From point Z draw line of longitude ZG, and from point Z [draw] perpendicular ZL, to which AM [extended] from point A is parallel. Extend line HZ until it intersects the latter, and let M be the point of intersection. From point M draw a [straight] line to B, and it will necessarily intersect line ZG, since it lies in the same plane with it. Consequently, since BH is parallel to GZ, HZM will lie in the same plane with them, and so MB [will lie] in that same plane, and if it intersects ZG at point G, then point G will be the point of reflection, and you can see this if you reverse the previous demonstration.[86]

[Convex Conical Mirrors]

[2.250] In convex conical mirrors, if the common section of the plane of reflection and the surface of the mirror is a line of longitude on the mirror, the image-location will be just as it was determined in the case of plane mirrors, and the same proof [applies].

[2.251] That the common section cannot be a circle is evident from the fact that the normal [must] fall orthogonally to the plane tangent to the mirror at the point of reflection, and [the plane of] the circle will necessarily be parallel to the [mirror's] base. But no [plane] parallel to [the plane of] the base will be orthogonal to a plane tangent to the mirror.

[2.252] If, however, the common section is a conic section, some images will lie on the surface of the mirror, some inside the mirror, and some outside it. And [all this] is determined by the same method as [was applied] in the case of the convex cylindrical mirror, and the same proof [applies]. Furthermore, just as was shown in the case of the convex cylindrical [mirror], the form of only one point on the surface of the eye is reflected orthogonally along the visual axis to the eye, and it does so from only one point on the mirror, and its image-location is continuous with the other image-locations [surrounding it], as was shown above.

[2.253] It remains [for us] to show that, in these sorts of mirror [i.e., conical convex], reflection may occur from only one point on the mirror, which we will prove as follows.

[2.254] **[PROPOSITION 30]** Let A [figure 5.2.30, p. 244] be the center of sight, B the visible point, G the point of reflection, and on point G produce a plane parallel to the base, a plane that will cut the cone along circle PG. Draw lines AG, BG, and AB, and from point G draw line GT to the center of the circle. Let the vertex [of the cone] be E, and from it draw axis ET. Then [in the plane of ETG] draw perpendicular HG to the plane [containing line CG that is] tangent to the mirror at point G, and, since [normal] HG bisects angle AGB, it will fall upon AB. Let Z be the point where it falls [on AB].

[2.255] From the vertex draw line of longitude EG on the mirror to point G, and draw a line from point A parallel to this line, and this parallel will necessarily intersect the plane of circle GP. Let it be AN, and let it intersect at point N. Likewise, from point B draw a parallel to EG, i.e., BM, and let it intersect the plane [of circle] PG at point M. From point N, as well, draw NF parallel to GT, and draw lines NG, MG, and NM.

[2.256] It is evident that TG will intersect NM. Let it intersect at point Q. It is also evident that MG will intersect NF, since it intersects [line TG] parallel to it. Let F be the point of intersection. Then, from point A draw AL parallel to HZ. It is clear that BG will intersect AL [since A, B, G, and Z lie in the same plane, and AL is parallel to HZ, by construction]. Let L be the [point of] intersection. Then draw GC, which is the common section of the plane [CGE] tangent to the mirror at point G and the plane of circle PG. It is clear that it will be orthogonal to GT, as well as to NF [because NF was constructed parallel to GT].

[2.257] In addition, take GD, which is the common section of the plane tangent [to the mirror at point G] and the plane of reflection [AGBZ], and GD will intersect AL, since it intersects GH [to which AL was constructed parallel]. Let D be the point of intersection, and GD will be perpendicular to AL [since it is perpendicular to ZG, which is perpendicular to plane CGED tangent to the mirror at point G and parallel to AL].

[2.258] It is clear from the foregoing that NF is parallel to GT [by construction], and AL is parallel to GH [by construction]. Therefore, the plane containing NF and AL i.e., FLAN] is parallel to plane G[E]TH. But EG is parallel to BM, so they lie in the same plane, and that plane intersects the aforementioned parallel [planes, i.e., GETH and FLAN], one along line [of longitude] EG, the other along line FL, so FL is parallel to EG. But AN is parallel to that same line [EG, by construction]. Therefore, FL is parallel to AN.

[2.259] However, the plane tangent to the mirror at point G intersects the same parallel planes [i.e., GETH and FLAN], one along line [of longitude] EG, the other along line CD. Therefore, CD is parallel to EG. It is therefore parallel to AN and LF, so AD:DL = NC:CF [by Euclid, VI.1, VI.2, and V.11].

[2.260] It is evident as well that angle [of incidence] BGZ = angle [of reflection] ZGA [by construction], and it also equals [alternate] angle GLA [between parallels HZ and AL], as well as angle GAL [which is alternate to angle AGZ between the same parallels], so [angles] GAL and GLA are equal. In addition [because triangle GAL is isosceles], GA and GL are equal, and GD is perpendicular to AL [by previous conclusions. So] AD = DL. Therefore, NC = CF [since AD:DL = NC:CF, as previously concluded], and GC is perpendicular [to FN. So] angle CFG = angle CNG. Hence, angle NGQ = angle MGQ.[87] Accordingly, [the form of] point M can be reflected to [point] N from point G on circle PG, barring interference from the cone [itself].[88]

[2.261] I say, then, that [the form of] point B is reflected to A only from G. For if it is claimed that it can be reflected from some other point, that point will either lie on line of longitude EG, or it will not.

[2.262] Let it lie on it, let it be X [figure 5.2.30a, p. 245], and from that point draw the normal to the plane tangent to the mirror at that point, and this normal will be parallel to ZG and therefore parallel to AL. AL will therefore lie in the plane of reflection containing this normal, and it will likewise lie in the plane of reflection containing normal ZG. Hence, those two planes of reflection intersect along line AL. But they intersect on point B, which is impossible, because B does not lie on line AL, which is evident from the fact that FL is parallel to BM.[89] It therefore follows that [the form of point] B can be reflected to A from no other point on line EG than G.

[2.263] But if [it were assumed to do so] from some other point [not on line of longitude EG], then let that point be U, draw line of longitude EUO, and take the plane parallel to base [PG of the mirror] passing through point U. It is clear that AN will intersect this plane. Let Y be the point of intersection. BM will intersect the same plane as well. Let K be the point of intersection, and draw lines KU, YU, and YK. Since that plane intersects the cone along a circle passing through U, draw line RU from point U to the center of this circle. Draw lines EK and EY, which [when extended] will intersect the plane of circle PG, and let I and S be the points of intersection. Now, draw lines IO and SO.

[2.264] Thus, just as it has been demonstrated [earlier in this proposition] for point M that, barring interference from the cone, it[s form] can be reflected to N from point G, so it is demonstrable for point K that it[s form] can be reflected to point Y from point U, and the proof is the same. And so angle RUY = angle RUK.

[2.265] It is therefore evident that BK is parallel to EG [by construction], and the common section of plane BGEK and the plane of circle PG is line MG. Therefore, since it lies in that plane [i.e., BGEK] and intersects the plane of circle PG, line EK will fall on common section MG. SMG will therefore be a straight line.

[2.266] By the same token, since plane NYEG intersects the plane of circle PG along line NG, line EY will intersect line NG [at point I]. Thus, ING is a straight line. It is also evident that plane IOE intersects the plane of circle PG along line IO, and it intersects the plane parallel to this one that passes through U along line YU. Therefore, YU is parallel to IO. Likewise, plane SOEK intersects those parallel planes along the two lines SO and KU. Therefore, SO is parallel to KU.

[2.267] Similarly, if the plane cutting the mirror along line of longitude EO and containing R, U, O, and M is taken, it will intersect those parallel planes along the two lines MO and RU. Therefore, these two lines are parallel. Hence, angle SOM = angle KUR, and angle MOI = angle RUY. But it has already been shown that angle KUR = [angle] RUY. Accordingly, angle SOM = angle MOI, so point S can be reflected to I from point O, barring interference from the cone.

[2.268] But it has already been shown [earlier in this proposition] that [the form of] point M can be reflected to I from point G, and so [the form of] point S, which lies on [straight] line SMG, can be reflected to I from point G. Therefore, [the form of] point S is reflected to I from two points on circle PG, which is impossible. It follows, then, that the main [supposition of this proposition], i.e., that [the form of] point B can be reflected to A from some point other than G on the mirror, is impossible, which is what was proposed.

[2.269] **[PROPOSITION 31]** Now, given a [convex] conical mirror, the point of reflection can be found [as follows].

[2.270] For instance, Let G [figure 5.2.31, p. 246] be the vertex of the cone, and at that point produce plane MNG parallel to the base of the cone. Let A be the visible point and B the center of sight. A and B will [both] lie above that plane [i.e., MNG]; or [they will both lie] below it; or [they will both lie] in the plane itself; or one [will lie] above it [while] the other [lies] below it; or one [will lie] in the plane [while] the other [lies] above or below it.

[CASE 1]

[2.271] Let them lie below the plane, and from point A produce a plane that cuts the cone parallel to the base, and from point G to point B draw a line, which, when extended, will fall on the plane produced from A [through the cone], since it lies between parallel planes [i.e., MGN and HEA]. Let H be the point where this line falls [on that plane].

[2.272] Now, it is proven according to the previous method [applied in proposition 30 above] that [the form of] A is reflected to H from some point on the circle formed on the cone by the plane [HAT] produced from points A and H. Let the point of reflection be found on that circle [by proposition 25], and let it be E. Then draw line AB, and draw the cone's line of longitude GE, as well as axis GT of the cone.

[2.273] From point E draw line ET to the center of the circle, and it will fall to the axis, and it will be orthogonal to the plane tangent to that circle at point E. Then, with lines AE and HE drawn in, it will bisect the angle formed by them, and it will divide line AH [in half]. Let R be the point of division.

[2.274] It is evident that GE and ET form a plane that cuts line AB. Let F be the point of intersection, and from point F draw normal FC to line GE, and it will be perpendicular to the plane tangent to the cone along line GE. Then, from point A draw AL parallel to line FC. FC, moreover, will intersect the axis at point K. From point A draw AS parallel to line RT, and from point E draw common section EO of plane AEH and the plane tangent to the cone along GE. It will fall orthogonally to AS, since it is orthogonal to ER [to which AS is parallel, by construction].

[2.275] Draw line BC, which, when extended, necessarily intersects line AL [which is parallel to FC, by construction]. Let the intersection be at point L, and from point C draw common section CP of the plane tangent [to the cone along GE] and plane ABL. Draw lines LS and PO.

[2.276] It is obvious that plane ALS is parallel to plane GEK [because AS is parallel to RT, by construction, AL is parallel to FC, by construction, and FC and RT lie in the same plane], and lines CE and PO lie in the plane tangent [to the cone along GE], that plane intersecting those parallel planes [ALS and GEK] along the two lines CE and PO. Hence, CE is parallel to PO.

[2.277] Furthermore, draw line HE until it intersects AS at point S. It is clear that line ES lies in plane HEG, and BL lies in the same [plane], and this plane cuts the aforementioned parallel planes [ALS and GEK] along the two lines EC and LS. Therefore EC is parallel to LS. Therefore PO will be parallel to LS [since it is parallel to CE], so AO:OS = AP:PL [by Euclid, VI.2].

[2.278] But it is clear that angle [of reflection] HER = angle [of incidence] REA [by construction]. Angle ESA = angle EAS [because angle of reflection REA = alternate angle EAS, and angle of incidence HER = alternate angle ESA], and EO is perpendicular [to AS, so] AO = OS [by Euclid, I.26]. Accordingly AP = PL. And CP is perpendicular to AL, since it is perpendicular to FCK. Thus [triangle CAL is isosceles, so] CL = CA, and angle CLA = angle LAC. Accordingly, angle BCF = angle ACF [because, given that FCK and APL are parallel, angle BCF = alternate angle CLA, and angle ACF = alternate angle LAC]. Therefore, A is reflected to B from point C, which is what was proposed.

[CASE 2]

[2.279] But if the center of sight and the visible point [both] lie in plane MGN [figure 5.2.31a, p. 247], let the former be at point M, the latter at point N, draw lines MG, NG, and MN, and bisect [angle] MGN with line UG. It is clear that [the form of] N is reflected from G to M.[90] It is also clear that line UG and the cone's axis lie in a plane that cuts the cone along a line of longitude [GE].

[2.280] From point U draw UE perpendicular to this line of longitude. Through point E produce the plane parallel to the base [of the cone], and this plane will cut the cone along a circle. Let ET be the common section of plane UEG and this circle. It is evident that it will fall on the axis as well as on the center of the circle.

[2.281] Then, from point M draw a line [MH] parallel to GE, a line that falls at point H on the plane of that circle. Likewise, from point N draw a line [NA] parallel to GE and falling at point A [on the plane of the circle]. Draw AH, and let ET intersect it at point R.

[2.282] It is obvious that MH, which is parallel to GE, lies in the same plane with it, and this plane [MHEG] intersects plane MGN and plane HEA along the two lines MG and HE. Therefore [because planes MGN and HEA are parallel, by construction], MG is parallel to HE. By the same token, AN and GE lie in the plane [ANGE] that cuts those parallel planes [MGN and HEA] along NG and AE. Hence, NG is parallel to AE. Likewise, plane UGE intersects the same planes [MGN and HEA] along the two lines RE and UG. Thus, UG and MG are parallel to HE and RE [in reverse order], so angle MGU = angle HER, angle UGN = angle REA, and angle HER = angle REA. And so [the form of] point A can be reflected to H from point E.

[2.283] Accordingly, if a line is drawn from point A parallel to UE, if another [is drawn] parallel to RE, if ME is extended until it intersects the line parallel to UE, if the common sections are drawn, as before, and if the preceding proof is repeated, it will be clear that [the form of point] N can be reflected to M from point E.[91] E will therefore be the point of reflection, which is what was proposed.

[CASE 3]

[2.284] Now, if both [the center of sight and the visible point] lie above MGN [figure 5.2.31b, p. 247], construct the cone opposite the original one. To do so, extend the lines of longitude of the previously constructed cone, and through point A pass a plane that cuts this latter cone parallel to the base, and it will cut the cone along circle YZ.

[2.285] B will either lie in this plane, or it will not. If it does, then carry out the procedure from point B. If not, then extend line GB until it intersects this plane. Let the intersection be at point D. It is evident that A is reflected to D from some point inside circle YZ.[92] Find that point (as we will later prove and explain [in proposition 38 below], it is not among those on the anterior [surface of the cone]), and let it be Z. Lines DZ and AZ will be drawn, and let line PZ bisect the angle [formed by them].

[2.286] Extend line ZG to the other cone, and it will reach its surface to form a line of longitude [on it]. Let it be line ZGE. It is obvious that plane PZE will intersect line AB. Let it intersect at point Q, and from point Q draw a perpendicular to line GE, and let it fall at point E.[93] It will also be perpendicular to the plane tangent to the cone along line GE. Through point E produce plane A'EH parallel to the base [of the cone], and from point D draw line DH parallel to ZE and intersecting that plane at point H. Let A'A be parallel to that same line.

[2.287] It is clear that DH is parallel to ZE, and they lie in the same plane [DHEZ], which intersects the parallel planes [A'EH and AZD] along the two lines DZ and HE. Accordingly, HE and DZ are parallel. Likewise, AZ and A'E are parallel. And it is evident that PZ passes through the center of circle YZ, and so does RET [pass] through the center of the other circle along which plane A'EH cuts the cone. Therefore, plane PZER intersects the two parallel planes [AZD and A'EH] along the two lines PZ and RE. PZ, then, is parallel to RE, so angle AZP = angle A'ER. And therefore angle A'ER = angle REH, so [the form of] A' is reflected to H from point E.

[2.288] Accordingly, if a line is drawn from point A' parallel to QE, if another [line is drawn] parallel to RE, and if the common sections [are drawn] as before, and if we repeat the previous method of proof, it will be evident that [the form of] point A is reflected to B from point E, which is what was proposed.[94]

[CASE 4]

[2.289] If, however, the center of sight lies in the plane parallel to the [cone's] base and passing through its vertex, i.e., [point] G, and if the visible point lies below this plane, the point of reflection will be found in the following way.

[2.290] Let M [figure 5.2.31c, p. 248] be the center of sight, A the visible point, and MGN the plane parallel to the base of the cone. From point A produce a plane parallel to the base of the cone, and it will cut the cone along circle DEK with centerpoint T. From point M draw MH perpendicular to this plane, and draw line HT. From point A draw line AEQ to HT within the circle such that EQ = QT, according to the earlier account [in proposition 23, lemma 5]. Then, draw line TEI, and from point H draw HB parallel and equal to TE. Draw lines MB and BE. It is evident that plane GTE will intersect line AM. Let F be the point of intersection, and from point F draw FOC perpendicular to line GE and intersecting it at point O. Then, draw lines MO and AO. I say that O is the point of reflection.

[2.291] It is clear that HB is parallel and equal to TE [by construction]. Therefore, HT is parallel and equal to BE [since they are cut by equal and parallel lines HB and TE so as to form a parallelogram]. But MH is parallel and equal to GT, since both are perpendicular [to parallel planes DEK and MGN]. Accordingly, HT is parallel and equal to MG. Therefore, MG is parallel and equal to BE [since BE is parallel and equal to HT, by previous conclusions], so MB is parallel and equal to GE.

[2.292] It is also clear that angle QTE = angle QET [because QT = EQ, by construction, so triangle EQT is isosceles], and so it is equal to angle AEI [which is the vertical angle of QET]. But it is [also] equal to angle IEB [which is alternate to QTE]. Thus, [angle] IEB = angle IEA, so [the form of] A is reflected to B from point E. And since MB is parallel to GE, if a line parallel to FOC, as well as to TE, is drawn from point A, and if the earlier figure and proof [based on figure 5.2.31, p. 282] are repeated, it is clear that A is reflected to M from point O, and so [we have done] what was proposed.

[CASE 5]

[2.293] Now, if M lies in the plane [of MGN], and A lies above that plane, then the cone opposite the original one will be produced. Pass a plane through A that [cuts the opposite cone] parallel to its base, and the point of reflection is found among points lying inside the circle formed by this plane. Draw the line to G from that point, and extend it. And the point of reflection will be found according to subsequent analysis [in proposition 38 below], and the same method of proof will apply.[95]

[CASE 6]

[2.294] But if the [relevant] points, i.e., the center of sight and the visible point, are so disposed that one of them lies above the plane of the [cone's] vertex and the other below it, let one of them be L [figure 5.2.31e, p. 249] and the other A, and let the plane at the vertex be MGN.

[2.295] Through point A extend a plane parallel to the base of the cone and intersecting it along circle DE with center T, and draw line LG. It will intersect plane AED. Let K be the [point of] intersection, and in circle DE find point E such that tangent SE drawn from that point bisects the angle formed by lines KE and AE [by proposition 22, lemma 4].

[2.296] Then, from point L draw a line [LB] parallel to [line of longitude] GE, a line that will necessarily intersect line KE. Let B be the intersection. It is clear that L lies in plane GEK, and LB, being parallel to GE, lies in that same plane. Draw line TEI. It is evident that plane GTE intersects line LA. Let it intersect at point U, and from there draw normal UOC to the plane tangent [to the cone at point O]. Finally, draw lines AO and LO.

[2.297] It is clear that [angle] AES = angle SEK [by construction], and since angle IES is a right angle, and [angle] SET is a right angle, [angle] IEA [which = IES + AES] = angle TEK [which = SET + SEK]. And therefore angle AEI = angle IEB [from which it follows that angle of incidence AET = angle of reflection TEB], so [the form of] A is reflected to B from point E. If, then, a line parallel to UO and a line parallel to IT are drawn from point A, and if the proof [from the previous cases] is repeated, it will be clear that [the form of] A is reflected from point O to L, and so [we have done] what was proposed.[96]

[2.298] It is therefore clear how the point of reflection can be found, and what has been discussed must be understood to apply to a single center of sight. But in [the case of] both eyes, the same thing happens, because the same form and the same location for the form is perceived by each eye, and, as has been claimed in the case of the convex spherical mirror, the forms perceived by both eyes in these sorts of mirror [i.e., conical convex] appear identical because of their proximity; sometimes they share precisely the same [image-]location, sometimes their [image-]locations overlap, and sometimes they are separated, but only a little bit.

[2.299] Furthermore, [any] form that reaches these mirrors along the normal returns along the same line, just as was shown earlier, and the form that is perceived by one eye along the perpendicular is perceived by the other eye along a line of reflection [oblique to that perpendicular]. Nevertheless, the locations of those forms are continuous [with one another], so the form appears identical to both eyes.

[Concave Spherical Mirrors]

[2.300] **[PROPOSITION 32]** In the case of spherical concave mirrors, the normal dropped from the visible point [to the mirror] sometimes intersects the line of reflection, and sometimes it is parallel to it. When it intersects, the image-location will sometimes lie on the [surface of the] mirror, sometimes behind the mirror, and sometimes in front of it. And when the image-location lies in front of the mirror, it will sometimes lie between the center of sight and the mirror, sometimes at the center of sight [itself], and sometimes beyond the center of sight. And we will demonstrate this [as follows].

[2.301] Let A [figure 5.2.32, p. 250] be the center of sight and D the center of the mirror, and through these points produce a plane that will cut the mirror along [great] circle HBFG. This plane will be a plane of reflection, because it is orthogonal to any plane that is tangent to the [great] circle [on the sphere]. Draw line AD, and from point A to the circle draw line AE, which is longer than AD. From point D to the circle draw [line] DH parallel to line AE, extend AD to points B and I [on the circle], and draw line DE.

[2.302] It is evident that angle AED is less than a right angle, because ED is a radius, and [by Euclid, III.31] any line in a circle forms an acute angle with the radius [or any other segment of the diameter]. At point E form angle DET equal to angle AED. It is clear that ET will fall inside the circle [since DET is acute] and will intersect line DH. Let T be the point of intersection. It is also clear that angle ADE > angle DET, so ET will intersect AB. Let it intersect at point Z.

[2.303] Then, from point A draw line AN to arc EH, draw line DN, and at point N form with line NM an angle [DNM] equal to angle DNA, and this line [NM] will necessarily fall inside the circle [because it cannot be tangent at N] and will intersect DH. Let it intersect at point M. It is clear, as well, that AN will intersect DH outside the circle [since DAN and ADH sum up to less than two right angles]. Let L be the intersection.

[2.304] In addition, draw line AG from point A to arc EF, draw DG, and let angle AGD = angle DGQ. It is obvious that QG will intersect DH. Let Q be the point of intersection. It is also obvious that AG will intersect DH on the side of F. Let O be the intersection. Moreover, since the arc [GY] that subtends GO within the circle is greater than arc GH, it is evident that GQ falls between D and H. For if line GH is drawn, angle HGD[P] will be subtended by a greater arc [i.e., HP] than angle [P]DGA[Y, i.e., arc PY].

[2.305] Furthermore, from point A draw line AC to arc FB so as to cut DH at point S in such a way that CS > SD, and draw DC. It is clear that angle DCA is acute. Form angle DCK equal to it. Since angle CDS > angle DCS, it is evident that CK will intersect DH. Let K be the point of intersection.

[2.306] According to the foregoing construction [making angle DET = angle AED], it is clear that [the form of] point T propagates to E and is reflected to A. TD is the normal dropped from point T, and this line, which is perpendicular to the plane tangent to the circle [at H], is [by construction] parallel to AE, the line of reflection, so it will not intersect it.

[2.307] [The form of] point Z, on the other hand, propagates to E and is reflected to A. AZ is the normal dropped from point Z, and it intersects AE at point A, so the image-location for point Z will be [at center of sight] A.

[2.308] On the other hand, [the form of] point M propagates to N and is reflected to A. Normal MD dropped from point M intersects [line of reflection] AN at point L, which lies behind the mirror, and the image-location for point M will be L.

[2.309] The form of point Q, however, propagates to G and is reflected to A, and its image-location [where line of reflection AG intersects normal QD] will be O, which lies beyond the center of sight.

[2.310] Finally, the form of point K propagates to C and is reflected to A, and the normal [dropped] from it is KD, and [so] its image-location is S [where KD intersects line of reflection CA].

[2.311] From the foregoing it is thus clear that some of the images lie behind the mirror, some between the center of sight and the mirror, some at the center of sight itself, and some beyond the center of sight, which is what was proposed.

[2.312] It is evident, moreover, that the visual faculty grasps forms that face it, so, when the image-location lies behind the mirror or between the eye and the mirror, the image is grasped according to how it is actually located. However, when the normal dropped from the visible point is parallel to the line of reflection, the image will appear at the point of reflection. For, since that [visible] point is [a] sensible [spot] represented by the point imagined [to lie] at its center, the image of any portion of that sensible spot taken beyond the midpoint [toward the mirror] will lie behind the mirror, whereas the image of the portion in front of the midpoint [away from the mirror] will lie between the eye and the mirror, and since the whole form appears as a continuum composed of the parts behind and the parts in front [of the mirror], the form of that sensible spot will necessarily appear in an image-location on the mirror itself.[97]

[2.313] But in the case of images that are located at the center of sight, they are not grasped according to how they are actually located, so [visual] deception often occurs in these sorts of mirrors [i.e., spherical concave]. To make this clear, stand a wooden rod less than half the radius of the mirror in length upright upon the surface of the mirror [along a normal]. Let the center of sight be situated directly above this rod, and look at a spot on the mirror that lies farther from the rod than the center of sight does [from the mirror] along the normal passing through the rod. The image of this rod will appear behind the object itself, but it will not be perceived properly; on the contrary, it will appear bowed when it is not.[98] Hence, in these sorts of mirrors the image is seen according to how it is actually located only when the image-location lies behind the mirror or between the eye and the mirror. But when the center of sight lies on the normal passing through the rod, it does not perceive the form of that rod clearly.

[2.314] On the other hand, if the center of sight is located on a diameter of the sphere and at its center, then, since every line dropped from it to the mirror is perpendicular to the mirror, the form of no point will be perceived except a point within the portion of the circle lying between the edges of the visual cone that is imagined to extend [to the mirror's surface] from the center of the circle [where the center of sight is located]. For the form of any other point will fall on the mirror along an oblique line, and it is necessarily reflected along an oblique line, so the line of reflection will not pass through the center, and so it will not reach the center of sight.[99]

[2.315] However, if the center of sight lies on a diameter but not at the center, it will not perceive the form of any point on the radius within which it lies. For the angle that the two lines from the given point on the radius to the center of sight will form at the same point on the mirror will not be bisected by the normal extended from that point on the mirror, since that normal is directed to the center of the mirror. But it can perceive the form of any point on [a] radius other [than the one opposite it on the diameter upon which it lies].[100]

[2.316] **[PROPOSITION 33]** Now, when a point is viewed in this sort of mirror, if the normal is not parallel to the line of reflection, the line extended from the center of the mirror to the visible point will have to the line extended from that same centerpoint to the image-location the same ratio that the line drawn from the visible point to the point we have called the [end]point of tangency has to the line extended from the [end]point of tangency to the image-location.

[2.317] For example, let E [figure 5.2.33, p. 251] be the center of the mirror, B the visible point, A the center of sight, G the point of reflection, and ZG the line of tangency. Either ZG will intersect EB, or it will be parallel to it.

[2.318] Let it intersect at point T. Line EB will intersect AG, but not at point G, since EB and BG are two [distinct] lines. Therefore, they will intersect behind G, or between G and A, or at A, or in front of A. Let [the intersection be] behind G, at point H. Accordingly, I say that EB:EH = BT:TH.[101]

[2.319] Draw normal EG, and from point H draw [line HL] parallel to line BG, so it will intersect EG. Let L be the intersection, and from point B draw [line BQ] parallel to GH, so it will necessarily intersect ZT. Let Q be the intersection.

[2.320] It is evident that angle BGE = angle AGE [by construction]. But angle BGE = [alternate] angle GLH, and angle AGE = [vertical] angle LGH [so triangle LGH is isosceles]. Therefore, LH = GH. Likewise, angle BGQ = [angle] AGZ [by construction], and angle AGZ = [alternate] angle GQB [making triangle GBQ isosceles], and so BQ = BG, from which it follows that BG:HL = BQ:HG [because BG = BQ, and HL = HG].

[2.321] But since angle GHT = [alternate] angle TBQ, triangle TBQ will be similar to triangle GHT [because they have two equal corresponding angles, i.e., angles TBQ and GHT and angle BTQ and vertical angle GTH]. Thus, QB:HG = BT:TH [by Euclid, VI.4], and so BG:HL = BT:TH [because we have established that BG:HL = QB:HG, and QB:HG = BT:TH]. But since triangle BGE is similar to triangle HEL [HL and GB being parallel, by construction], then BG:HL = EB:EH, and so EB:EH = BT:TH, which is what was proposed.

[2.322] The same proof will apply if the image-location [H] lies between A and G [figure 5.2.33a, p. 251], at A [figure 5.2.33b], or beyond A [figure 5.2.33c, p. 252].

[2.323] If, however, line of tangency ZG is parallel to normal BH [figure 5.2.33d, p. 252], draw perpendicular GE, which, since it is perpendicular to ZG, will be perpendicular to BH. And [right] angle BEG = [right] angle HEG, and angle BGE = angle EGH. It follows that triangle BGE is similar to triangle EGH. Therefore, BE:EH = BG:GH, which is what was proposed, because in this case, no [end]point of tangency other than G can be assumed, according to the way we defined the [end]point of tangency earlier [in proposition 6, where the endpoint of tangency lies at the intersection of the normal dropped from the object-point and the line tangent to the point of reflection].

[2.324] **[PROPOSITION 34]** Now, let DGT [figure 5.2.34, p. 253] be the [great] circle [cut from the mirror by the plane of reflection], A the center of sight inside the circle, E the center of the mirror, and B the visible point. Draw diameter DAG.

[2.325] If B lies on radius EG, there can be reflection from some point on semicircle GTD as well as from some point on the semicircle opposite [and below] it. For if a line is drawn to some point [F] on semicircle GTD from some selected point [B] on radius EG, and if another line is drawn from point A to the same point [F on semicircle GTD], those two lines will form an angle that the radius [EF] extended from point E will bisect at that point. The same holds in the case of the opposite semicircle.

[2.326] But if B lies outside diameter DAG [figure 5.2.34a, p. 253], draw the diameter passing through B, and let it be TBQ. I say that [the form of] B can be reflected to A from the arc lying between the radii in which A and B lie, and likewise from the arc opposite it, i.e., from arc TD and from arc GQ, but it cannot be reflected from any point on arc GT or arc QD.

[2.327] For instance, take some [potential] point [of reflection] K on arc GT, and draw lines AK and KB until KB intersects diameter DG at point O. Since O and A lie on the same side of E, which is the center of the circle, the normal dropped from point K to E will not divide angle OKA, and so [the form of] B is not reflected to A from point K. Likewise, if another [potential] point [of reflection] F is chosen [on that same arc], it will be obvious that normal EF will not divide angle AFB, and so [the form of] B is not reflected to A from point F.

[2.328] That reflection can, however, take place from a point on arc TD or on arc GQ is clear from the following. Let M be a [potential] point [of reflection] on arc DT, and draw lines AM and MB to form quadrilateral AMBE. Therefore, normal EM will divide angle AMB.

[2.329] By the same token, let H be a [potential] point [of reflection] on arc GQ. Line AH will intersect diameter TQ at point C, and line HB [will intersect] the same [diameter] at point B. And these two points lie on different sides of the centerpoint [E], so line EH will divide that angle.

[2.330] Likewise, if B lies on the surface of the mirror, or outside the mirror, as long as A lies inside the mirror, the same method of proof will apply as before. And the same holds if A lies on the surface of the mirror, and B lies inside or outside it.[102]

[2.331] But if AP is drawn from A parallel to TE [figure 5.2.34b, p. 254], the locations of images reflected from points [such as M] on arc TP will lie outside [and behind] the mirror, whereas the locations of images [reflected from points such as N] within arc PD will lie [at points, such as S] beyond the center of sight A, and the locations of images [reflected from points such as H] within arc QG lie [at points, such as X] between the center of sight and the mirror. And the same thing that has been just said about image-locations must be understood [to apply] when AM is drawn parallel to line TQ.

[2.332] On the other hand, if A lies outside the mirror and B inside it [figure 5.2.34c, p. 254], what we claimed will be evident [as follows]. From point A draw lines AH and AZ tangent to circle GTD, draw the two diameters AEG and TEQ, and let B lie on diameter TEQ. [The form of] B is reflected to A from some point on arc TD, but it is obvious that [it is] not [reflected] from any point on arc ZD. Therefore, [it is reflected] from some point on arc TZ, and likewise from some point on arc GQ opposite TD. However, according to the previous method [of demonstration], reflection will not occur from arc TG or [arc] DQ.

[2.333] If, however, B lies outside this diameter [TEQ] and on another, which is likewise [designated] T'EQ', reflection will occur from arc T'D, but only within portion T'Z on it, and [it will occur] from GQ', the arc opposite it. Reflection will not occur, however, from arc T'G or [arc] DQ'.

[2.334] **[PROPOSITION 35]** Furthermore, if a diameter is taken on a [great] circle within a spherical concave mirror, any point on that diameter, no matter how far it is extended, can be an image-location.

[2.335] For instance, let AG [figure 5.2.35, p. 255] be a diameter of circle AMG, whose center is D. Take point Z on this diameter, [and take] E [as] the center of sight. I say that Z can be an image-location.

[2.336] For instance, draw line ETZ, T being a point on the circle. Draw line DT. Angle ETD will be acute. Form [angle] DTL equal to it. It is clear [from the equality of angles ETD and DTL] that [the form of] L is reflected to E from point T and that its image will be Z.

[2.337] Similarly, if point L is selected, it will be clear that it is an image-location. Draw line EL to point B on the circle, and draw line DB. Angle EBD will be acute. Form [angle] DBC equal to it. [Accordingly, the form of] point C is reflected to E from point B, and its image-location will be L, and so whatever other point is chosen, the same proof will hold.

[2.338] Now, among the points that are perceived in these sorts of mirrors, some of their images are distributed in four locations, some in three, some in two, and some in [only] one. A point whose image lies at four locations is reflected from four distinct points, and from no others nor from more [than four]. A point whose image occupies three locations is reflected from three points on the mirror, and from no more than three; one whose [image occupies] two [locations is reflected] from two points [only]; whereas one whose image lies in a single location may have its reflection occur from one point only, and it may [have it occur] from some point on a specific [great] circle, but from no other [point within the mirror].

[2.339] **[PROPOSITION 36]** For example, let E [figure 5.2.36, p. 255] be the center of sight, let H be a visible point on the same diameter, and let D be the center of the circle. Draw diameter ZEHA. ED is either equal to DH, or not.

[CASE 1]

[2.340] Let it be equal, and from point D on EH draw diameter GDB perpendicular [to EH], and draw lines HG, GE, HB, and BE. It is evident that triangle HGD = triangle EGD, and it equals triangle HBD, as well as triangle EBD. Since angle HGE is bisected [by diameter GDB], it is clear that [the form of point] H is reflected from point G to E, and its image-location is E [where line of reflection GE intersects normal HD dropped from visible point H]. Likewise, [the form of point] H is reflected to E from point B, and its image-location is E.

[2.341] Therefore, if diameter ZEHA remains stationary and semicircle AGZ, or only triangle HGE, is rotated [around AZ, as an axis] throughout [the circumference of] the sphere, point G will describe a circle in the course of its motion, and from any point on that circle [the form of point] H is reflected to [point] E, and its image-location will always be point E, and so [we have demonstrated] what was proposed.

[2.342] That reflection of [the form of] point H to E cannot take place from any point other than one on that circle [of revolution] is evident from the following. Take point C. Line EC > line EG, and line HC < line HG, so EC:HC ≠ ED:DH [which means that angle ECD ≠ angle DCH, by Euclid, VI.3]. Therefore, line DC will not bisect angle ECH, so [the form of point] H cannot be reflected to E from point C. The same disproof will apply if C is taken between G and Z.

[CASE 2]

[2.343] But if ED > DH, adjust the figure [accordingly], and add line HQ to line EH [figure 5.2.36a, p. 256] so that EQ,QH = DQ^2. Accordingly EQ:DQ = DQ:HQ [by Euclid, VI.17], so EQ:DQ = ED:DH, as Euclid demonstrates [in V.19].[103]

[2.344] Produce a circle with a radius of QD having its center at Q, let G and B be the points where the two circles intersect, and draw lines EG, EB, QG, QB, DG, DB, HG, and HB. Accordingly, it is evident that EQ:QG = QG:QH [because radius QD = radius QG, and, as established earlier, EQ:QD = QD:QH], and angle GQH is common to both triangles EQG and HQG [whereas angles QHG and QGE are right]. Therefore, those two triangles are similar. So [by Euclid, VI.4] EQ:QG = EG:GH. Therefore, ED:DH = EG:GH [since we established that EQ:QG = EQ:QD = ED:DH], so line DG will bisect angle EGH [by Euclid, VI.3].

[2.345] Consequently, [the form of] point H is reflected to E from point G, and its image-location is point E. Likewise, [the form of point] H is reflected from point B to E, and its image-location is E.

[2.346] Hence, if points E and H are held stationary and triangle EHG is rotated [around QE, as axis], point G will describe within the sphere a circle from any point of which [the form of] H is reflected to E, and E will always be the image-location.

[2.347] Moreover, as before, it is clear that [the form of point] H cannot be reflected to point E from any point [within the sphere] other than one on that circle. For if C is taken between G and A, EC > EG, and HC < HG, so EC:HC ≠ ED:DH, and so [by Euclid, VI.3] DC does not bisect angle ECH. Likewise, if C is taken between G and Z, [then the supposition that reflection can occur from C] can be disproved.

[2.348] And so what was set out [has been demonstrated], as long as it is borne in mind that E is an imaginary point, and the circle [of revolution] with E as pole is an imaginary circle, and H is an imaginary point. Therefore, what has been claimed according to geometrical demonstration is not to be understood according to ocular proof, because imaginary objects are invisible to sight. But since the form of H appears continuous with the forms of other points [in close proximity within the visible spot represented by H], the form that sight will see is a form whose midpoint is H, and the midpoint of that form [when reflected to E] will be E, and this form will be reflected from a circular segment of the mirror within which the aforementioned circle [created by the rotation of G] will represent the midline with E as its pole.

[2.349] Moreover, if ED > DH, insofar as it can be enough longer that [the form of point] H does not reflect to E from point G, it must be understood that, unless EA:AH > ED:DH, [the form of] H cannot be reflected to E.

[2.350] For if it could be reflected, let it be reflected from point G. Angle GDH will be less than a right angle, since it is subtended by an arc less than a quarter of the circle. From point G draw the tangent, which will necessarily intersect EA. Let Q′ be the intersection. By the [33d proposition[104] we know that] EQ′:Q′H = ED:DH, but EA:AH > EQ′:Q′H. Therefore, EA:AH > ED:DH, and so, by necessity, if [the form of point] H is reflected to E, EA:AH > ED:DH.[105] The claims that have been made are therefore obvious when the center of sight and the visible point lie on the same diameter.

[2.351] **[PROPOSITION 37]** Furthermore, if the visible point and the center of sight do not lie on the same diameter, and if they lie outside the mirror, the [form of the] visible point is reflected to the center of sight from only one point on the mirror.

[CASE 1]

[2.352] For instance, let T [figure 5.2.37, p. 257] be the visible point, H the center of sight, and D the center of the sphere, and draw lines HD and TD. Plane HDT intersects the sphere along circle EBQG.

[2.353] It is clear that [the form of] T is not reflected to H except from some point on this circle. According to earlier discussion [in proposition 34 above], it is also clear that it is not reflected from arc QG or BA. It is therefore reflected from either arc GB or AQ.

[2.354] Bisect angle TDH with line LEDZ, and draw tangent KEF from point E. If points T and H lie on that tangent, [the form of point] T will not be reflected to H from any point on arc BG. For if a line will be drawn from point T to some point on this arc [on the] inner [side of the circle], the line drawn from point H to the same point will fall to it on the outer side [of the circle], not the inner side, and so there will be no reflection.[106]

[2.355] Moreover, it will be clear from the following that reflection may occur from only one point on arc AQ. Draw lines TZ and HZ. Since angle TDH is bisected, [angle] TDZ = angle HDZ.

[2.356] Either lines TD and HD are equal, or they are not equal. If they are equal, and if line DZ is common, then triangle TDZ = triangle HZD, and angle TZH is bisected by line DZ, and so [the form of point] T is reflected to H from point Z.

[2.357] That it cannot [be reflected] from another point [on arc AQ] will be established as follows. Take point O, draw lines TO and HO, and let line ODM bisect the angle [formed by them]. It is clear [by Euclid, III.8] that TZ < TO and that HO < HZ, and [by Euclid, VI.3 it is also clear that] TZ:HZ = TL:LH [since angle HTZ is bisected by ZL, leaving triangles HZL and TZL equal] and TO:HO = TM:MH [since angle TOH is supposedly bisected by OM, leaving triangles HOM and TOM equal]. But HO:TO < HZ:TZ [since TO > HO, and HZ = TZ]. Accordingly, HM:MT < HL:LT, which is impossible [because, according to the bisection of angle HOT by HM, HM:MT = HO:TO, whereas, according to the bisection of angle HZT by ZL, HZ:TZ = HL:HT, so it should necessarily follow that HM:MT = HL:LT, because M and L should cut equiproportional segments from HT].

[2.358] It is therefore obvious that, if T and H lie the same distance from the center [of the mirror], and if they lie above the tangent, [the form of point] T is reflected to H from only one point on the mirror, and it will have a single image-location.[107]

[CASE 2]

[2.359] Now, let BDQ and ADG [figure 5.2.37b, p. 258] be two diameters within the sphere [from which the mirror is formed], let diameter EDZ bisect angle BDG, and from point E draw the two [lines] ET and EH perpendicular to the two diameters BD and GD.

[2.360] It is obvious that triangle ETD = triangle EHD, since ED is common to both [and angles EHD and ETD, as well as EDH and ETD are equal, by construction]. Thus, [the form of point] T is reflected to H from point E. By the same token, [it is reflected to point T] from point Z. It is also obvious [from proposition 34 above] that it is not reflected to [point] E from any point on arc AB or arc GQ, nor is it reflected from any point on arc AQ other than from point Z, according to the previous demonstration [in case 1 above]. But that it cannot be reflected from any point on arc BG other than from point E will be shown as follows.

[2.361] Given point O [from which reflection supposedly does occur], draw lines TO, HO, and DO. Construct a circle the size of DE [as diameter] that passes through the three points T, D, and H, and line DE will be the diameter [by Euclid, III.31], since the angle ETD that it subtends is a right angle [by construction]. Therefore, that circle will pass through point E.

[2.362] Accordingly, since E is common to each circle [i.e., TDHE and AZGE], and since it lies on the same diameter [within each circle], the smaller circle will be tangent to the larger circle at point E, as Euclid demonstrates [in III.11]. Therefore, this [smaller] circle will intersect line DO.

[2.363] Let it intersect at point I, and draw lines TI and HI. It is already evident that TD = DH [since they are corresponding sides of equal triangles TED and HED]. Thus, angle TID = angle DIH, since they are subtended by equal arcs [DT and DH in circle HTE]. It follows that [being adjacent to angle TID] angle TIO = angle HIO [which is adjacent to angle DIH]; angle IOT = angle IOH, by construction, and IO is common. [Hence] triangle TIO

= triangle HIO [by Euclid, I.26], and TO = HO, which is impossible, since [by Euclid, III.7] HO > HE, [while] TO < TE, and TE = HE, as has been proven earlier. It therefore follows that [the form of point] T may not be reflected to H from any point other than E or Z.

[2.364] Moreover, from point E draw line EM to diameter TD, from line HD cut off a segment ND equal to MD, and draw EN and EM. It is evident that angle EMD is greater than a right angle [because it is exterior to right angle ETM in triangle ETM]. Cut from it [an angle] equal to a right angle with line CM, which will intersect DE. Let C be the point of intersection, draw NC, and construct a circle according to the size [of diameter] CD that passes through the three points M, D, and N. Since CMD is a right angle [by construction], CD will be the diameter, and the circle will pass through C. It is therefore obvious that [the form of point] M is reflected to N from point E [because triangle NCE = triangle MCE, so angle NED = corresponding angle MED], and likewise [it is reflected] from point Z, but from no other point on arc AB or QG; and it is obvious that [it does not reflect] from any point on arc AQ other than point Z.

[2.365] That [it does not reflect] from any point on arc BG other than from point E is evident from the earlier procedure. For if [some such] point is taken, if lines are drawn to that point from points T, D, and H, if a point is taken where the last circle will cut the diameter, and if lines are drawn to points T and H from the point of intersection, the same disproof as before will apply.

[2.366] From the foregoing, then, it is clear that, if a third diameter bisects the angle formed by two [other] diameters, and if from the endpoint of that [third] diameter perpendiculars are drawn to those [other two] diameters, the points on the diameters where those perpendiculars fall are reflected to one another from only two points on the mirror. In addition, any of the points taken on the perpendiculars to the diameters below these endpoints, i.e., toward the center, is reflected from only two points [on the circle], and the one is reflected to the other that lies equidistant from the center, and for all such points the image-location is twofold.[108]

[CASE 3]

[2.367] Furthermore, given the two diameters BQ and AG [figure 5.2.37c, p. 259], and given that EZ bisects the angle formed by them, take point T on BD beyond the point [between D and T] where the perpendicular dropped from point E falls [upon DB], take DH on DG equal to DT, and draw TE and HE. [Thus, according to the preceding case, the form of point] T is reflected to H from point E, and likewise from point Z, [but] not from any other point on arc AQ, or from any point on arcs AB or GQ.

[2.368] Then, from point T on TD draw a perpendicular, which will intersect DE outside the circle on the [mirror's defining] sphere, because angle DTE is acute. Accordingly, let it intersect at point O, and draw lines TO and HO. Construct a circle passing through the three points T, D, and H. That circle will necessarily pass through point O, and DO will be its diameter. Draw lines TO and HO, and draw line KE tangent to circle BZG at point E. It is clear that the latter circle will intersect the first one, i.e., BZG, at two points. Let those points be L and M, and draw lines TL, HL, LD, TM, DM, and HM.

[2.369] Therefore, since arc TD = arc HD [by construction], angle TLD = angle DLH [by Euclid, III.27], and so [the form of point] T is reflected to H from point L. Likewise, angle TMD = angle DMH, and so [the form of point] T is reflected to H from point M. It is therefore evident that [the form of point] T is reflected to H from four points, i.e., E, Z, L, and M, and its image-location will be fourfold.

[2.370] But [the form of point] T cannot be reflected to H from any point other than these. For, let F be given [as such a point], draw lines TF, HF, and DF, extend DF until it intersects tangent KE, and let K be the intersection. Then draw lines TK and HK. Hence, angle TFD = angle DFH, by construction; and it follows that [being adjacent to angle TFD] angle TFK = angle KFH [which is adjacent to angle DFH]. But angle TKF[D] = angle [D]FKH, because they are subtended by equal arcs, and FK is common. [Therefore] triangle [TKF] = triangle [HFK], and so TK = KH, which is impossible, because HK > HO [since arc HK > arc HO, whereas] TK < TO, yet TO = HO [by construction].

[2.371] It is evident, then, that reflection does not occur from any point other than from the four [previously defined] points on the circle.

[2.372] [In summary] therefore, if two points, i.e., T and H, are taken on different diameters [at locations on those diameters that are] equidistant from the center, and if they lie at the points on the diameters where the perpendiculars drawn from the end of the diameter that bisects the angle formed by those two diameters fall, or if they lie between the center and those points, i.e., below the perpendiculars, as long as they lie equidistant from the center, then [the form of point] T will be reflected to H from two points only [as demonstrated in case 2 above].

[2.373] If, however, T and H lie at spots beyond the perpendiculars up to the interior surface of the circle, [the form of point] T will be reflected to [point] H from four points. But if they lie on the circle or outside it, but still below tangent KE, [the form of point] T will be reflected to H from two points only [i.e., E and Z]. And if they lie upon the tangent, [the form of point] T will be reflected to H from only one point [i.e., Z].[109] And these phenomena occur as long as T lies the same distance as point H from the center.

[2.374] **[PROPOSITION 38]** Now, if T and H lie on different diameters, and if they lie unequal distances from the center, they will be reflected to one another from one point [only on the opposite arc].[110]

[2.375] For instance, draw diameters ADG and BDQ′ [figure 5.2.38, p. 260], and let EZ bisect the angle formed by them. Let [object-point] T be nearer centerpoint D than [center of sight] H. Take [some arbitrary] line LQ and cut it at point M so that QM:ML = HD:DT [by Euclid, VI.10]. Bisect LQ at point N, from point N draw perpendicular NK, and at point L form with line FL an angle [FLQ] equal to half of angle ADT. Angle FLQ will be acute, so FL will intersect NK. Let it intersect at point F, and from point M draw a line to side FL intersecting side NK at point K. Let that line cut side FL at point C such that KC:CL = HD:DZ [by proposition 21, lemma 3 above].

[2.376] Then, on point D form angle IDA equal to angle LCM, and let I be a point on the circle lying above or below Z. At point I form angle O′ID equal to angle CLM, to the resulting line O′I drop perpendicular HC′ from point H, draw line C′F′ equal to line C′I, and draw lines HF′ and HI.

[2.377] According to previous arguments, it is clear that from point M no line other than line MCK can be drawn to side FL to cut it in the way that line MCK does. For if [such a line] could [be found], let it be MPO. It is evident that $PO < CK$, which will become evident if line PY is drawn parallel to CK, that line being shorter than CK and longer than PO. But $PL > CL$. Thus, PO:PL ≠ KC:CL, so PO:PL ≠ HD:DT. It therefore follows that no line like MCK, other than MCK [itself], can be drawn from point M [to side FL].

[2.378] Now, since [angle] O′DI = angle LCM [by construction], and angle O′ID = angle CLM [by construction], triangle CLM will be similar to triangle IO′D. Therefore angle IO′D = [corresponding] angle LMC. It follows that [being adjacent to angle IO′D] angle C′O′H = angle KMN [which is adjacent to angle LMC], and right angle HC′O′ = [right] angle KNM [both being right angles by construction]. [So] it follows that angle NKM [in triangle NKM] = angle C′HO′ [in triangle C′HO′, so triangles NKM and C′HO′ are similar, by Euclid, VI.4].

[2.379] Now, if line DI is extended until it intersects HC′ at point R, angle RDH = angle KCF [because angle KCF = vertical angle LCM, angle RDH = vertical angle O′DI , and angle LCM = angle O′DI, by construction]. Triangle RDH will [thus] be similar to triangle CKF [since corresponding angles RHD and CKF are also equal, by previous conclusions]. Therefore, RD:DH = FC:KC. But HD:DI = KC:CL [by construction]. So RD:DI = FC:CL [by Euclid, V.22]. Accordingly, RI:DI = FL:CL [by Euclid, V.18]. But DI:IO′

= CL:LM, since triangle DIO′ is similar to triangle CLM [by previous conclusions]. Therefore, RI:IO′ = FL:LM [by Euclid, V.22]. But RI:IC′ = FL:LN, because triangle RIC′ is similar to triangle FLN [since right angle RC′I = right angle LNF, and angle RIC′ = corresponding angle FLN]. Hence, IO′:IC′ = LM:LN [by Euclid, V.22]. So QM:LM = F′O′:IO′.[111]

[2.380] Now, if line UI is drawn from point I parallel to HF′, and if line DA is extended until it intersects UI at point U, triangle O′UI will be similar to triangle HO′F′. Therefore, HO′:O′U = QM:ML [because HO′:O′U = F′O′:O′I], and so HO′:O′U = HD:DT [because QM:ML = HD:DT, by construction]. But since triangle HC′I = triangle HC′F′, because HC′ is perpendicular [to equal bases F′C′ and C′I], then angle HF′C′ = angle C′IH, so [angle] C′IH = angle UIO′, and so [by Euclid, VI.3] HO′:O′U = HI:UI [because angle UIH in triangle UIH is bisected by IO′, insofar as angle UHI is composed of equal angles UIO′ and C′IH], and so HI:UI = HD:DT.

[2.381] But angle UID > angle DIH [since angle UID > angle UIO′, which = angle O′IH, of which DIH is a part]. From it [i.e., angle UID] cut angle PID equal [to DIH], and draw line TP. Let P be a point on diameter DA.

[2.382] It is evident that the ratio HI:UI is compounded from HI:IP and IP:UI, while HI:IP = HD:DP [by Euclid, VI.3], because DI bisects angle PIH [in triangle PIH]. Therefore, HI:UI, which is as HD:DP, is compounded of the ratios HD:DP [which = HI:IP] and PI:UI. But the ratio HD:DT is compounded of HD:DP and DP:DT. Thus, DP:DT = PI:UI.[112]

[2.383] Now, angle O′IH is half of angle UIH [by previous conclusions], whereas angle DIH is half of angle PIH [by construction]. It follows that angle DIO′ is half of angle PIU, whereas angle DIO′ is half of angle TDP, since it is equal to angle FLM. Therefore, angle PIU = angle TDP,[113] and DP:DT = PI:UI [by previous conclusions]. Hence, triangle UIP is similar to triangle TPD [by Euclid, VI.4], and [so] angle UPI = [angle] TPD. TPI will thus be a straight line, because angle DPT + angle TPO′ = two right angles, and so angle O′PI + angle O′PT = two right angles [since O′PI is alternate to DPT]. So [the form of point] T is reflected to [point] H from point I, and this proof will apply whether T lies outside or inside the circle, and [it will apply] likewise if point H is taken outside or inside [the circle], as long as [they are] unequally distant from the center.[114]

[2.384] **[PROPOSITION 39]** Now, having produced diameters BQ and AG, and [given] diameter EZ bisecting angle BDG, I say that, no matter what point other than Z is taken on arc AQ, an infinite number of point-pairs that are not equidistant from the center can be reflected [to one another] from that point.

[2.385] For example, take point H [as the potential point of reflection in figure 5.2.39, p. 261], and take point L on diameter GD. On diameter BD cut off [segment] MD equal to LD, and draw lines LM, LH, MH, and DH. Let F be the point where EZ cuts LM; [and so] LF = FM [since triangles FMD and LDF are equal, by Euclid, I.4].

[2.386] Draw [line] HD until it falls on LM at point N. Hence, LN < MN. But, since angle MDF = [angle] FDL, as well as [vertical] angle QDZ, since angle MDA = [vertical] angle LDQ, and since angle ADH = [vertical] angle NDL, then angle LDH > angle MDH.[115] Therefore [by Euclid, I.24], LH > MH, since MD and DH = LD and DH [respectively]. Thus, angle DHL < angle DHM, for if they were equal [then DHN would bisect angle MHD, and so, by Euclid, VI.3], LH:MH would be as LN:NM, which is impossible. If, on the other hand, it were greater [i.e., if DHL > DHM], then cut from it [an angle] equal [to DHM], and it will be disproven in this [same] way.[116] Therefore it is smaller [i.e., DHL < DHM].

[2.387] Accordingly from angle MHD cut off [angle] THD equal to it [i.e., DHL]. [The form of point] T is therefore reflected to [point] L from point H, and TD < LD [i.e., T and L lie unequal distances from D].

[2.388] So, too, if points other than L and M that lie equidistant from point D are selected on diameters HD and GD, it is proven in a similar way that from point H there occurs a mutual reflection of points that lie at different distances from the center. And so the proof will be the same for an infinite number of points chosen on these diameters, as well as for any point chosen on arc AQ other than point Z [because, in order for reflection to occur from Z, the points must be equidistant from the center, as shown in proposition 37].[117]

[2.389] **[PROPOSITION 40]** Moreover, if T and L [figure 5.2.40, p. 261] are taken on [different] diameters at unequal distances from the center, and if they are reflected to one another from point H, [the form of point] T will not be reflected to [point] L from any point on arc AQ other than from point H.

[2.390] For if [it were reflected] from another [point], let it be K, and draw TK, LK, DK, LT, TH, LH, and NDH. Then extend DK until it falls to point C on LT. It is clear that LH:TH = LN:NT [by Euclid, VI.3].

[2.391] Likewise, since angle TKC = angle LKC, by supposition, then LK:TK = LC:CT. But [by Euclid, III.7] LH > LK, and TH < TK. Hence LH:TH > LK:TK, so LN:NT > LC:CT, which is patently impossible. It follows that [the form of point] T cannot be reflected to L from any point on arc AQ other than from point H. What pertains to arc AQ is therefore evident.

[2.392] **[PROPOSITION 41]** To continue, let A [figure 5.2.41, p. 262] be the center of sight, B the center of the mirror, and draw diameter DABG. Take the plane in which AB lies so that it will somehow intersect the sphere along [great] circle DLG. I say that points that do not lie the same distance from the center as A are reflected [to A] from any point on semicircle DLG.

[2.393] For instance, take point E, and draw lines EA and EB. It is obvious that angle AEB will be acute, because it will be subtended by a smaller arc than a semicircle. Form [angle] OEB equal to it, and draw line OE as far as you like. It is evident that [the form of] every point on that line is reflected to A from point E.

[2.394] Now, if a perpendicular is dropped from point B to line OE, that perpendicular will be equal to, longer than, or shorter than BA. If it is equal, then every line, other than the perpendicular, that is drawn from point B to line OE will be longer than line BA, and so, with one exception [i.e., point O], every point on line OE will lie a different distance from the center than point A.

[2.395] If the perpendicular is longer [than AB], then all the points on that line [i.e., OE] will lie farther from the center than point A [and will therefore lie different distances from B than A]. But if the perpendicular is shorter, two lines equal to BA can be drawn from [points on] opposite sides of the perpendicular, whereas all the remaining lines will be either shorter or longer [than BA and will therefore lie at different distances from B]. It is therefore evident that [the forms of] points that do not lie the same distance from the center as [point] A are reflected to [point] A from point E, which is what was proposed.

[2.396] From these considerations it is clear that, if [center of sight] A is taken outside the circle—let it be at H [figure 5.2.41a, p. 262]—and if diameter HDBG is drawn along with tangents HT and HQ, then reflection of points that do not lie the same distance from the center as H can take place to H from any point on arc TG other than from T or G. And the proof will be the same [as the previous one].[118]

[2.397] **[PROPOSITION 42]** On the basis of these determinations, it will be established that, if reflection occurs to [point] A from point E or from another point not lying the same distance as point A from the center, the diameter in which the point [whose form is] reflected lies forms two angles with diameter ABG, one of them opposite the reflected angle, the other adjacent to it, and that adjacent angle will sometimes be greater than the reflected angle and sometimes smaller.[119]

[2.398] For instance, draw FB [figure 5.2.42, p. 262] perpendicular to EO. BA is either perpendicular to it or not.

[2.399] Let it be perpendicular. Accordingly, EA will be parallel to FB, and the two angles FBA and FEA will sum up to two right angles. Now, when line BO is drawn, the two angles OBA and OEA will sum up to less than two right angles. Hence, angle OBG > angle OEA, which is the reflected angle. And since triangle EBF = triangle EBA, BF = BA, and so OB > BA.

[2.400] Moreover, when line BN is drawn, the two angles NBA and NEA will sum up to more than two right angles. Thus, angle NBG < angle NEA, and NB > BA, and so [the forms of points] N and O will be reflected to [point] A from point E. Also, they lie at different distances than point A from the center, and diameter OB forms an angle with ABG that is greater on the side of G than the reflected angle [OEA], while diameter NB is longer [than diameter AB], so [we have demonstrated] what was proposed.

[2.401] If, however, BA is not perpendicular to EA, then draw the perpendicular BK, which either lies above AB [figure 5.2.42a, p. 262], or lies below it [figure 5.2.42b, p. 262]. The proof will be the same [in both cases].

[2.402] Let BF be perpendicular to EO, draw FT equal to AK, and draw TB. It is evident that in triangle KEB right angle EKB = [right angle] EFB, and angle KEB = angle FEB [by construction]. It follows that the third [angle, i.e., EBK] = the third [angle, i.e., EBF], and since side EB is common to both triangles [EBF and EBK], the triangles will be equal, and FB = KB. But AK = FT [by construction. And since triangle BAK = triangle BTF] AB = BT, and angle ABK = angle FBT.

[2.403] With common angle [KBT] added to [both in order to yield angle] FBA, [angle] KBF = [angle] TBA.[120] But [angles] KBF and FEA sum up to two right angles,[121] so [angles] TBA and TEA sum up to two right angles, and so [angle] TBG = angle TEA, which is the reflected angle.

[2.404] Therefore, if a line is drawn from point B to [fall on] line ET beyond T, it will form an angle with BG on the side of G that is smaller than the reflected angle. And that line will be longer than AB, because TB = AB.

[2.405] Furthermore, any line drawn from point B to [fall on] line ET in front of T [i.e., between T and E] will form an angle with BG that is greater on the side of G than the reflected angle, and it will be unequal to AB, so [we have demonstrated] what was proposed.[122]

[2.406] **[PROPOSITION 43]** Now, let B be the center of sight and G the center of the sphere. Draw diameter ZBGD [figure 5.2.43, p. 263], and produce the plane that contains the diameter and cuts the sphere along [great] circle ZEH. I say that, if point A is reflected to point B from some point on the circle, and if the distance from point A to the center is not the same as that of point B from the center, diameter AG will form an angle [AGD] with diameter GD on the side of D that cannot possibly be equal to the reflected angle.

[2.407] Let it be equal, let T be the point of reflection, and let AG ≠ BG. Draw lines TA, TG, and TB, and construct a circle passing through the three points A, G, and B, and this circle will necessarily pass through point T. For if the circle lies beyond it, and if lines are drawn from points A and B to the corresponding point on that outlying circle, they will form an angle smaller than angle ATB. And it will be proven that it is equal.

[2.408] For [angle AGD] along with angle AGB will sum up to two right angles, and angle ATB along with angle AGB sums up to two right angles, since angle ATB = angle AGD, by construction [and within the circle angle ATB + angle AGB = two right angles by Euclid, III.22], and so it is impossible [for T to lie outside the circle]. Likewise, if the circle were to pass below T, the same disproof would hold [insofar as the new angle ATB would be greater than ATB within the circle].

[2.409] It follows from this that it must pass through point T, but since angle ATG = angle BTG [by supposition], arc AG = arc BG, and so [chord] AG = [chord] BG. Yet they have been assumed to be unequal, and so [we have demonstrated] what was proposed [i.e., that angle ATB cannot equal angle AGD when AG ≠ BG].[123]

[2.410] **[PROPOSITION 44]** Furthermore, if the two points A and B [figure 5.2.44, p. 264] are taken on the two diameters EGH and ZGD so that BG > AG, I say that, if [the form of] point A is reflected to [point] B from two points on arc EZ, the angles of reflection will not both be smaller than angle AGD.

[2.411] For, on arc EZ select two points, T and Q, from which [the form of point] A is reflected to [point] B, and draw lines BT, GT, AT, BQ, GQ, and AQ. If angle ATB < angle AGD, I say that angle AQB will not be smaller than angle AGD.

[2.412] Let it in fact be smaller, draw line GN bisecting the angle formed by the diameters [i.e., angle BGA], and draw line AB, which GN intersects at point F. It is evident that BG:GA = BF:FA [by Euclid, VI.3]. But BG > GA; [so] BF > FA.

[2.413] Bisect AB at point K, and construct a circle passing through the three points A, B, and T, a circle that will not pass through G, because [if it did] angles AGB and BTA [in quadrilateral BGTA] would be equal to two right angles [by Euclid, III.22], and it is obvious that they are less [than two right angles], since angle BTA < angle AGD [by supposition, and angle AGD + adjacent angle AGB = two right angles]. It will therefore pass above G.

[2.414] By the same token, it will not pass through Q. For if the point, i.e., M, is taken on the circle where line GQ intersects it, arc AM would be equal to arc MB, since [by Euclid, III.26] they would subtend equal angles [MQA and MQB] at Q, which is impossible, because, if point O, where line GT intersects this circle, is taken, arc AO = arc OB, since they subtend equal angles at T [and so GM, which supposedly bisects circle BMA, would cut a smaller arc on the side of A than on the side of B]. It follows that this circle passes above Q, for if it passed below, the same disproof would apply.

[2.415] Now, draw a line from point O to point K [which is the midpoint of BA, by construction], and since it bisects chord AB and likewise bisects arc AB, this line will be perpendicular to AB. But [by Euclid, I.18] angle BAG > angle ABG, since BG > GA. Also, [exterior] angle BFG [of triangle GFA] equals the sum of the two [interior] angles FAG and FGA [by Euclid, I.32], while [exterior] angle AFG [of triangle BFG] equals the sum of the two [interior] angles FBG and FGB [by Euclid, I.32].

[2.416] But [angle] AGF = [angle] FGB [by construction], and [angle] FAG > [angle] FBG [because it is subtended by BG, which is longer than AG]. Thus [for the same reason], angle BFG > angle AFG. Accordingly, [angle] AFG is less than a right angle, so [vertical angle] NFB is less than a right angle. But OK forms a right angle on FB. Therefore, when it is extended, it will intersect GN above BF, never below it.

[2.417] Now, if a circle is produced to pass through the three points A, Q, and B [figure 5.2.44a, p. 265], it will pass above G, and GQ will bisect its arc A[Q]B [since angles AQG and BQG are presumed equal]. But K bisects chord AB [by construction]. Hence, KO will intersect GN below BF and above point G. Therefore, it will intersect GN sooner [than it should, i.e.,] below FB, and it has already been disproven [that it can intersect below FB].

[2.418] It therefore follows that angle AQB may not be smaller than angle AGD, or else [the form of point] A is not reflected to B from point Q. The same disproof will hold if any [other] point is taken on arc EN.[124]

[2.419] Now, if [some] point C is taken on arc NZ [figure 5.2.44c, p. 267], and if reflection of [the form of] point A to B occurs from point C so that the reflected angle at C is less than angle AGD, just as the reflected angle at T is less than the same angle, [the supposition that the reflected angle at C can be less than angle AGD] is disproved in the following way.

[2.420] Draw AC, BC, and GC. It follows necessarily that GC intersects KO on arc A[O]B, and line GC will bisect that arc on circle ABT, as will line KO. Accordingly, let point L be where lines GC and KO intersect. When line TC is drawn, then, since the two lines GC and GT are equal [being radii of circle ZHDE], the two angles GCT and GTC will be equal, and both of them will be acute.

[2.421] Hence, if the line [TX] perpendicular to GT at point T is drawn, it will be tangent to the circle on the mirror, and when it is extended it will fall upon the endpoint of the smaller circle's diameter [by Euclid, III.31], since the angle it forms with TG is subtended [by an arc equal to] a semicircle on the smaller [circle]. Since TO falls upon KO, and since KO, when extended, passes through the center of the smaller circle [by virtue of its bisecting line AB along the orthogonal], that perpendicular [TX] will necessarily fall upon the endpoint of KO, when it is extended [by Euclid, III.31], and TC lies below that perpendicular when it is taken with respect to N.

[2.422] Therefore, no matter what line is drawn [from G] to line TC so as to intersect diameter OK of that circle, it will fall to a point on line TC but below that perpendicular [TX]. Therefore, since GC falls to C and intersects OK, C will lie below the perpendicular and beneath the arc [on circle AOBT passing] through that perpendicular.

[2.423] Therefore, if a circle is produced to pass through the three points A, B, and C, it will pass through C, and it will intersect circle ABT at the two points A and B. And when it continues past point B, it will pass on to point C, and although it lies below that circle [i.e., ABT], it will necessarily intersect it at a third point, which is impossible.[125]

[2.424] It follows, then, that [the form of point] A may not be reflected to B from two points on the arc extending between the diameters on which they lie, i.e., arc EZ, in such a way that both angles of reflection are less than angle AGD, which is what was proposed.

[2.425] **[PROPOSITION 45]** I say, moreover, that two points lying different distances from the center can be reflected from two points on the arc facing them, that is, the arc lying between the diameters in which those points lie.

[2.426] For instance, taking two diameters, i.e., BD and GD [figure 5.2.45, p. 268] in the [great] circle of the sphere, let the angle formed by them be bisected by diameter ED, and take point M on BD beyond the point where the perpendicular dropped from point E to BD will fall. Then take ND equal to MD, and construct a circle passing through the three points D, N, and M. That circle will necessarily pass beyond E, for if [it passed] through E, it would create a quadrilateral at the four points D, N, E, and M, and the two opposite angles of that quadrilateral are equal to two right angles, which would not be so [in this case], since line EM lies beyond the perpendicular, and angle EMD is acute.

[2.427] Likewise, its counterpart at N is acute, because EN [also lies] beyond the perpendicular [so EMD and END sum up to less than two right angles]. The disproof will be similar if the circle is [assumed] to pass below

E. Therefore, it will pass beyond, and it will intersect the [great] circle of the sphere at two points, e.g., T and L.

[2.428] Draw lines MT, DT, NT, ML, DL, and NL, and draw line MN to intersect [line] TD at point F and line ED at point P. It is evident that, since MD = ND [by construction], since PD is common, and since the [subtended] angle [MDP] = the [subtended] angle [NDP, by construction], triangle [MDP] = triangle [NDP], and angle FPD will be a right angle. Hence, angle PFD is acute.

[2.429] From point F draw KF perpendicular to TD. It is clear that any point on line NL will lie below point K [since FK intersects ML between M and L]. Taking the position of [some such] point [on NL] below [K] with respect to T, let that point be Z, and draw line TZ until it reaches point C on the circle. Arc NC is either shorter than arc TL or not.

[2.430] If it is not shorter, then take an arc on it [i.e., on arc NC] that is shorter, and draw a line from point T to the endpoint of that arc, and it will be the very line [TC that we wanted in the first place].

[2.431] So let NC < TL. It is obvious that angle TNL > angle CTN, since it is subtended by a longer arc [i.e., TL as opposed to CN]. Cut an [angle] equal [to CTN] from it, let that angle be INZ, and at point T form angle OTM equal to angle CTN. Since angle TML > angle MTO,[126] line TO will intersect line LM. Let it intersect at point O.

[2.432] Hence, since angle LMT [exterior to triangle MOT] = [the sum of] the two [interior] angles MOT and MTO [by Euclid, I.32], since angle LNT = angle LMT, because they are subtended by the same arc [TL], and since angle INZ = angle MTO [by construction], angle INT [which = LMT – INZ] = angle MOT [which = LMT – MTO], and so triangle MOT is similar to triangle INT, and likewise triangle INZ is similar to triangle TNZ. So NT:TO = NI:MO, and likewise TN:TZ = IN:NZ.

[2.433] But TZ > TO, which becomes clear as follows. Let R be the point where TZ intersects KF. Angle TFR is a right angle [by construction], so angle FTR is acute. Hence, angle OTF, which equals it, is acute.[127] Moreover, KF is perpendicular to TD [by construction], so when it is extended it will intersect TO, and the line [TX] drawn from point T to that point of intersection, a line that includes TO as a segment, will be equal to line TR [because triangles TRF and TXF are equal, by Euclid, I.26]. And thus TO < TZ, so NT:TO > NT:TZ.

[2.434] Therefore [since we have already established that NT:TO = IN:MO and that NT:TZ = IN:NZ], IN:MO > IN:NZ, so MO < NZ. Accordingly, from NZ cut off segment NS equal to MO.

[2.435] Since angle LND + angle LMD = two right angles [by Euclid, III.22], angle LND = angle OMD [which is adjacent to LMD], and SN + ND

= OM + MD [since MD = ND, and SN = OM, by construction]. Hence OD = SD [because triangles SND and OMD are equal, by Euclid, I.4].

[2.436] But ZD > SD, because angle LND + angle LMD = two right angles [by Euclid, III.22]. Angle LMD is acute, however, since angle EMD is acute. Therefore, angle LND > a right angle. Hence, ZD > SD [because it intersects LN farther from vertex N than does SD], so ZD > OD [because SD = OD, by previous conclusions].

[2.437] Accordingly, [the form of point] O is reflected to Z from the two points T and L [given that OTD = ZTD, by previous conclusions, and ZLD = OLD, since they are subtended by equal arcs], and O and Z lie unequal distances from the center, and they lie on different diameters.

[2.438] That they do not lie on the same diameter is evident from the fact that angle SDN = angle ODM. Thus, when the common angle SDM is added [to each] angle, [angle] NDM = angle SDO. But angle NDM < two right angles, so angle ZDO < two right angles [which means that Z, D, and O cannot lie on a single straight line]. Therefore, O and Z do not lie on the same diameter but on different ones.

[2.439] **[PROPOSITION 46]** Furthermore, given two points O and K [figure 5.2.46, p. 269] that lie different distances from the center [of the mirror], one will be reflected to the other from two points on the arc facing the radii in which those point lie, but they will not reflect from any point on that arc other than those two.[128]

[2.440] For example, let D be the center [of a great circle on the mirror], let K lie farther from D than O does from D, let GD and OD be diameters, and let T be one point of reflection. It is clear from earlier discussions [in propositions 43 and 44 above] that the two reflected angles will not [both] be smaller than, nor equal to, angle ODA. Hence, one of them will be greater. Let the reflected angle at point T be greater [than angle ODA], and draw lines OT, DT, and KT.

[2.441] Then, from that reflected angle [OTK] cut off [angle] OTF equal to angle ODA, and bisect angle FTK with line TE. From point K draw [line KZ] parallel to TF, a line that will intersect TE. Let it intersect at point Z, draw line OK, bisect angle ODK with line DU so that it intersects line OK at point C, and let KD > OD. Therefore, since KD:DO = KC:CO [by Euclid, VI.3], KC > CO. Then, let line DT intersect line OK at point N. I say that C lies between N and K, not N and O, which will be shown as follows.

[2.442] Angle KCD [exterior to triangle CDO] = the two [interior] angles CDO + COD [by Euclid, I.32], and angle OCD [exterior to triangle CDK] = the two [interior] angles CKD + CDK. But angle CDO = angle CDK [by construction], and [by Euclid, I.19] angle KOD > angle OKD [since KD >

OD]. Hence, angle KCD > angle OCD, so angle KCD > a right angle. Also, angle KND is acute, which will be established thus.

[2.443] If a circle is constructed through the three points O, T, and K, it will pass below D, because, since angle OTK > angle ODA [by construction], the two angles OTK + ODK > two right angles, and line ND will bisect arc OK of that circle below D.[129]

[2.444] If a line is drawn from the point of bisection [*s*] to the midpoint [*x*] of line OK, which is the chord on that arc, that line will be perpendicular to OK [by Euclid, III.3], and it will fall between C and K, since CK > CO [by previous conclusions]. Moreover, the angle at [point] N beyond that perpendicular on the side of C [i.e., DNC] will be acute, and the angle at C on the side of O [i.e., OCD] is acute [since angle KCD is obtuse, by previous conclusions]. Accordingly, if C were to fall between N and O, it would be impossible for that perpendicular to fall between N and C, because it would intersect DC and would form a triangle with one angle right and the other obtuse.[130]

[2.445] Hence, it [i.e., the perpendicular] will fall between N and K, and the angle at N on the side of the perpendicular will be acute, so this [same angle] will be obtuse on the side of C [if C lies between N and O], and so there will be a triangle with two obtuse angles.[131]

[2.446] It is evident that angle KTD is half of angle KTO [by construction], but [angle] KTE is half of angle KTF [by construction]. It follows that [angle] ETD is half of angle FTO [since ETK = FTK, by construction], but [angle] FTO = angle ODA [by construction]. Thus, [angle] ETD is half of angle ODA.

[2.447] But angle ODA + [adjacent] angle ODF = two right angles, and the three angles of triangle ETD sum up to two right angles. With common [angle] EDT subtracted, [angle] TED remains equal to half of angle ODA + angle ODN. But angle ODC + half of angle ODA is a right angle [since 2ODC (which = ODK) + ODA = two right angles]. Therefore, angle TED is acute [since TED = ODC – NDC + half of ODA], so its vertical angle [KEZ] is acute.

[2.448] Thus, if a perpendicular is dropped from point K to TZ, it will fall between E and Z. For if it were to fall above E, then, since angle TEK is obtuse, it would follow that the triangle [formed by KD and the perpendicular intersecting TE above E] would have two [of its three] angles [consisting of] a right angle [formed by the line from K that intersects TE above E] and an obtuse angle [KET]. Let KQ be the perpendicular, then. I say that KT:TF = KD:DO.

[2.449] The proof is as follows. Either TO is parallel to KD, or it intersects it. Let it be parallel [figures 5.2.46a and 5.2.46b, p. 270]. Angle ODA =

[alternate] angle TOD, and so [angle] TOD = angle OTF [which = angle ODA, by construction]. Either OD and TF are parallel, or they will intersect.

[2.450] If they are parallel [figure 5.2.46a, p. 270], then, since they fall between parallels, they will be equal. If, however, they intersect [figure 5.2.46b, p. 270], then they form a triangle [two of] whose sides [OP and TP] are equal, because they subtend equal angles [since angle DOT = alternate angle ODA, and angle OTF = angle ODA, by construction], and because FD intersects those sides parallel to the base [TO]. Therefore, the ratio of one side to DO will be the same as the ratio of the other side to TF, and so TF = DO.

[2.451] {I claim that this is so if they [i.e., DO and TF] intersect below KD. And if they intersect below TO, the same proof will apply, because they will form a triangle one of whose sides is TO and the other two of which are equal, and the ratio of one [of those equal sides] to DO will be the same as the ratio of the other to TF.}[132] Furthermore, angle TDK = [alternate] angle DTO, because DT lies between parallels. Thus, it [i.e., angle TDK] = angle DTK [which = angle of reflection DTO], so DK and TK are equal. Hence, TK:TF = KD:DO.

[2.452] If, on the other hand, TO intersects KD, let it intersect at point P on the side of A [figure 5.2.46c, p. 270]. We know that KT:TF is compounded of KT:TP and TP:TF [i.e., KT:TF = (KT:TP):(TP:TF)]. But KT:TP = KD:DP [by Euclid, VI.3], because DT bisects angle KTO. Moreover [given that triangles TPF and DPO are similar], TP:TF = DP:DO, because angle ODP [i.e., ODA] = angle PTF [by construction], and the angle at P is common. Part [ODP] of the [larger] triangle [PTF] is [therefore] similar to the whole. Hence, KT:TF is compounded of KD:DP [which = KT:TP] and DP:DO [which = TP:TF]. But KD:DO is compounded of the same [ratios, i.e., KD:DP and DP:DO], so KT:TF = KD:DO.

[2.453] If, however, TO intersects KD on the side of G [figure 5.2.46d, p. 270], let L be the [point of] intersection, and from point D draw [line] DR parallel to line KT so as to intersect TO at point R. Accordingly, angle KTD = [alternate] angle TDR, but it is [also] equal to angle DTO [by supposition], so DR = TR. However, since triangle LTK is similar to triangle LRD [because TK and RD are parallel], DR:RL = KT:TL, and so RT [which = DR]:RL = KT:TL. But RT:RL = DK:DL. Thus, KT:TL = KD:DL.

[2.454] Yet, since angle FTO = angle ODA [by construction], angle ODL [adjacent to ODA] = angle FTL [adjacent to FTO], and the angle at L is common. [Thus], triangle ODL will be similar to triangle FTL. Hence, TL:TF = DL:DO, and so [by previous conclusions] KT:TL = KD:DL. Moreover [by previous conclusions], TL:TF = DL:DO, so KT:TF = KD:DO, which is what was proposed.[133]

[2.455] However, since KZ [in figure 5.2.46, p. 269] is parallel to TF [by construction], angle KZE = [alternate] angle ETF [and vertical angle ZEK = vertical angle TEF], so triangle KZE is similar to triangle ETF, and so KE:EF = KZ:TF. But [by Euclid, VI.3] KE:EF = KT:TF, because the angle at T is bisected. Thus, KZ = KT.

[2.456] Since KQ is perpendicular to EZ, though, all of its angles [of intersection with TZ] will be right angles. But angle ETD is acute, because it is half of angle [FTO, by earlier conclusions]. Therefore, KQ will intersect TD. Let H be the intersection, draw line EH, and from point E draw EC′ parallel to KH, and extend it to DH [which it intersects at point C′].

[2.457] Adjust the figure according to the interrelationship of lines [figure 5.2.46e, p. 271], and construct a circle that passes through the three points C′, T, and E. Extend KD until it reaches the circle at point M, and draw MT. Angle TME = angle TC′E, since they are subtended by the same arc, and angle TC′E = angle C′HK [since EC′ and KH are parallel, by construction]. [Therefore, angle] TME = angle C′HK.

[2.458] Cut from angle TME angle F′MD equal to angle DHE, and let I be the point where F′M intersects TC′. It is evident that triangle IMD is similar to triangle EHD [because all their corresponding angles are equal], so HD:DM = EH:IM.

[2.459] Likewise, triangle TMD is similar to triangle KHD [because angle TMD (i.e., TME) = angle TC′E (which = angle THK), by construction, and angle HDK = vertical angle TDM], and [therefore, in similar triangles TMD and KHD, the corresponding sides are proportional, so] KD:DT = HD:DM, and so [given that HD:DM = EH:IM, by previous conclusions] KD:DT = EH:IM.

[2.460] But KD:DT is known, since it remains one and the same throughout, no matter where point of reflection T might lie on arc EG, because TD remains unchanged, and so does KD. Line EH also remains one [and the same] no matter the reflection, so it does not change its length, and so line IM will always be one [and the same], and so point F′ is known and determinate.[134]

[2.461] Therefore, if reflection could occur from three points on arc BG, three lines equivalent [to FM] could be extended from point F′ to circle TC′E, because KD:DT would be as EH is to any one [segment IM] of them. But it is clear from earlier discussion [in proposition 20, lemma 2 above] that only two equal lines can be [so] drawn, so reflection will occur from only two points, which is what was proposed.[135]

[2.462] **[PROPOSITION 47]** Furthermore, given two points K and O [figure 5.2.47, p. 275] lying on different diameters and at different distances from the center, to find [either] point of reflection.

[2.463] For example, take line ZT′, and cut it at point E so that ZE:ET′ = KD:DO. Since KD > DO [by construction from the previous proposition], ZE > ET′. Bisect ZT′ at point Q, from point Q draw a line [K′QH] perpendicular to ZT′, and form angle ET′D′ equal to half of angle ODA. It will be acute. Accordingly, T′D′ will intersect the perpendicular [KQ].

[2.464] Let the intersection be at point H, and [by proposition 24, lemma 6] draw line D′EK′ so that K′D′:D′T′ = KD:[DT], the radius of the sphere. Then, form angle KDT in the mirror equal to angle K′D′T′ that we have [in the construction based on T′Z]. I say that T is a point of reflection, and if you repeat the previous proof, you will see this clearly.[136]

[2.465] **[PROPOSITION 48]** Moreover, if two points are taken on different diameters, if they lie at different distances from the center, and if they lie outside the circle and are reflected from some point on the [concave] arc facing the diameters, they will not be reflected from any other [point] on the same arc.

[2.466] For example, let A and B [figure 5.2.48, p. 276] be points lying outside the circle on different diameters, G the center [of the great circle on the mirror], and T the point of reflection, and draw BT, AT, and GT. BT will intersect the [convex] arc on the circle. Let Q be the point of intersection. So too, AT will intersect the [convex] arc on the circle. Let M be the point of intersection.

[2.467] Since angle BTG = angle ATG [by construction], they are subtended by equal arcs on the circle, which will be evident if diameter TG is drawn. Accordingly, arc QT = arc MT [and so, therefore, do chords QT and MT]. Thus, if [the form of point] B is reflected to [point] A from a point other [than T], let it be H, and draw lines BH, AH, and GH. Let BH intersect the circle at point L, AH at point N.

[2.468] According to the previous reasoning, HL = NH [because they subtend supposedly equal arcs]. But we have just [established] that QT = TM, which is impossible [if HL = NH]. It follows that [the form of point] B may not reflect to [point] A from point H or from any point other than T on the arc facing [either] diameter.

[2.469] Likewise, if one of the points lies on the circle, while the other lies outside, then the one can be reflected to the other from only one point on the arc.

[2.470] Moreover, if the line extending from one of the two points [A or A′ in figure 5.2.48a, p. 276] to the other [B] is tangent to the circle or lies entirely outside it, then, if some point [T] is taken on the [convex] arc facing the diameters, one of the lines [i.e., either AT or A′T] extended from [each of] the two points to that point will lie entirely outside the circle [because it will strike its outer rather than its inner surface]. And so neither of the points [A and B, or A' and B] will be reflected to the other from any point on that arc [CE], and [it will be reflected] from only one point on the opposite arc [KD] of the mirror, and so [it will be reflected] from only one point on the [entire] mirror.

[2.471] **[PROPOSITION 49]** On the other hand, if the line extending from one point to the other [i.e., from object-point to center of sight] cuts the [great] circle [on the mirror], construct the circle [passing] through the center of the mirror and those two points. That [second] circle will lie entirely within the circle [of the mirror], or it will touch it [at one point], or it will intersect it.

[CASE 1]

[2.472] Let it lie entirely within [figure 5.2.49, p. 277], and draw two lines [AT and BT] from the two points to some point [T] on the facing arc [of the mirror]. The angle they will form [ATB] will be smaller than the angle one diameter forms with the other on the adjacent side of the [mirror's] center [i.e., angle DGB]; and no matter what angle is formed in this way on the facing arc, it will be smaller than the latter angle.

[2.473] For the angle [ACB] formed within the inner circle by the lines drawn from the points to the arc on it lying between [the two points] will be equal to that latter angle [DGB], because, combined with the angle [AGB] formed by the diameters above the center, it sums up to two right angles [by Euclid, III.22]. But the angle [ACB] within arc [ACB] in the smaller circle is greater than the angle [ATB] within arc [CTH of the circle] on the mirror.

[2.474] Therefore, in the arc of the circle [on the mirror], reflection will occur from only one point, since it has already been claimed [in proposition 44 above] that it is not possible for reflection to occur from two points such that both [reflected] angles are smaller than the angle formed by the diameters on the adjacent side of the center.

[CASE 2]

[2.475] If, however, that [second] circle touches the circle of the mirror at one point [T in figure 5.2.49a, p. 277], the angle formed by the lines drawn from those points [A and B] to the point of contact [T] will be equal to the angle [DGB] formed by the diameters on the adjacent side of the center [because AGB + DGB = two right angles, by Euclid, III.22, and so do AGB + ATB], so no reflection will occur from that point of contact [according to proposition 43 above]. Moreover, the angle formed at any other point [e.g., T'] on the arc [CH] of the larger circle will be smaller than that one, so, according to previous claims [in proposition 44 above], reflection will not occur from two points on that arc.

[CASE 3]

[2.476] If, on the other hand, the inner circle cuts the circle of the mirror, the two points [A and B] will lie outside the circle [of the mirror], or [they will both lie] inside it, or one [will lie] inside and the other outside, or one [will lie] on the circle and the other outside or inside it.

[SUBCASE 3A]

[2.477] If they [both] lie outside [as represented by A and B in figure 5.2.49b, p. 277], or if one lies on the circle and the other outside [as represented by A and B′ in the same figure], then the [second] intersecting circle will not cut the arc of the mirror's circle between the diameters, and so, no matter what angle is formed on that arc [e.g., ATB or ATB′], it will be larger than the angle [DGB] formed by the diameters on the adjacent side of the center. And it has already been demonstrated in the preceding figure [i.e., proposition 48] that [the forms of] these points can be reflected from only one point on the [concave] arc [KD] lying between [the diameters].[137]

[SUBCASE 3B]

[2.478] But if the two points lie inside [the mirror's circle, as represented in figure 5.2.49c, p. 277], the inner circle will cut the arc lying between [the diameters] at two points [E and F], and there will be two arcs [CE and FH] left over on opposite sides [of arc CH facing the diameters].

[2.479] If one of the points lies inside the circle and the other on the circle or outside it [as represented in figure 5.2.49d, p. 278], the [second] circle will cut the arc [CB′] lying between [the diameters] at a single point [E], and only one arc-[segment, CE] will be left over.

[2.480] If it intersects [the arc between the diameters] at two points, all the angles [e.g., ATB in figure 5.2.49c, p. 277] formed upon the arc lying between the two points of intersection [E and F] will be greater than the angle [DGB] formed by the diameters on the adjacent side of the center, and from this arc [EF] reflection may occur from only one point, or it may occur from two [by proposition 46 above].

[2.481] And from the two arcs [CE and FH] that are left over from the entire arc on opposite sides of [points E and F of intersection], all the angles [e.g., AT′B] will be smaller than the angle [DGB] formed by the diameters [on the adjacent side of the center], and reflection will occur from only one point on them [by proposition 44 above].

[2.482] And [so] in this case [when two arc-segments are left over on opposite sides of intersection-points E and F], reflection can occur from two points on the arc lying between the diameters, or from three points [in the whole arc CH, i.e., two from EF and one from CE or FH].

[2.483] Moreover, it is clear [from proposition 38 above] that reflection will occur from only one point on the opposite arc [DK], so in this case it may occur from three [points], or it may occur from four.[138]

[2.484] But if it [i.e., the second circle] cuts the arc lying between the diameters at only one point on the larger circle, all the angles formed on the segment of that arc [EB' in figure 5.2.49d, p. 278] included within the smaller circle will be greater than the angle [DGB'] formed by the diameters [on the adjacent side of the center], and [so] reflection can occur from two points, or from one point, on that part [CB' of the arc facing the diameters].

[2.485] All the angles in the other side [CE] of the arc lying between [C and B'] will be smaller than the angle formed by the diameters [on the adjacent side of the center], and reflection will occur from only one point on that segment, and so, given that reflection always occurs in this case from one point on the opposite arc [DK of the mirror], reflection may occur from three [points], or it may occur from four, but never from more.

[2.486] It is therefore clear that points lying different distances from the center may at times be reflected from only one point, at times from two, at times from three, or at times from four, but never from more [than four]. Moreover, if the [two] points lie the same distance [from the center], reflection can occur from one point only, or from two, or from four, [but] never from three [alone].[139]

[2.487] When reflection occurs from one point, one image appears; when [it occurs] from two, two [images appear]; when [it occurs] from three, three [images appear]; when [it occurs] from four, four [images appear]. But if the visible point and the center of sight lie on the same diameter, reflection will occur from an entire circle [within the sphere of the mirror], and the image-location will be [at] the center of sight [by proposition 36 above]. If the center of sight lies at the center of the mirror, though, it sees nothing [other than itself]. On the other hand, if the visible point lies at the center of the mirror, it[s image] will not be seen, because its form will reach the mirror along the normal and can be reflected only along the normal.

[2.488] Yet when the center of sight and the visible point lie outside the center on different lines, those lines, when extended to the center, will cut two arcs on different sides of the circle within the sphere. Reflection will occur from only one point on one [of those arcs], but [it may occur] from three [points] on the other. But if the center of the sphere lies on one side, while the center of sight and the visible point lie on the other, the arc that the diameters cut [on one side or the other in the circle] will be blocked by the [viewer's] head, so reflection will then occur from only three points [at most]. And if in that case the eye is directed to the arc where only a single reflection occurs, the other arc [from which] three [reflections occur] will be blocked, and only one image will appear.

[2.489] Furthermore, if the [surface of the] mirror is fully closed, no image will be perceived in it. Hence, there must be some opening in it, and it will sometimes happen that the arc lying between the diameters is open, in which case nothing can be seen in it, so it will rarely turn out that four images are seen in this [sort of] mirror. Hence, if one wants to see this multiplicity of images, he should position his eye inside the mirror [but] near its surface so that only a small part of it is blocked by the mass of his head and so that he can scan the entire surface of the mirror with his eye.

[2.490] When something is perceived in this [sort of] mirror with both eyes, if the line of reflection is parallel to the normal, the image-location will lie at the point of reflection [by proposition 32 above], and since the points of reflection are separated from one another with respect to the two eyes, two images of the same point will appear to the two eyes [and will be melded at the point of reflection]. On the other hand, if the line of reflection is not parallel to the normal, and if the visible point lies the same distance from one eye as from the other, or if the difference [in distance] is slight, the image-location will be the same for both eyes, or it will be different [for each eye] but only slightly divergent. Therefore, either a single image, or virtually a single image, will appear, as was demonstrated in the case of convex spherical mirrors [in paragraph 2.221 above].

[Concave Cylindrical Mirrors]

[2.491] In the case of concave cylindrical mirrors, the common section [of the plane of reflection and the mirror] is sometimes a straight line. When the plane of reflection intersects the axis, the common section is sometimes a circle—[i.e.,] when that plane is parallel to the bases [of the cylinder]—[and] the common section is sometimes a cylindric section [i.e., an ellipse]. When [the common section] is a straight line, image-location and the analysis of reflection will be the same as in plane mirrors. When it is a circle, the analyis will be the same as in concave spherical [mirrors]. However, when the [common] section is cylindric [i.e., elliptical], the image-location will lie behind the mirror, or beyond the center of sight, or at the center of sight, or between the mirror and the center of sight, or on the mirror itself, which will be demonstrated in the following way.

[2.492] **[PROPOSITION 50]** Let ABG [figure 5.2.50, p. 280] be the [elliptical common] section. Draw normal DG within this section, and, according to previous discussion [in book 4], it is clear that this normal is a diameter of the circle [parallel to the cylinder's base and coincident with the section], and it must be unique, because from no other point on the section can a normal be drawn to the plane tangent [to both the circle and the section].[140] Select another point [on the section], let it be B, and from it draw a line within the section that is normal to the line tangent to the section at point B, and, as claimed earlier [in proposition 26], this line will necessarily intersect normal [GD]. Let it intersect at point D, and let [point] B be chosen near enough to point G that angle BDG is acute.

[2.493] Then, from point G, draw line GH parallel to BD within the section, and it should lie within the cylindric section, because angle HGD will be acute, since it is equal to [alternate angle] GDB. From point G draw a line between D and H, and it will necessarily intersect BD. Let it intersect at point N, and between N and G select some point O. Beyond point N [on line GN] select point T. Furthermore, from point G draw another line GZ above GH, [but] still within the section, and it will necessarily intersect BD on the other side [of G]. Let E be the [point of] intersection. Draw line GQ so that angle QGD = angle ZGD, and form angle LGD equal to angle HGD and angle MGD equal to angle NGD.

[2.494] It is evident that, if the center of sight lies at point Z, [the form of] point Q will be reflected to it from point G [since QGD = ZGD by construction], and point E [behind the mirror on normal QD will be the location] of its image. If the center of sight lies at point H, [the form of] point L will be reflected to it from point G [since LGD = HGD by construction], and its image-location will be [point] G [on the mirror's surface, because normal LD is parallel to line of reflection GH]. If the center of sight is at point O, [the form of] point M will be reflected to it [from point G, since MGD = NGD by construction], and its image-location will be [point] N [behind the eye on normal MD]. If [the center of sight] lies at N, the image-location [for the form] of point M will be at the center of sight [itself], i.e., at N [where normal MD intersects line of reflection GN]. And if [the center of sight] is at T, the image-location [for point M] will lie between the eye and the mirror, because it lies at N, and so [we have demonstrated] what was set out [to be proven].

[2.495] These conclusions must be understood [to apply] when the visible point does not lie on the [same] normal as the center of sight, for in that case, since an infinite number of planes can all be imagined to lie on that normal such that each one is orthogonal to the plane tangent to the mirror, and since all of them lie on the normal, any one of those planes will form a rectilinear common section [with that tangent plane]; and reflection will only occur along the same normal, the center of sight [will constitute] the image-location, and no point will be seen unless it lies on the surface of the eye.

[2.496] However, one of those planes forms a circular common section [with the mirror], and in that case, when the center of the mirror [D in figure 5.2.50a, p. 281] lies between the visible points [e.g., C] and the eye [E], each of those points can be reflected to the eye from two points [e.g., H and L] on the circle, since lines may be drawn from each of them to form an angle with the plane tangent [to the point of reflection such] that the diameter [HDL] drawn [from that point of reflection] to the center [of the circle] bisects [that angle]. I say this, of course, about points that lie on that normal, and their image-locations lie at the center of sight.[141] The other points on that normal [i.e., between D and E in figure 5.2.50a] will not be reflected to the eye except for the point that lies on the surface of the eye, and that one [is reflected] along the normal.

[2.497] On the other hand, when the common section is a cylindric section, the points on the normal cannot be reflected from any [other] points on the section, because the form reaching along the normal must be reflected along the normal, and in [such] a section the normal is unique [as shown in book 4], so reflection will occur only along this normal, and only the point on the surface of the eye [will be so reflected], and the image-location will be at the center of sight.

[2.498] If, however, the center of sight lies at the center of the circle, the portion of the eye that the normals extending from the center of sight to the circle [on the mirror] cut off will be reflected from the corresponding portion of the circle [on the mirror] that the normals cut. Since any line extending from the center of sight to the circle is normal, reflection will occur along the normal, and the image-location will be [at] the center of sight, which is the center of the circle.

[2.499] Now, at point A [figure 5.2.50, p. 280] form acute angle FAG of some kind. It is clear that FA will intersect GZ. Let the intersection be at point Z, and form angle CAG equal to angle FAG. AC will intersect GQ. Let the intersection be at point C. It is evident that [the form of point] C is reflected to [point] Z from point G, and it is also reflected to [point] Z from point A, but not from any other point on the section, because it cannot reflect except from the endpoint of the normal, and there is only one such normal in the section, namely, GA.

[2.500] **[PROPOSITION 51]** Moreover, if two points are taken on the axis of the cylinder, [the form of] one will be reflected to the other from one full circle in the cylinder, and the image-locations will lie on a given circle outside the cylinder.

[2.501] For example, Let EZ [figure 5.2.51, p. 282] be the axis, T and H two points selected on the axis, and AG and BD the bases of the cylinder. Bisect TH at point Q, and construct a circle with Q as its center, i.e., LM, that will be parallel to the [cylinder's] bases, LM being its diameter, and BLA and DMG being sides of the cylinder. Also, construct circle KC with H as its center and CK its diameter, and draw lines TL, TM, HL, and HM.

[2.502] It is evident that each of the four angles at Q is a right angle, that TQ = QH, and that QL = QM. Those triangles [i.e., TLM and MLH] will be similar, so angles TLQ and QLH will be equal; likewise, angles TMQ and QMH will be equal. Therefore, if H is the center of sight, [the form of] point T will be reflected to point H from point L, and likewise from point M. Accordingly, if triangle TLH is rotated while axis TH remains stationary, point L will describe a circle, the two angles TLQ and QLH will remain constant throughout, and throughout this motion [the form of] T will be reflected to H.

[2.503] Now, draw line CHK until it intersects line TL, and let F be the [point of] intersection. It is evident that F will be the image-location, and as triangle TLH revolves, triangle TFH will revolve, and during this motion point F will describe a circle outside the cylinder. That entire circle will be the location for [all] images, and this is what was proposed.[142] The same method of proof will apply for any two points [chosen] on the axis.

[2.504] **[PROPOSITION 52]** Furthermore, some points that are selected outside the normal [extending] from the eye have one image, some [have] two, some [have] three, and some [have] four, but none [has] more [than four].

[2.505] For instance, let A [figure 5.2.52, p. 283] be a visible point outside the normal [extending] from the center of sight, and construct a plane passing through A and parallel to the bases of the mirror. It will, of course, form a circle on the cylinder. Let H be the center of that circle, and choose another point B within the plane of the circle, and draw diameters AH and BH [B thus serving as a center of sight within this plane].

[2.506] From what has been claimed about spherical concave mirrors [in proposition 49 above], it is clear that [the form of point] A may be reflected to [point] B from [at least] one point on the arc [passing through points E, D, and G] that subtends those two diameters, [or] perhaps from two or three, but from no more [than three]; and from the opposite arc [reflection can occur] from only one point. Accordingly, let [the form of point] A be reflected to [point] B from three points on the subtending arc [that passes through points E, D, and G], let those points be G, D, and E, and draw lines AG, HG, BG, AD, HD, BD, AE, HE, and BE.

[2.507] From point A in the same plane draw three lines AK, AF, and AN parallel to the three diameters HG, HD, and HE [respectively]. Hence, since AK is parallel to HG, BG will intersect AK. Let it intersect at point K. Likewise, BD will intersect AF. Let the intersection be at point F. So, too, BE [will intersect] AN. Let the intersection be at point N.

[2.508] Then, from point H erect axis HU, and from point B erect a line perpendicular to the plane of the circle. Let it be BT, and it will be parallel to the axis. Take some point T on it, draw the three lines TK, TF, and TN, and from the three points G, D, and E, erect three lines GM, DL, and EQ [respectively] perpendicular to the plane of the circle [each line thus being a line of longitude on the cylinder's surface]. They will be parallel to TB. Hence, EQ will lie in the plane of triangle TBN. So EQ will intersect TN. Let it intersect at point Q. Let DL intersect TF at point L, and [let] GM intersect TK at point M. These three perpendiculars will constitute lines of longitude on the cylinder.

[2.509] From point Q draw a line parallel to line NA, and it will intersect axis UH, because it will be parallel to EH [which is parallel to NA by construction]. Let the intersection be at point U, and draw line TA, which QU will intersect, because QU is drawn from one side of the triangle [i.e., TNA] parallel to the base [AN, which is parallel to HE by construction]. Let I be the point of intersection, and draw line QA.

[2.510] It is clear that angle BEH = [alternate] angle ENA [since HE and AN are parallel, by construction], while angle HEA = [alternate] angle EAN [for the same reason], and angle [of reflection] BEH = angle [of incidence] HEA [by construction. Consequently] angle EAN = angle ENA, so EN = EA.

[2.511] Moreover, EQ is perpendicular [to the plane of the circle. Accordingly] triangle QEA = triangle QEN [by Euclid, I.4]; [and so] QN = QA, and angle QNA = angle QAN [since triangle QNA is isosceles]. But angle TQI = [alternate] angle QNA [between parallels QI and NA], and angle IQA = [alternate] angle QAN [between parallels QI and NA]. [Hence], angle IQT = angle IQA, so [the form of point] A is reflected to [point] T from point Q on the cylinder.[143]

[2.512] In the same way it will be demonstrated that [the form of point] A is reflected to [point] T from points L and M, and thus from three points on the same side of the cylinder.[144]

[2.513] Nor can it be reflected from more points, for let another [such point] be given. If a side [i.e., a line of longitude] is drawn from that point, it will fall on the circle that we have, and, by repeating the proof, it will be demonstrated [according to proposition 49 above] that from the point where the side falls to the circle it is impossible for [the form of point] A to be reflected to [point] T.

[2.514] [But the form of point] A can be reflected to [point] B from one point on the opposite arc of the circle. Let Z [in figure 5.2.52, p. 283] be that point, and draw diameter HZ as well as line AC parallel to it. Then draw BZ, and let it intersect AC at point C. Erect perpendicular OZ, which will be a line of longitude and [thus] parallel to TB, and draw TC, which will be intersected by line OZ. Let the intersection be at point O. It will be proven according to the previous method that [the form of point] A is reflected to [point] T from point O.[145] And if another point from which reflection can [supposedly] occur is chosen on that side of the cylinder, it will be demonstrated by repeating the proof [according to which the line of longitude is dropped from that point to the circle] that it is impossible for [the form of point A] to be reflected [to point B] from any point on that side of the circle other than Z [as demonstrated in proposition 38 above].

[2.515] Therefore, if [the form of point] A is reflected to [point] B from one point on a given side of the circle, it is reflected from one [point] on the same side of the cylinder; if [it is reflected] from two [points on the circle, it will be reflected] from two [points on the cylinder]; if [it is reflected] from three [points on the circle, it will be reflected] from three [points on the cylinder]; but [such reflection] can [occur] from no more [than three points on that side]; whereas on the opposite side [it can occur] from only one point on the circle and [thus] from only one point on the cylinder.

[2.516] Furthermore, TB is parallel to UH, and there can be no plane chosen, other than plane TBUH, in which T lies with UH. Likewise, there can be no plane other than AUH in which A lies with UH, and it is perpendicular [to the plane of the base-circle]. Hence, T does not lie in the same perpendicular plane with A, nor on the same circle [forming its plane], nor is it on the axis, because it lies on a line parallel to it. Accordingly, the plane in which [the form of point] A is reflected to [point] T constitutes a cylindric section [because it is oblique].

[2.517] Now, let TA be extended beyond T and A on both sides to form RP. Since there are four planes of reflection, because [reflection occurs] from four points, and since the two points T and A lie in each of them, RP will be common to the four planes of reflection. Moreover, each of these planes cuts the plane tangent to the mirror at a point on its own common section [with the mirror], but not on the same common section [as any other plane of reflection].[146] Line RP is normal to one of the four common sections, but not to two, for if it were perpendicular to the tangent plane [common to two or more common sections], it would thus reach the axis. Hence, the normals [dropped] from point T to these four common sections are distinct [from one another], and there is only one that passes through A.[147]

[2.518] Moreover, the normal [within any of the planes of reflection] will either be parallel to the line of reflection or will intersect it beyond or inside the mirror. If it is parallel, the point of reflection will be the image location, as has been demonstrated [in proposition 32], and since there are four points of reflection, there will be four images. If it intersects, then, since there are four normals there will be four intersections and four images.

[2.519] Furthermore, given a visible point and a [center]point of sight, the point of reflection will be found [as follows]. For example, let A be the visible point. Construct a plane passing through A cutting the cylinder parallel to the base, and it will form a circle. Either [center of sight] B is in the plane of this circle, or it is not. If it is, we will find the point of reflection on that circle as has been shown for the spherical concave [mirror in propositions 38, 47, and the variation on proposition 38 provided in note 135]. If it is not [in the plane of the circle], draw a perpendicular to the plane of this circle from point B, repeat the previous proof, and the point of reflection will be found.[148] Moreover, when both eyes are looking, one image will actually form two, but they will abut or overlap, so they will appear single.

[Concave Conical Mirrors]

[2.520] In concave conical mirrors, the common section of the plane of reflection and the surface of the mirror will be a line of longitude along the mirror, or it will be a conic section. If it is a line of longitude, the image-locations will lie in [i.e., behind] the mirror['s reflecting surface].[149] If it is a conic section, the image-locations will sometimes lie beyond the center of sight, sometimes at the center of sight [itself], sometimes between the center of sight and the mirror, and sometimes behind the mirror, just as was shown in the case of the concave cylindrical mirror.

[2.521] Furthermore, if a physical spot is taken on the normal extending from the center of sight to the plane tangent to the mirror [and if it lies] between the center of sight and the mirror, its form will not be reflected to the center of sight along the normal, because that spot will block the endpoint of the normal [at the eye], and for that reason it will not be reflected from it. However, if there is no such [physical] spot on that normal, [the form of] a visible point will be reflected to the eye along this normal, that point, and that point only, being the one on [the surface of the eye] that intersects the normal.

[2.522] On the other hand, if the center of sight lies on that normal as well as on the axis, it will form a circle, and the line drawn to any point on it from the center of sight will be normal to the plane tangent [to the mirror at that point], so from any point on that circle reflection can occur to the eye along the normal. And the portion of the eye that the two normals cut off to form the greatest [visible] angle on it will be reflected.

[2.523] If, however, the axis lies between the center of sight and the mirror, there will be no reflection to it along the normal except for that point on it that the normal intersects.[150]

[2.524] **[PROPOSITION 53]** Now, if the center of sight and the visible point lie on the axis, [the form of] the latter will be reflected to the former.

[2.525] For instance, let H [figure 5.2.53, p. 286] be the center of sight, and T the visible point. Construct a plane that cuts the cone along the length of the axis, and let it be ABGH, with AH the axis and AB and AG edges of the cone. From point T draw TQ perpendicular to line AB, and extend it to QL. Let [QL] = QT. Then, from point H draw a line to point L, and it will intersect line of longitude AB. Let it intersect at point B, and from point B draw a line parallel to line TQ, which will necessarily reach the axis. Let it reach [the axis] at point D, and draw line TB.

[2.526] Since TQ is perpendicular to AB, and since TQ = QL, it is clear that triangle BTQ = triangle BQL, and angle QLB = angle QTB. But angle QTB = [alternate] angle TBD, and angle DBH = [alternate] angle QLB. Therefore, angle TBD = angle DBH, and so [the form of point] T is reflected to H from point B, and L is the image-location.

[2.527] Accordingly, if triangle TLH is rotated [about axis TH], point B will describe a circle on the cone, and from any point on that circle [the form of point] T will be reflected to [point] H. Meanwhile, outside this circle, L will describe a circle that will constitute in its entirety the image-location for point T.

[2.528] **[PROPOSITION 54]** Now, in this [sort of] mirror, having selected two points, i.e., Z and E [figure 5.2.54, p. 287] outside the normal [extending] from the center of sight and outside the axis, construct a plane on [point] Z parallel to the base [of the cone]. It will produce a circle in the mirror. E will lie in this circle or in another plane.

[CASE 1]

[2.529] Let it lie in the plane of that circle, and draw line EZ. It is evident [from proposition 49, case 3 above] that [the form of point] Z is reflected to E on one side of that circle from one point, or from two, or from three; and on the other side [it is reflected] from one [point].

[2.530] Take a point on the circle from which [the form of point Z] is reflected [to E], let it be H, and [let] T [be] the center of the circle. Draw lines ZH and EH. Diameter TH will bisect the angle [formed by them], and it will intersect line EZ. Let it intersect at point Q, let A be the vertex of the cone and AH a line of longitude.

[2.531] From point Q draw line QM falling orthogonally to line AH, and let it reach the axis [AT]. Let it fall at point D on the axis, and draw lines ZM and EM. From point Z in the plane of the circle draw line ZL parallel to line QH. Let EH intersect it. Let the intersection be at point L, and from point H draw HC perpendicular to LZ.

[2.532] Then, in the plane of triangle EMZ draw line ZO parallel to line QM. Let EM intersect it at point O, and draw line LO. From point C draw CN parallel to LO, and draw line NM.

[2.533] It is clear [by construction] that angle EHQ = angle QHZ as well as [alternate] angle HLZ [since ZL is parallel to QH by construction], and angle QHZ = [alternate] angle HZL. [Thus] HL = HZ, and HC is perpendicular to LZ [by construction. Since, therefore, HC is common to triangles LCH and CHZ] triangle LCH = triangle CHZ [by Euclid, I.26], and [so] LC = CZ.

[2.534] CN is parallel to OL [by construction, so, by Euclid, VI.2] LC:CZ = ON:NZ, so ON = NZ. Furthermore, since OZ is parallel to QM [by construction], plane ZLO will be parallel to plane QMH. Plane EOL intersects these two [planes] along [rectilinear] common sections, i.e., MH and OL, that will be parallel, so HM and CN are parallel [because CN is parallel to OL, by construction]. And because HC falls between parallels LZ and HQ, and since it is perpendicular to LZ, it will be perpendicular to HQ, so CH will be tangent to the circle.

[2.535] Therefore, plane AHC is a plane tangent to the cone. CN and NM lie in this plane, and line DM is perpendicular to this plane. It is therefore perpendicular to line NM, so NM is perpendicular to OZ, and [it has already been established that] ON = NZ. [Therefore, by Euclid, I.4] MO = MZ, and [so] EM:MO = EM:MZ [by Euclid, V.7].

[2.536] But EM:MO = EH:HL [by Euclid, VI.2], EH:HL = EH:HZ [by Euclid, V.7, because HL = HZ by previous conclusions], and EH:HZ = EQ:QZ [because HQ bisects angle EHZ]. Thus, EM:MZ = EQ:QZ [since EM:MZ = EM:MO = EH:HL = EH:HZ], so [because the respective sides of triangles EMQ and ZMQ are proportional, making the two triangles similar] angle EMQ = angle QMZ, [and] so [the form of point] Z is reflected to E from point M. Hence, if [the form of point] Z is reflected to E from point H on the circle, it is reflected to the same point from point M on the cone. And if [it is reflected] from two [points] on the circle, [it is reflected] from two [points] on the cone; if [it is reflected] from three [points on the circle, it is reflected] from three [points on the cone]; and if [it is reflected] from more [points on the circle, it is reflected] from more [points on the cone].[151] On the other side of the circle, the proof that [reflection occurs] from one point on the cone just as from one [point] on the circle will be constructed in the same way.

[CASE 2]

[2.537] However, if E does not lie on the circle that passes through Z parallel to the base [of the cone], E will lie above or below it. Let it lie above, since the same proof applies to both cases. Draw line AE [figure 5.2.54a, p. 288] until it touches the plane of that circle, and let H be the point of contact, Q being the center of the circle. It is obvious that [the form of point] H can be reflected to Z from some point on the circle. Let it be T, and draw diameter QT [normal to the point of reflection]. Line HZ will intersect this diameter at point N. Draw [line] EZ and line of longitude AT.

[2.538] Since point Z lies on one side of diameter QT, and since [point] E lies on the other, it is evident that line EZ will intersect plane AQT. Let it intersect at point O, and from point O draw [line] OC perpendicular to line AT, and it will necessarily fall upon the axis. Let it fall at point D, and draw lines EC and ZC. I say that [the form of point] E is reflected to Z from point C.

[2.539] [Here is] the proof. From point Z draw line ZF parallel to [line] QT, and extend line HT until it intersects it. Let the intersection be at point F. Likewise, from point Z draw [line] ZK parallel to line OC, and extend line EC until it intersects it. Let the intersection be at point K.

[2.540] Since line ZF is parallel to [line] QT [by construction], while [line] ZK is parallel to [line] OC [by construction], it is clear that plane ZKF will be parallel to plane OCT, which lies within plane AQT [since O is where HK intersects plane AQT]. Plane HFK intersects these two planes along lines CT and KF. Hence, CT and KF are parallel.

[2.541] From point T draw [line] TP perpendicular to line ZF. Since it falls between two parallels [i.e., TQ and ZF], it is obvious that it will be parallel to line NZ, and so it will be tangent to the circle [at point T].[152] Hence, plane ATP is tangent to the cone along line [of longitude] AT, and line OC is perpendicular to this plane. Accordingly, plane ATQ will be orthogonal to plane ATP, and plane ATP intersects the two planes ATQ and ZKF, which are parallel. Thus, the common sections, one being CT, the other PI, are parallel. But it has already been shown that CT is parallel to KF. Hence, PI is parallel to KF.

[2.542] But it is evident that [since ZT falls between parallels NT and ZF] angle NTZ = [alternate] angle TZF, whereas angle HTN = [alternate] angle TFZ, and TP is perpendicular [to ZF. Therefore, by Euclid, I.26] FP = PZ. But FP:PZ = KI:IZ, [so, by Euclid, VI.2] KI = IZ.

[2.543] Now, if line CI is drawn, then, since plane ATPI is perpendicular to plane ZKF [which is parallel to plane ATQ], CI will be perpendicular to ZK, and angle CKZ = angle KZC. But angle ECO = angle CKZ [by Euclid, I.29, because OC and KZ are parallel, and EC cuts them both], and angle

OCZ = [alternate] angle CZK, so angle ECO = angle OCZ. Hence, [the form of point] E is reflected to [point] Z from point C, which is what was proposed.

[2.544] Moreover, if another point is taken on the circle from which [the form of point] H is reflected to [point] Z, it will be demonstrated that [the form of point] E is reflected to Z from some point other than C on the cone. And if [the form of point] H is reflected to [point] Z from three points on the circle, [the form of point] E will be reflected to [point] Z from three [points] on the cone; if [it is reflected] from four [points on the circle, it will be reflected] from four [points on the cone].

[2.545] Furthermore, the point of reflection from which [the form of point] E is reflected to [point] Z is easy to find when the point on the circle from which [the form of] point H is reflected to [point] Z is found, and it will be found in the preceding way.

[2.546] If, however, it were claimed that [the form of] point E could be reflected to [point] Z from more than four points on the cone, it would be possible by repeating the earlier proof to show that [the form of] point H is reflected to [point] Z from more than four points on the circle, and in the case where [the form of] point E will happen to be reflected to [point] Z from however many points on the circle, or from one only, [the form of] point E will happen to be reflected to [point] Z from that many points on the cone, or from one only, and vice-versa. But if [this] contrary claim is made, it can be disproven in the preceding way [i.e., according to proposition 49].

[2.547] Hence, it is clear that some of the points [that are reflected] have a single image, some [have] two, some three, and some four, but no [more] than four are possible. In addition, when the mirror is exposed to both eyes, the same image will have different locations, but, because of its imperceptibility, this difference [in location] does not cause an error [in visual perception].

NOTES TO BOOK FIVE

[1]Here Alhacen establishes the agenda for the first topical segment of book 5: to verify the cathetus-rule of image-location on the basis of both empirical observation and natural philosophical principles. Specifically, what is to be shown is that the image of any point-object invariably lies at the intersection of the line of reflection and the normal dropped from the point-object to the reflecting surface. As already established in book 4, these lines and their constituent points lie in a single plane that is orthogonal to the reflecting surface at the point of reflection, and with that in mind Alhacen sets out in the next thirty-nine paragraphs to show that the cathetus-rule applies to any reflection, no matter the shape of the reflecting surface or the location of the viewpoint.

[2]Obviously, in the case of the cone, unlike that of the straight thin rod, the fact that the point-to-point connection is orthogonal to the mirror cannot be seen directly; it must be inferred through what Alhacen later refers to as "sense-induction" (see note 19, p. 490 below). Alhacen's choice of a cone both here and in subsequent experiments is dictated by his analysis in book 4 of the radiation of light according to cones.

[3]The observations discussed to this point are illustrated in figure 5.1, p. 215.

[4]The image of the rod will be compressed in a spherical mirror, and the sharper the curvature of that mirror, the greater the compression. Accordingly, the apparent distance of the mark on the rod's image from the mirror's surface will be less than the distance from that surface of the mark on the actual rod, but both distances will be proportional to the lengths of the two rods.

[5]In other words, even though the narrowing of the actual rod may be fairly gentle, the compression of the rod's image will make that narrowing appear more radical than it actually is. Such distortions, which fall under the category of visual illusions for Alhacen, are dealt with in book 6 of *De aspectibus*.

[6]What Alhacen seems to have in mind here is moving the cone on the mirror until some portion toward its base is blocked from view by the mirror itself—i.e., so that it falls behind the "horizon" of the mirror. In that case, the image of the portion that remains visible over the mirror's horizon will be visible, and it will appear to lie directly in line with that portion.

[7]The illusion to which Alhacen alludes here involves both the compression of images and the bowing of those images under most circumstances in convex cylindrical mirrors. Accordingly, if the rod is placed aslant to the mirror's surface, or if a cone is placed on its surface, the relationship between the object itself and its image will be skewed by curving in the mirror. Adding to the complexity of image-formation in convex cylindrical mirrors is the fact, already discussed in book 4, that the plane of reflection can form three different sections on the cylinder's surface

when it cuts it: a straight line along the longitude, a circle parallel to the bases of the cylinder, and an ellipse.

[8]As constructed earlier, the panel is 6 digits high, and the cylindrical mirror inserted into it is 3 digits high. Moreover, that mirror was inserted so that its midpoint along the midline of longitude on the panel lay precisely 3 digits above the bottom of the panel. Thus, the top of the mirror lies 4.5 digits above the bottom of the panel. But by construction the distance from the top surface of the bronze plaque to the top surface of the ring is 5 digits plus half a grain of barley. Accordingly, if the panel were inserted into the hollow of the ring to rest at the level of the bronze plaque, the top of the mirror would lie half a digit plus half a grain of barley below the top surface of the ring. It is clear from the subsequent experiment, however, that the mirror must protrude above that surface, so presumably we are meant to fill the hollow with wax an adequate distance above the level of the bronze plaque to raise the panel with its inserted mirror high enough that the top of the mirror will stand above the top surface of the cylinder.

[9]Presumably, the kind of ruler Alhacen has in mind is the one represented in the left-hand diagram of figure 4.3.5, p. 194.

[10]In fact, Alhacen has already established this point empirically and descriptively in 4, 5.19-5.24, pp. 47-50 above.

[11]The experiment just completed is illustrated in the top diagram of figure 5.2, p. 216. The mirror is represented by the segment of the cylinder in which DRE represents the midline of longitude. Imagine this segment inserted into its panel so that the mirror protrudes above the top surface of the ring, the inner edge of which is represented by arc FG. The circle passing through point R on the cylinder represents the projection of the plane of this surface through the cylinder, so this circle and FG are in the same plane. Along the midline of the top surface of the ring we pose a ruler so that it reaches midpoint R of the mirror and so that its sharpened edge touches the surface of the mirror at point R, where tangent XRY and midline of longitude DRE intersect. The surface of the panel is therefore tangent to the mirror along line of longitude DRE. Then, along the edgeline of the ruler, we place a needle with a small white object O at its endpoint. The needle, which is clearly meant to serve as a guideline for observation, is therefore posed perpendicular to the plane of the panel, so it lies directly in line with point R and centerpoint C of the projected circle on the cylinder. That point, of course, lies on axis HU of the cylinder. Under these conditions, if we establish a line of sight along the needle with one eye that is raised slightly above the needle, we will see that object O and the needle lie directly in line with their images in the mirror. Hence, the image of O lies on the normal along which the needle lies. In that case, moreover, the plane of reflection will be defined by line OR and midline of longitude DR on the mirror, so the plane of reflection will be orthogonal to the plane tangent to the mirror along that line of longitude—i.e., the plane of the panel—which includes tangent XY to point of reflection R. If we shift our point of view to B or A and establish a line of sight in the plane of the top surface of the ring, we will see that the images of O and the needle lie directly in line with their actual counterparts. In this case, the plane of reflection is defined not by OR and midline of longitude DR but by OR and

tangent XY. Nonetheless, since the needle is perpendicular to the plane of the panel, which is orthogonal to the plane of reflection, the image lies on the normal to its surface, and, no matter where we place the center of sight along the top edge of the ring, the image of OR will lie directly in line with OR itself. With this experiment, then, Alhacen has verified the cathetus-rule in two of the three cases according to which the plane of reflection cuts the cylinder: i.e., when that cut forms a line of longitude on its surface or when it forms a circle.

[12]In this experiment, Alhacen instructs us to place the panel atop a wax triangle, as represented in the bottom diagram of figure 5.2, p. 216, where the base of the panel is attached to triangle K. This triangle is thick enough to hold the panel firmly upright at the appropriate tilt when it is inserted into the hollow of the ring. As before, the mirror protrudes above the top surface of the ring, which is represented by FG, but in this case the plane projected through FG cuts an elliptical section on the mirror's surface because of its slant. The rest of the experiment is the same as illustrated in the top diagram. Thus, the needle with its object is placed along OR, which is normal to the mirror at point R and passes through point C on the cylinder's axis. If we then look with one eye along OR, we will see the images of the object and the needle directly in line with their actual counterparts. And if we shift the line of sight to B or A, the images and the objects will still be in perfect alignment, so the cathetus-rule is verified for the third case according to which the plane of reflection cuts the cylinder: i.e., when that cut forms an ellipse on its surface.

[13]In other words, the perpendicular is the shortest possible line between the object-point and the mirror, and this is the line through the point of reflection and the axis. The relevance of this point becomes clear later on, when Alhacen explains why the image should appear on the normal rather than on some other line, an explanation that appeals to both natural economy and symmetry.

[14]Thus, as Alhacen showed in book 4, all planes other than those that cut circles on the cone's surface will be planes of reflection.

[15]As will become clear shortly, Alhacen considers only those images that spread out on the mirror's surface not to be perceived according to reality, because they do not represent their object in any recognizable way. Later on, Ahacen will explain that there are five cases of reflection in concave mirrors that involve five different relationships between the normal dropped from the object-point and the line of reflection. One of them involves no intersection, both the normal and the line of reflection being parallel. Of the remaining four, one intersection occurs at the center of sight itself, and another occurs behind the center of sight, where the image cannot be seen.

[16]Obviously, the more brightly illuminated the cone, the brighter its image, especially in iron mirrors.

[17]In this case, the eye is displaced only slightly from the vertex of the cone; otherwise, the image will appear not behind but in front of the mirror.

[18]Alhacen's point about images that are perceived in front of the mirror is illustrated in figure 5.3, p. 217. Arc AB represents a section of a spherical concave mirror. The cone with its vertex at centerpoint C of the sphere is placed to the side so

that rod DF can be stood directly along line ECDF, endpoint D of that rod lying below centerpoint C of the circle, and center of sight E lying above it. Accordingly, the viewer is to direct his attention to point R, where line CR falls between D and E until he sees the image of endpoint D of the rod in front of the mirror. The form of point D will have reached E by reflection along RE, and it will lie directly in line with centerpoint C of the sphere and object-point D, so line ECDF will form a diameter of the circle and will therefore be orthogonal to the mirror's surface at F. Accordingly, the image of D will be seen at the intersection of line of reflection ER and normal FE. This, of course, is a gross simplification of a much more complex situation, but the empirical results are roughly consonant with the geometrical idealization.

[19]By "sense-induction" Alhacen clearly means a process of trial-and-error sighting rather than a process of precise instrumentation.

[20]Figure 5.4, p. 217, illustrates the experiment to this point. The mirror, whose midline of longitude is DRE, forms a segment of the cylinder. FG is a segment of the inner edge of the ring, and the middle circle passing through R and centered on C is the projection of the plane of the top surface of the ring onto the cylinder containing the mirror. Thus, centerpoint C of that circle lies between R and segment FG of the ring. The needle is placed along line OR within the plane of the circle centered on C and perpendicular to midline DRE of the mirror at point R. OR is therefore perpendicular to tangent XY at point R. If the center of sight is stationed in the plane formed by OR and DRE, then reflection will occur anywhere along line of longitude DRE. Thus, if the eye is raised just a bit above object O at the end of the mirror, it will see that object, the needle, point R on the mirror, the needle's image, and O's image directly in line with one another along perpendicular OR and its extension behind the mirror. On the other hand, if the eye is stationed to the side, along AR, for instance, then the image of O will appear in front of the mirror but along line OR, so that O, the needle, O's image, and point R in the mirror will appear directly in line with one another. Thus, in both these cases, the image will appear at the intersection of the line of reflection, which is necessarily the line along which the image is seen, and normal OCR.

[21]In this case, if the center of sight is still fixed at A in figure 5.4, p. 217, and if another needle, like OR, with a small white object on its end is placed on the top surface of the ring so that its point lies directly in line with R and perpendicular to line of longitude DRE, then the image of the object will appear in front of the mirror and in line with the object, the needle, and point R on the mirror.

[22]See esp. 2, 3.86-3.91 and 2, 3.135-3.146, in Smith, *Alhacen's Theory*, 454-456 and 475-479.

[23]As Alhacen remarked earlier, we perceive images as if they were objects. Hence, our judgment of their size and distance depends on size-distance invariance, according to which we judge the distance of a given object according to its perceived size, which decreases as it gets farther away; see, e.g., 2, 3.137, in ibid., 475. So it is essentially by a judgment of the amount of the visual field subtended by the image, correlated with a knowledge of the actual size of the object represented by the image, that we come to a conclusion about its distance. That is why, for instance,

when we see ourselves in a plane mirror, our image appears to lie as far behind the mirror as we actually lie in front of it. This distance is twice the distance between our face and the mirror, since the image reaches us along both lines of incidence and lines of reflection.

[24]Alhacen is arguing from the equality of triangles OAR and IAR in the top, left-hand diagram of figure 5.5, p. 218, O being the object-point and I being the image-point. Thus, since OA and IA are equal, by supposition, since AR is common, and since angles OAR and IAR are right, it follows by Euclid, I.4 that OR and IR are equal.

[25]Alhacen's argument here is a simple one based on symmetry. Let E in the top left-hand diagram of figure 5.5, p. 218, represent the center of sight, O the object, AR the mirror's surface, and R the point of reflection. We have already established that the image is seen along the extension of line of reflection ER. If the image is located at I, where normal OA intersects ER extended, then the two triangles OAR and IAR will be equal, so sides RI and OR on them will be equal. Thus, I will be perceived to be the same size and lie the same distance from E as the object would if it were viewed from a distance composed of OR and RE. If the image lay beyond the normal at I', it would appear smaller and farther away than it should, whereas if it lay in front of the normal at I', it would appear larger and nearer than it should.

[26]In referring to the center of the eye here, Alhacen means not the centerpoint of the eyeball, which constitutes the center of sight, but rather the point on the cornea through which the visual axis falls.

[27]Let the circle centered on E in the top, right-hand diagram of figure 5.5, p. 218, represent the eye, the circle centered on C the mirror, arc AB the portion of the eye visible in the mirror, and arc A'B' the image of the eye seen in the mirror. If the form of the eye's center, where line EC intersects arc AB, were to radiate to the mirror along a line other than normal EC and reflect back along it, then the image of E seen along that line and through that point in the mirror would be asymmetrical with respect to the image of the eye as a whole. On the other hand, if the image of A were to be seen at A" outside of normal AC, then the resulting image A"B' would be asymmetrical with respect to the image of the centerpoint. Hence, it is only when the image of each point on the eye lies at the intersection of the line of reflection and the normal through that point that the image maintains its symmetry with respect to the eye itself. As Alhacen remarks, if the form reflects back along the normal, then there is no point of intersection between the two coincident lines, which means that there is no defined image-location. At the end of proposition 2, p. 400, however, he resolves this problem by claiming that the points surrounding the center of the eye do have a definite image-location, so we infer the image-location of the eye's centerpoint from the context of the image-locations of those surrounding points.

[28]In providing this caveat Alhacen appears to acknowledge the tentative nature of his argument from symmetry, which is essentially an aesthetic argument. In so doing, he also seems to be acknowledging the limits of certainty in human reason, even when it is carried out punctiliously.

[29]The claim that an image in a spherical convex mirror can appear beyond or outside the mirror seems at first glance to indicate that such images are actually

projected outside the mirror in the way real images are projected from concave mirrors. In fact, Alhacen means no such thing. What he does mean is that, in certain cases, the intersection-point of the line of reflection and the normal dropped to the mirror's center from the object-point falls beyond the great circle formed on the mirror by the plane of reflection on the invisible side of the mirror. Thus, even though it may be technically formed outside the mirror, such an image is still seen within the visible portion of the reflecting surface. As Alhacen points out in the very next paragraph, the same holds for convex cylindrical and conical mirrors. These points will be clarified in detail later on in various propositions. What Alhacen means in saying that the faculty of sight does not perceive the location of any image that coincides with the mirror's surface by deduction (*syllogistice*) is that in such a case it does not need to estimate its distance behind or in front of the surface according to the process discussed earlier in paragraphs 2.33-2.34.

[30]In all these cases, the image-location, whether beyond, at, or inside the mirror, is a function of where the image lies with respect to the common section of the mirror's surface and the plane of reflection. In all the manuscripts collated, the word I have translated as "inside" is *citra* rather than *infra*. This use of the term is puzzling, because *citra* is generally applied by Alhacen to those images that appear to lie "in front" of concave mirrors.

[31]Adapted from Ptolemy's analysis in *Optics*, IV.71 (Smith, *Ptolemy's Theory*, 195), the lower diagram in figure 5.5, p. 218, illustrates these points. Alhacen's analysis later in proposition 32 below is somewhat more complex. Let E be the center of sight, C the center of the mirror, and R the point of reflection. When point O1 is chosen on line of incidence O5R, its form reflects to E along RE, and its image I1, where normal CO1 intersects line of reflection ER, lies behind the mirror. When point O2 is chosen on the same line of incidence, its form reflects to E along the same line of reflection, but its normal O2C is parallel to the line of reflection, so there is no intersection. The reflection of the form of point O3 yields an image at I3, which is behind the center of sight, whereas the reflection of the form of point O4 yields an image at center of sight E itself. Finally, the reflection of the form of O5 yields an image at I5, which lies between the center of sight and the mirror. Of these images, only I1 and I5 appear clearly; the rest are nebulous, so they do not represent their objects "as they exist in reality," according to Alhacen's brief discussion in paragraph 2.26 above.

[32]That angle BGD = angle AGH follows from the supposition that angle of incidence BGE and angle of reflection AGE are equal. Since angles EGD and EGH are both right, by construction, then the remaining angles BGD and AGH must be equal.

[33]This is essentially the rationale provided in paragraphs 2.38 and 2.39 above.

[34]In this case, of course, line AB forms the common section of two intersecting planes, one of which intersects the mirror's surface orthogonally, the other of which necessarily intersects that surface obliquely. Hence, no line drawn from AB to the mirror's surface in the plane that intersect that surface obliquely can be orthogonal to that surface.

[35]In other words, given a single line of incidence to a given point on the mirror, there cannot be two different lines of reflection from that point.

[36]As applied to plane mirrors, these four propositions encapsulate the four main points that Alhacen intends to address for the remaining six mirrors in order from convex spherical, through convex cylindrical, convex conical, concave spherical, and concave cylindrical, to concave conical. Those points are: 1) how to find the image-location given the center of sight, the point-object, and the point of reflection; 2) how to find the point of reflection given the center of sight and the point-object; 3) how many images of a given point-object can be formed for a given center of sight; and 4) how the resulting image or images are perceived in binocular vision.

[37]In other words, line HG, which is supposed not to reach the circle's center, and NG, which does, must both be perpendicular to line PE, so they must coincide.

[38]The purpose of establishing that AN (the distance between object-point A and the mirror's centerpoint N) is to AE (the distance between object-point A and endpoint of tangency E) as DN (the distance between image-point D and the mirror's centerpoint N) is to DE (the distance between image-point D and endpoint of tangency E) become clear later in proposition 16, where the proportionality is instrumental in proving that, for any given object-point and any given center of sight facing a convex spherical mirror, there can be only one point of reflection. Hence, the endpoint of tangency is of ancillary—and, indeed, very limited—significance in comparison to the four cardinal points (i.e., the center of sight, the object-point, the point of reflection, and the endpoint of the normal) listed in 4, 5.10, 4, 5.26, and 4, 5.39, pp. 329, 335, and 340 above.

[39]In other words, as shown at the end of proposition 2, the image-location of the point on the anterior surface of the eye where normal GD intersects it will be determined by the image-locations of the points on the eye's surface surrounding it.

[40]The reason for proving this point is that, if line of incidence OH and KD did intersect on the side of O and K when extended, then KD would be the normal dropped from their point of intersection, so its image would lie at point K on the mirror's surface, where the line of reflection GH meets KD. The image would thus not be inside the mirror, as proposed.

[41]As in the previous case, so in this one, if line of incidence PI were to meet SD on the side of P and S, the image of their point of intersection would be at S on the mirror's surface rather than inside the mirror.

[42]In other words, when GHK rotates about axis GD, point H on the sphere's surface will describe a circle with its center on GD, and for any point of reflection on the portion of the mirror cut off by that circle, up to the circumference described by H, the image seen from center of sight G will lie inside the sphere that defines the mirror.

[43]In this case, then, the segment of the mirror cut off by the rotation of GB will be defined by point B of tangency, so the relevant section of the mirror is contained by the circle described by B on the outside and the circle described by H on the inside. Whether the image lies inside, outside, or on the surface of the sphere that defines the mirror depends upon both the location of the point of reflection within that segment and the location of the object-point.

[44]Thus, as pointed out in note 29 above, although the image may lie outside the circle on the mirror defined by the plane of reflection, that image will always be seen in the visible portion of the mirror, never outside it.

[45]In other words, if ED is to serve as the normal for some object-point radiating to the mirror along line of incidence QF, then ED must intersect QF at that point.

[46]As specified in the previous proposition, this limit is determined by the equality of segment AT on diameter DA and FT on line of reflection BFT in figure 5.2.11, p. 225. As was demonstrated in that proposition, no point between T and A can serve as an image-location because none of the lines of incidence falling to points between F and H will meet normal AD on the side of D.

[47]As defined previously, O is the inner limit according to the equality of MO and BO. T is the outer limit insofar as proposition 9 established that all image-locations lying at the mirror's surface or outside it must lie on diameters below point G of tangency, and it is obvious that no reflection can occur from G, since lines GT and AG of incidence and reflection can form no angle, being part of one continuous straight line.

[48]That is, in order to get to N, it would either have to cut the circle somewhere below N or loop through tangent AGT below N.

[49]The reasoning behind this *reductio ad absurdum* seems to be as follows. We have established by construction that RL < BL. Given that supposition, the object is to project a line of reflection from A to normal LB through arc GD such that the segment of that line between the arc and the normal is equal to the segment of the normal between B and the point where the line of reflection intersects it. This former segment will either fall to the left of RL, or it will fall to the right. Suppose it falls to the right, and let it be HF in figure 5.2.13, p. 226. By supposition, then, HF = BF, so HF > LB. But HF < MO, which = LB. Therefore, HF is both longer and shorter than BL, which is impossible. Consequently, the sought-after segment must lie to the left of RL. That there is such a segment to the left of RL follows from the fact that, as the line of reflection from A sweeps from R to D along arc RD, and from L to B along LB, the segments on it lying between the arc and normal LB increase to at least the length of DB. As those segments lengthen, however, the segment they cut off from BL on the side of B shorten. So there must be a point at which the one segment's lengthening counterbalances the other's shortening until the two segments become equal. The main thrust of this proof is to preempt the possibilty that this point might lie outside the circle, because if it did, then from that point inward toward B, the relevant segment of the line of reflection projected from A to normal BL through arc GD would be longer than the resulting segment of the normal. It would therefore follow that no point on FB could be an image location for center of sight A, according to how the limit of image-location is defined in proposition 10. But that contradicts propositions 9-11, where it is established that, if reflection occurs from any point on arc GD, there will be images inside the mirror.

[50]What Alhacen means here is that the visual faculty does not perceive where the various image-locations lie within the invisible portion of the mirror in relation to the great circle produced in that portion of the sphere by the plane of reflection. All it perceives is that the images lie behind the visible portion of the mirror.

[51]Consequently, by Euclid, I.4, triangles BNG and ANG are equal, so angle BGN = angle AGN, and so their adjacent angles are equal, one being the angle of incidence, the other the angle of reflection.

[52]This operation can be done as follows according to figure 5.2.19b, p. 230. Construct a circle whose diameter = QE. Take some random diameter XY, and to endpoint X apply tangent ZX = AG. Then, from endpoint Z of that tangent draw line ZECQ through centerpoint C of the circle. From Euclid, III.36, it follows that the rectangle contained by diameter EQ + segment ZE and segment ZE (i.e., ZQ,ZE) is equal to the square on the tangent, which is equal to the square on AG, since tangent ZX was constructed equal to AG.

[53]That AH = EZ follows from the fact that DA = QZ and that DA,AH = AG^2, as just concluded. But we established earlier by construction that QZ,ZE = AG^2. Hence, DA,AH = QZ,ZE, and since DA = QZ, by construction, then AH = ZE. Moreover, since DA = DH + AH, since DA = QZ, and since AH = ZE, it follows that DA – AH = DH = QZ – ZE = QE.

[54]For the relevant proposition in Apollonius' *Conics*, see R. Catesby Taliaferro, *On Conic Sections: Books I-III* (Encyclopedia Britannica, 1952), 685-686.

[55]The relevant proposition is II.8, in ibid., 687.

[56]As previously established, AD^2 = AD,DN + AD,AN; but GD^2 = AD,DN; thus, AD^2 = GD^2 + AD,AN; also, AD^2 = BD,DG = GD^2 + BG,GD; thus, AD^2 = GD^2 + BG,GD = GD^2 + AD,AN; so GD^2 + BG,GD = GD^2 + AD,NA; subtracting GD^2 from both sides, we get BG,DG = AD,AN or, reversed, AD,AN = BG,DG.

[57]In fact, Euclid does not demonstrate this explicitly, but in III.36 he proves that, if any line, such as DA in figure 5.2.19d, p. 232, cuts a circle from a point outside it, the rectangle formed by the segments DH and HA produced by the cut will be equal to the square on the tangent dropped to the circle from that outside point. Since both DA and GB are projected from the same point D, their respective rectangles, DH,HA and DG,GB will be equal to the same tangent squared, so they will be equal.

[58]We have established that AD,DH = BD,DG and that AD,DH = HD,DN + DH,AN. Therefore, BD,DG = HD,DN + DH,AN. But we have also established that HD,DN = GD^2. Therefore, BD,DG = GD^2 + DH,AN, which is to say that GD^2 = BD,DG – DH,AN. But it has also been established that BD,DG = BG,GD + GD^2, which is to say that GD^2 = BG,GD + GD^2 – DH,AN, which finally translates to GD^2 + DH,AN = GD^2 + BG,GD. If we drop the common term GD^2 from both sides, we are thus left with DH,AN = BG,GD.

[59]The point here is illustrated in figure 5.2.20, p. 233, by the two lines TC and TC′ dropped from point T on section TP to section CU. In the first case, TC′ = BG represents the minimal distance between point T on section TP and its matching branch CU. Hence, if the circle with radius TC′ is produced from T, it will touch section CU at only one point. In the second case, the minimal distance between T and section CU is less than BG, so the circle with radius TC = BG produced from T will intersect section CU at two points, one such intersection being at point C on section CU. The two intersections at C and C′ are represented in figure 5.2.20a, p. 234. The shortest possible distance between any two hyperbolic sections is the

apex-to-apex distance along the axis, so if T lies at the apex of section TP, and if the shortest distance between the two sections is equal to BG, then TC = BG must touch facing section CU at its apex, in which case TC will pass through the intersection of the two asymptotes. If, on the other hand, T lies at the apex of section TP but the apex-to-apex distance between TP and CU is less than BG, then TC = BG will intersect CU at two points. Finally, if T does not lie at the apex of TP, as represented in figure 5.2.20a, then the minimal distance between T and section CU must be no more than BG if TC = BG is to touch that section. If it is less, of course, then the circle with radius TC = BG produced from T will intersect section CU at two points.

[60]As given in the actual manuscripts, figure 5.2.20 misrepresents this case, since all of them show CT intersecting section CU in such a way that the circle with radius TC = BG will cut CU at two points. Thus, contrary to what is specified here in the text, CT is not the minimal distance between T and section CU. There is no provision in the Latin text for insuring that the minimal distance between the hyberbolic sections produced not be greater than TC = BG, but according to Sabra, "Ibn al-Haytham's Lemmas," 310, the Arabic text refers the reader to V.34 and V.61 of Apollonius' *Conics*.

[61]We established earlier that GB:BD = LM:MH, so by reversal BD:GB = MH:LM. Hence, by Euclid, V.16, BD:MH = GB:LM. We just concluded that BD:DE = MH:HZ, so again, by Euclid, V.16, BD:MH = DE:HZ. Therefore, GB:LM = DE:HZ, and so, by Euclid, V.16, GB:DE = LM:HZ.

[62]Alhacen's point here is illustrated in figure 5.2.20b, p. 234, which is an extension of figure 5.2.20. In addition to the original angles and the resulting line AED produced on the basis of line TC = BG passing above the intersection of asymptotes HL and NZ—as represented by the fainter lines—another line AED can be produced according to TC' = BG passing below the intersection of asymptotes HL and NZ—as represented by the darker lines. According to the resulting triangles, then, the previous proof will apply in this case as well.

[63]GM will be a diameter since angle GDM is right, by construction, and is therefore subtended by a semicircular arc; so GM bisects the circle.

[64]The sense of this passage in the Latin text is confusing, since it implies that the whole of line NL = H, but it becomes clear later in the proof of case one that it is not NL but its segment CL that is equal to H.

[65]That these two triangles are similar follows from the fact that angle DMN + angle DCN = two right angles, as does angle TCL + angle DCN, so angle TCL = angle DMN, which = angle TAD, by construction. But vertical angles CTG and ATD are equal, so the remaining angles of each triangle—i.e., TLC and ADT—must be equal.

[66]Although Alhacen's proof for case 2 is based on having D lie on BG beyond point B—i.e., beyond the right angle of triangle ABG—D can also lie on BG beyond point G, as illustrated in figure 5.2.21b, p. 235. In that case, the construction and proof are slightly different from those provided by Alhacen in cases 1 and 2. As before, we start by drawing DM parallel to AB so that angle GDM is right. We then extend AG to meet it at point M. Then, we form angle DMN equal to angle GAD and produce the circle through points M, N, and D. GM will thus be a diameter on

that circle, since it subtends right angle GDM. From point N we extend line NCL to meet line MGA at L such that CL = H according to the required proportion AD:H = E:Z. Finally, we draw line CG and extend line DC to meet AB at point Q. DQ will intersect AG at T. It therefore remains to demonstrate that TQ:TG = AD:LC (which = H) = E:Z.

The proof is as follows. Angle QAT = alternate angle GMD, since BA and MD are parallel by construction. Angle GCD + angle GMD = 2 right angles, by Euclid, III.22. But angle QAT + angle QAL = 2 right angles, by Euclid, I.13. Hence, angle QAL = angle GCD, so their adjacent angles GCT and QAT are equal. Since corresponding angles GCT and QAT in triangles GTC and QAT are equal, and since vertical angles GTC and QTA in those same triangles are equal, the remaining angles are equal. Thus, the triangles are similar, by Euclid, VI.4, so it follows that TA:CT = QT:TG. Meantime, angle NMD = angle TAD, by construction. But angle NMD = angle NCD, since they are both subtended by the same arc ND, and angle NCD = vertical angle TCL. Thus, angles TCL and TAD in triangles TCL and TAD are equal, and angle CTL is common, so the remaining angles TLC and TDA are equal. Triangle TCL is therefore similar to triangle TAD, from which it follows that TA:CT = AD:LC. However, as established earlier, TA:CT = QT:TG, so AD:LC = QT:TG. But, since LC = H by construction, and since AD:H = E:Z by construction, then QT:TG = E:Z.

[67]This follows from the previously established conclusion that triangles AEL and ELQ are similar, since they are similar to similar triangles KMN and IKN. Thus, angle EAL = angle EQZ. But, by construction, angle EQZ = angle QAT (i.e., LAT), since EQ and AT are parallel, so angle EAL = angle LAT. Therefore, since angle GAU = angle GAE, by construction, and since angle EAL = angle LAT, angle UAT = 2GAE + 2EAL = 2GAL, so angle GAL is half of angle UAT.

[68]Although a definition of compound ratio is provided in book 6 (def. 5) in some of the manuscripts of Euclid's *Elements*, both Heiberg and Heath concluded with good reason that this definition was interpolated later and is thus not authentically Euclidean. The definition, as given in English by Heath, is as follows: "A ratio is said to be compounded of ratios when the sizes of the ratios multiplied together makes some (?ratio, or size)." In VI.23 Euclid does advert to the compounding of ratios on the basis of three magnitudes, K, L, and M, magnitude L being the mean proportional. Accordingly, Euclid claims that "the ratio of K to M is compounded of the ratios of K to L and that of L to M" (T. L. Heath, trans., *The Thirteen Books of Euclid's Elements*, vol. 2 [New York: Dover, 1956], p. 248), which would translate to K:M = (K:L):(L:M). Within the context of the theorem, which involves the ratio of two areas, what Euclid seems to be getting at is that K:M = parallelogram K,L:parallelogram L,M, so that K:M = K,L:L,M, which is to say that the ratio of the respective sides is the same as the ratio of the areas formed by each of those sides with a given length L. Therefore, the full expression of the compound ratio K:M = (K:L)(L:M) seems to be reducible to K:M = K,L:L,M.

[69]That OU is bisected by FL follows from the fact that triangles QFT and OFU are similar, by construction. Hence, since FL bisects base QT of triangle QFT, it must also bisect base OU of triangle OFU.

[70]It has already been established that EC:GD = TQ:PO and that GD:DI = PO:OU. Hence, by Euclid, V.22, it follows *ex aequali* that EC:DI = TQ:OU. Another rationale for this conclusion is based on the already-established fact that GD = PO. But EC = TQ, by construction. Hence, EC:TQ = GD:PO, which is to say that EC:GD = TQ:PO. But we just established that GD:DI = PO:OU, so, by Euclid, V.22, EC:DI = TQ:OU. At bottom, of course, this means that DI = OU.

[71]We have already established that EC:DI = TN:UH and that DI:GD = UH:UP. Thus, from Euclid, V.22, it follows that EC:GD = TN:UP.

[72]We know from previous conclusions that EC:GD = TN:UP, and we have just concluded that GE:EC = PT:NT, so, by Euclid, V.22, GE:GD = PT:UP.

[73]The second case just described is illustrated in figure 5.2.24a, p. 238, where LN is still equal to H, but CLN intersects diameter MG above the circle's centerpoint, rather than below it, as in figure 5.2.24. The two cases thus reflect the situation illustrated in figure 5.2.20b, p. 234, where two lines can be dropped from point A to diameter GB such that both the segments ED produced between the diameter and the circle below the diameter are equal to given line HZ. The proof for this second case remains the same as it is for the first: triangles QNL and DQA will be similar, as will triangles TQA and NQG, so it will follow that TQ:QG = AD:NL = AD:H, since H = NL, by construction. This lemma is actually a subcase of proposition 21, lemma 3, case 1. In this instance, however, instead of intersecting side AB of triangle ABG on the side of A (as illustrated in figure 5.2.21, p. 235), the requisite line from D intersects it on the side of B—i.e., on the opposite side of AB. Hence, point Q, where it intersects side AG, lies inside rather than outside the circle. These differences notwithstanding, the aim of the construction—to cut side AG of triangle ABG in such a way that TQ:GQ = AD:NL (= H) = E:Z—is precisely the same as in proposition 21, lemma 3, case 1.

[74]We established earlier that BD:HT = BG:GH, so it follows from the equality of DH and HT that BD:DH = BG:GH. We also established that HD:DL = HG:X. Therefore, given that BD:DL = (BD:DH):(DH:DL), we can substitute the relevent ratios in the compounded expression to get BD:DL = (BG:GH):(HG:X).

[75]The gist of Alhacen's argument here is that, since there actually is a point of reflection D when center of sight A and object-point B are positioned as illustrated, then there will necessarily be a line SP that will fulfill all the appropriate conditions for the construction. This line, however, will not be S'P', which forms S'KC less than a right angle. If we do suppose that the relevant angle is less than a right angle (e.g., S'KC in figure 5.2.25a, p. 240), then let A and B lie in the same locations with respect to the mirror as in figure 5.2.25. According to the construction given at the beginning of the proposition, we start by dividing line MK at F so that BG:AG = FM:FK, as in figure 5.2.25a. Next, we bisect MK at O, and draw line CO perpendicular to it. From point K we draw KC to form angle OCK equal to half of angle BGA. Through F we draw line S'FP' so that S'P':P'K = BG:GD. We then connect S'K to form triangle S'P'K. Finally, from angle BGA we cut off an angle BGD equal to angle S'P'K. Point D, where GD intersects the circle, will be the sought-after reflection-point. However, as is clear from the figure, point D lies below AG, so

there can be no reflection from B to A. In order, therefore, for D to be located on the mirror between GA and GB, angle CKS must be greater than a right angle.

[76]In this case, then, we can produce line SP so that the angle SKC it forms with CK is greater than a right angle and so that the appropriate conditions are fulfilled for the selection of reflection point D between AG and BG on the mirror.

[77]In other words, if two angles, CKS and CKS′ could be formed at K, both of them greater than a right angle, then it would be possible to cut from BGA two angles BGD and BGD′ equal to angles SPK and S′P′K, respectively, such that both D and D′ would lie on the mirror between AG and BG, which is impossible. At bottom, the size of angle CKS depends upon the size of angle AGB and the relative lengths of AG and BG, since the proportions that dictate the configuration of triangle SKC depend on those variables. Accordingly, if A and B are positioned in such a way that the line connecting them is tangent to, or cuts, the circle—in which case, of course, there can be no reflection—then angle CKS will never be greater than a right angle.

[78]As will become clear shortly, Alhacen's qualification here ("by sense-deduction") is crucial, because, according to mathematical analysis, most object-points will not share precisely the same image-location for both eyes in spherical convex mirrors. However, the disparity in location is generally small enough that the two images are perceived as one by the visual faculty.

[79]Alhacen's point in this paragraph is illustrated in figure 5.2.25b, p. 240. In the left-hand diagram, the circle centered on G represents a single plane of reflection through the mirror. That plane contains the two centers of sight, A, and A′, the visible point B, and centerpoint G. BG is the normal, and D and D′ represent the points of reflection for the respective centers of sight A and A′. Accordingly, for both eyes the image will lie at I on the normal. In the right-hand diagram, there are two planes of reflection intersecting along normal BG, so the image for both eyes will still lie at point I on the normal.

[80]Several points are at issue in these brief paragraphs. First, in spherical convex mirrors, the arrangement of the parts of the image is the same as that of the object, although of course the image is distorted by the mirror's curvature—a point left unspoken by Alhacen. Second, when both eyes see something in a spherical convex mirror, the constituent images of points on that object may not lie at precisely the same location for each eye, but when the point viewed is disposed the same way for both eyes, then its image will lie at precisely the same location for each eye. Orientation is the key. If a cross-section of the object faces the eye directly so that the visual axes and the common axis converge on a point on it, then the images in both eyes will lie at precisely, or very nearly precisely, the same spot on the normal dropped from the object point to the mirror's center. Likewise, for any cross-section of an object that coincides with or is at least in line with the common axis, the object-points on that line will appear at precisely the same point on the normal for both eyes. For any other cross-section, however, the image location for a given point-object will not lie at precisely the same point on the normal for both eyes. Hence, as illustrated in figure 5.2.25c, p. 241, if O is a point on a cross-section of the visible surface of some visible object, if E and E′ are the centers of the right and left eye, respectively, and if the given cross-section lies off to the side, then the angle of

reflection will be different for E than it is for E′, as already demonstrated in proposition 17 above. That being the case, points I and I′, where the respective lines of reflection intersect normal OC, will be different for both eyes. In order, therefore, for the image-location to be precisely the same for both eyes, the angle of reflection must be precisely the same for each eye. Nevertheless, as Alhacen points out, the disparity in image-locations is generally so small that the visual faculty ignores it, fusing the images into one image, the result, at worst, being a slight blurring or indistinctness in the resulting perception. For Alhacen's discussion of this process of image-fusion in direct vision, see *De aspectibus*, 3, 2.1-2.22, in Smith, *Alhacen's Theory*, 562-573.

[81]As was pointed out in book 4, the phrase "cylindric section" translates the Latin phrase *sectio columpnaris* (or *columpnalis*). This, of course, is the figure produced in a cylinder by an oblique planar cut—i.e., one that intersects the axis of the cylinder at a slant. In proposition 20 of *On the Section of a Cylinder*, the fourth-century mathematician Serenus demonstrates that such a cut yields a true ellipse, not simply an ellipse-like oval. Altogether, then, there are three different ways in which a cylinder can be cut by a plane of reflection that also cuts the axis: through a line of longitude (so that both the line of longitude and the axis lie in the same plane), along a circle (so that the plane is perpendicular to the axis), and along an ellipse (so that the plane cuts the axis at a slant). Rather than translate *sectio columpnaris* as "ellipse," I have chosen the more literal "cylindric section" in order to distinguish it from what Alhacen calls the *sectio piramidalis*, or "conic section" proper.

[82]This, of course, is the same proportionality that obtains for spherical convex mirrors, as demonstrated in proposition 7 above.

[83]As already remarked, when Alhacen refers to images "outside" the mirror, he means images that lie beyond the common section of the plane of reflection and the mirror on the invisible side of the mirror. The image is still seen behind and within the outer boundary of the visible portion of the reflecting surface.

[84]Reflection within the plane of a line of longitude is therefore governed by the rules pertaining to reflection from plane mirrors.

[85]The supporting logic of this abbreviated *reductio ad absurdum* is as follows. If we assume that there is a point L of reflection on the cylinder other than G, and if, as just demonstrated, this point L cannot lie on circle GH itself, then it must lie outside the plane of circle GH. If a normal LI is dropped from this point to point I on axis CE, then, when extended beyond the cylinder, it must intersect line A′B′, because it must lie in the same plane of reflection as that line. LI will therefore intersect A′B′ at some point K. But the normal from any point, such as K, on line A′B′ will necessarily also meet the axis at point E and will therefore lie in the plane of GH. Two normals KE and KI will have thus been dropped from point K to axis CED, which is impossible, so reflection can occur from A′ to B′ at no point other than G on the cylinder.

[86]Implicit in this method for finding the point of reflection on convex cylindrical mirrors is Alhacen's general method for finding the point of reflection on convex conical, as well as concave cylindrical and conical mirrors. Given object-point B and center of sight A in figure 5.2.28a, p. 243, we pass a plane through A to cut the

cylinder along circle ZEI. Then, from point B we drop line BH parallel to the edge of the cylinder—and thus orthogonal to the plane of circle ZEI—so as to meet that plane at point H. By proposition 25 above, we find point Z of reflection on circle ZEI according to which H will reflect to A at equal angles. We then produce a line of longitude from Z and pass a plane through line AB orthogonal to the cylinder's surface. Point G, where this plane cuts the line of longitude will be the sought-after point of reflection.

[87]Angle CFG = angle CNG, and right angle GCF = right angle GCN, so angle CGF = angle CGN. But angle CGF = alternate angle GMQ, and angle CGN = alternate angle GNQ. Thus, angle GMQ = angle GNQ, and right angle NQG = right angle MQC, so angle NGQ = angle MGQ.

[88]In other words, if we take the circle in the abstract, not as an actual section of the cone, then, assuming that the plane tangent to point of reflection G is perpendicular to the plane of the circle, the form of M will reflect from G to N, because the normal TGQ will be orthogonal to that tangent plane. The ulterior purpose of the theorem to this point is to establish that, if the form of point B is assumed to reflect to A from point G, then, when a line parallel to line of longitude EG is dropped from B to point M on the plane of circle PG, and when a line parallel to line of longitude EG is likewise dropped from A to point N on the same plane, the form of M *would* be reflected to N from point G on the abstracted circle, which is what is intended by the qualification "barring interference from the cone" (*non impediente piramide*). The procedure just outlined, which involves dropping line AN parallel to line of longitude EG from the center of sight to the plane of circle PG, and dropping its counterpart BM to that plane from object-point B, is, of course, the extension of the method described in note 86 above for convex cylindrical mirrors.

[89]In other words, if the two planes containing A and the two normals intersect on AL, then, since B supposedly lies within the same two planes, and since it lies in the same plane as FL, the two planes would have to intersect along both AL and FL, which is impossible.

[90]Here, of course, Alhacen is treating G abstractly as a point of reflection on some surface to which the plane of reflection is normal.

[91]In other words, if we generate the equivalent of triangle SAL in figure 5.2.31, p. 246, and then connect the appropriate lines from M and N, we can create the equivalent of triangle CAL within the equivalent of triangle BAL and, on that basis, prove that angle MEU = angle UEN. The equality of these angles is of course obvious from the construction, since the plane GTERU that passes through the line of longitude GE and axis GT bisects the entire figure HANGM, so all the component triangles, GMN, MEN, and HEA are bisected at their respective vertices G and E.

[92]Here, again, Alhacen is treating the plane of reflection, within which the circle lies, in the abstract, as though it were normal to a reflecting surface, even though it is not normal in the actual case of the cone.

[93]If center of sight B lies at point D, then of course the normal to line of longitude GE will be dropped from P, in which case the point where that normal will intersect ZGE—i.e., the point of reflection—will fall below E. The farther beyond D point B lies, the farther below point E the point of reflection will fall on GE. On the

other hand, as B descends along BG toward G, the point of reflection ascends above E along GE. Whatever the case, if reflection is to occur from A to B, it must be possible to pass a plane through them such as to intersect one of the lines of longitude on the lower cone along the orthogonal. As Alhacen demonstrates later in book 5, there will be only one point of reflection in the concave segment of the circular section formed in the upper cone. The actual procedure for determining that point is given in proposition 38, pp. 458-459.

[94]As in the previous case, so in this one, it is obvious that plane PZER passing through the line of longitude and the axis of the cone bisects angles AZD, AED, and A'EH. Clearly, then, no matter where B lies on DBG, the resulting angles BEQ and AEQ will be equal.

[95]The actual construction and proof of this proposition is as follows. Let M in figure 5.2.31d, p. 248, lie in plane MGN, which passes through vertex G of the cone parallel to its base-circle. Let A lie above that plane. Draw AM, generate the complementary cone GZY, and pass a plane through A parallel to plane MGN so as to produce circle ZY. From point M erect a line perpendicular to plane MGN, and let it intersect plane AZY at point B'. Find point Z from which B' will reflect to A, and draw line of longitude ZGX along both cones. From M draw a line parallel to ZGX, and let it intersect plane AZY at point B. From A draw a line parallel to ZGX, and let it intersect plane MNG at point N. Draw AB and NM, and produce ZR and GS to bisect angles BZA and MGN respectively. The plane RZGS formed by these lines and ZG intersects line AM at point Q. The normal dropped from point Q to line of longitude ZGX will intersect it at point E, which is the sought-after point of reflection for M and A. As Alhacen remarks, the proof that follows from this construction is essentially the same as the earlier ones based on the equality of the respective angles of all the triangles formed by planes AZGN, BZGM, and RZGS—culminating, of course, with the equality of angles AEQ and MEQ.

[96]At this point Alhacen's general method for finding the point of reflection in a convex conical mirror is clear. First, the relevant line of longitude must be found. This depends on finding the point of reflection on the circle projected from the center of sight either on the cone from which reflection occurs, or on the upper extension of this cone. Second, the point where the plane containing the line of longitude and the cone's axis intersects the line joining the center of sight and the visible point must be determined. From that point a normal is dropped to the line of longitude on the cone, and the point where that normal intersects the line of longitude will be the point of reflection.

[97]This point is illustrated in figure 5.2.32a, p. 250. In this case, MQ represents the visible spot, exaggerated in size for the purposes of clarity. As just demonstrated, the form of MQ's central point T reflects to center of sight A along line AE, which is parallel to normal TD dropped from T, so there is no definite image-point for T. On the other hand, the form of point M is reflected to center of sight A along NA, which intersects normal MD at L. Meantime, the form of endpoint Q reaches the center of sight A along GA, which intersects normal QD at O, beyond the eye. And finally, the form of point T reflects to center of sight A from the opposite side of the mirror along CA, which intersects normal TD between A and the mirror's

surface. The same holds for every point between M and T and T and Q. Accordingly, the visual faculty averages out the various image-locations behind and in front of the mirror to locate the image on the mirror's surface in much the same way that it locates the image of the eye's centerpoint seen along the visual axis according to the context of surrounding points on the eye's surface.

[98]In theory, of course, since the eye lies on the normal passing through each point on the rod, the image-location for all those points will be the center of sight. In practice, however, the rod has breadth, like the sensible spot just discussed. Accordingly, the resultant overall image of the rod will be located, part by part, according to the average of composite image-locations. On that basis, the rod will be seen as a confused blur on the mirror's surface, thus taking the curved shape of that surface.

[99]In other words, if the center of sight lies at the center of the mirror, then reflection to that point can only occur along the normals dropped from that point to the mirror's surface, and such normals lie within the visual cone with its vertex at the center of sight.

[100]Hence, as illustrated in figure 5.2.32b, p. 250, if A is the center of sight, then the form of no point, such as O, on radius AD will reflect to A, because no matter what theoretical point of reflection R is chosen, the normal DR to that point will lie outside the angle ORA and will therefore not bisect it. On the other hand, the form of point O' on radius O'D can reflect to A, because some point R' of reflection can be found such that angle O'R'A can be bisected by normal DR'. But its image-location will lie at A, so it will not be properly perceived. The form of point O" on radius O"D can also reflect to A, because some point R" of reflection can be found such that angle O"R"A can be bisected by normal DR", and its image-location will lie beyond the center of sight at I, so it too will not be properly perceived by the eye at A.

[101]In other words, what is to be demonstrated is that the distance EB between object-point B and the mirror's centerpoint E is to the distance between image-point H and centerpoint E as the distance between object-point B and endpoint of tangency T is to the distance between image-point H and endpoint of tangency T. This, of course, is the very same proportionality that was derived for convex spherical mirrors in proposition 7 (see figure 5.2.7, p. 223)—i.e., AN:DN = AE:DE—where A is the object-point, N the mirror's centerpoint, D the image-point, and E the endpoint of tangency. The significance of this proportionality as applied to concave spherical mirrors becomes clear later in proposition 36, pp. 452-454.

[102]As in the earlier analysis of image-locations in spherical, cylindrical, and conical convex mirrors, so in this analysis of concave spherical mirrors, "outside the mirror" means outside the sphere defining the mirror, not simply outside the reflecting surface. The same will of course hold for concave cylindrical and conical mirrors. The claim that reflection can occur from arcs DT and GQ whether A or B lies on the mirror's surface or outside it, as long as the other point lies inside the mirror, reveals the true intent of this theorem: to prove not that reflection *will* occur from arcs TD and GQ but that it *may* do so. That it may not occur becomes clear if we assume that A in figure 5.2.34a, p. 253, lies closer in toward E and that B lies far

above the mirror along the extension of diameter TE. For in that case, no matter where the line of incidence from B strikes the mirror, the angle it forms with the normal from E will be greater than the angle formed by the line of reflection from that point to A. Thus, whether reflection will or will not actually occur from arcs TD or GQ depends on the relative position of A and B vis-à-vis their respective diameters.

[103]Since EQ = QD + ED, and QD = QH + HD, then EQ:QD = (QD + ED):(QH + HD). But EQ:QD = QD:QH. Therefore, by Euclid, V.19, EQ:QD = remainder ED (from QD + ED):remainder HD (from QH + HD), so EQ:QD = ED:HD. Since EQ = QH + HD + ED, and QD = QH + HD, the lengths of both EQ and QD are ultimately contingent on the length of QH, which is determined as follows. For the sake of convenience, let QH = a, HD = b, and ED = c. Thus, EQ = a + b + c, and QD = a + b, from which it follows that (a + b + c):(a + b) = c:b. By Euclid, VI.16, rectangle (a + b + c)b = rectangle (a + b)c, which translates to $ab + b^2 + bc = ac + bc$. Dropping the common term bc, we are left with $ab + b^2 = ac$, which can be expressed in the form $b^2 = ac - ab$, which is to say that the square on HD is equal to the difference between the rectangle formed by QH and ED and the rectangle formed by QH and HD—i.e., $HD^2 = QH(ED - HD)$. Since HD and ED are given, their difference ED – HD is given. Call that difference X. Therefore, $HD^2 = QH,X$, so it follows by Euclid, VI.17, that HD is the mean proportional between QH and X. Consequently, QH:HD = HD:X. Given both HD and X, then, QH can be found by Euclid, VI.11, and once it is found, then QE will be formed by adding it to HE.

[104]I have changed the text here to reflect my numbering of the propositions. In the Latin manuscripts, the propositional designation is confused, in great part because none of the manuscripts actually numbers the propositions. Thus, three of the manuscripts seem to refer to the "26th" proposition, and three seem to prefer "ZG" to "26." One manuscript has "36" instead of either "26" or "ZG," and this designation in fact refers more or less appropriately to the diagram numbered 36 in its text. The fact that all the manuscripts have something here, whether the number "26" or the letter designation "ZG" (which closely resembles "26" in medieval orthography) indicates that either the Arabic text or the original Latin translation derived from it had some sort of equivalent designation at this point.

[105]The issue Alhacen addresses at the end of this analysis is whether a given arrangement of E and H will allow the form of H to reflect to E. His answer is that, as long as EA:AH > ED:DH, such reflection can occur. The actual demonstration of this point, which is unclear as presented in the text, is contingent on whether a tangent Q'G can be drawn from point G to intersect the point where diameter ZDAQ meets the circle produced at Q. If so, then EA:AH > Q'E:Q'H, which is to say that EA:AH > ED:DH, since Q'E:Q'H = ED:HD. When reversed to read ED:HD = Q'E:Q'H, this latter proportionality follows directly from proposition 33, where it is established that in reflection from concave spherical mirrors the distance between the object-point and the mirror's centerpoint is to the distance between the image-point and the mirror's centerpoint as the distance between the object-point and the endpoint of tangency is to the distance between the image-point and the endpoint of tangency. In this case—which reflects the situation illustrated in figure 5.2.33c,

p. 252—Q′ is the endpoint of tangency, T the object-point, H the image-point (which coincides with the center of sight), and D the mirror's centerpoint.

How this stricture (i.e., that EA:AH must be less than ED:DH if reflection is to occur) actually applies is easily understood in the context of figure 5.2.36b, p. 256. In the upper diagram of that figure, E and H are situated so that point Q lies closer to the mirror than before, in this case coinciding with point A on it. Point G, where the two circles intersect, should be the point of reflection, and, according to the construction, the line of incidence EG will be tangent to the circle centered on Q. The tangent to point G on the mirror will intersect diameter ZA at point Q′, which is thus the endpoint of tangency, and it is clear that EA:AH > Q′E:Q′H, which = ED:HD. Hence, the stipulation will have been met, so G is indeed a legitimate point of reflection. In the lower diagram, however, the limiting case is illustrated. Here, Q coincides with H, leaving tangent GE parallel to diameter AZ (i.e., angles HDG and DGE sum up to a right angle). Accordingly, point E can never fall on AZ, so there can be no reflection. Moreover, the stipulation that EA:AH > ED:HD is no longer met, because EA and ED would be infinitely long, and there is no proportionality between finite and infinite magnitudes. In practical terms, what this means is that, as long as point Q falls to the left of point H, there will be some point E on diameter AZ to which the form of E will reflect. If, on the other hand, point Q falls at or to the right of H, there will be no such point E on diameter AZ.

[106]The point here is illustrated in figure 5.2.37a, p. 257, where T and H both lie on tangent KEF. In this case a line of incidence can reach from T to any point on arc EG, such as R, on the concave side of the circle (assuming no blockage on arc EB), but the resulting line of reflection RH will lie on the convex side of the circle, so reflection cannot occur.

[107]Like proposition 18 above, which deals with reflection from convex spherical mirrors, this proposition is based on the special case in which the visible point and the center of sight are equidistant from the mirror's center, which is to say that they lie on the extensions of two intersecting diameters of the circle. Accordingly, the point of reflection is found simply by bisecting angle HDT. If, on the other hand HD ≠ TD, and if the resulting angle HDT is bisected by ED, which is continued to point Z, then clearly angle HZL ≠ angle TZL. The next two cases deal with special cases of this special case, when the two points lie on intersecting diameters inside the mirror at points on those diameters that are equidistant from the center. The more general case, in which HD and DT are unequal, is addressed in proposition 38, where Alhacen applies a method for determining the reflection-point that is equivalent to the one adduced earlier in proposition 25 for convex spherical mirrors.

[108]In other words, for all equidistant point-pairs on HD and TD, including H and T, which are defined by the perpendiculars dropped from E to BD and GD, reflection can occur from E and Z only. As will become clear in the next proposition, this restriction is lifted for equidistant point-pairs except H and T lying on HG and TB.

[109]At this point the significance of the circle produced through centerpoint D and the point-pairs chosen on BD and DG (e.g., circle HDTO) becomes clear. As

long as that circle intersects the circle of the mirror within the arc BG defined by the intersecting diameters, there will be four points of reflection within the mirror. If that circle is tangent to the mirror, or if it intersects the mirror at or beyond B and G, then there will only be two points of reflection. In the latter case this is obvious from the fact that the intersection-points will lie either on arcs AB and QG, from which no reflection can occur, according to proposition 34, or within arc QA, from which only one reflection (i.e., from Z) can occur, according to case 1 above. As will become clear later on, when Alhacen deals with points on the intersecting diameters that are not equidistant from the center of the mirror, the number of possible points of reflection is contingent on whether and how the circle passing through those points and the center of the mirror intersects arc BG.

[110]Although the text fails to make this specification, this theorem is intended to prove that, if the two points T and H are taken on diameters ADG and BDQ′ in figure 5.2.38, p. 260, and if they lie at different distances from centerpoint D of the mirror, the form of point T will reflect to H from only one point on arc AQ′. As will become clear in fairly short order, the form of T can reflect to H from as many as three points on arc GB. Altogether, then, the form of point H can reflect to T from as many as four points within mirror ABGQ′, one such reflection occurring from arc AQ′.

[111]That QM:LM = F′O′:IO′ is explained as follows. First, it has already been established that triangle LMC is similar to triangle O′ID, and triangle C′O′H is similar to triangle KMN. Therefore, corresponding sides IO′ and LM of triangles O′ID and CLM are proportional, as are corresponding sides MN and C′O′ of triangles KMN and C′O′H. Therefore, LM:MN = IO′:C′O′. So, too, LM:(MN + LN) = IO′:(C′O′ + C′I). By construction, however, QN = LN, and F′C′ = IC′. Therefore, by Euclid, V.18, LM:(MN + QN) = IO′:(F′C′ + C′O′). Accordingly, since LM + QN = QM, while F′C′ + C′O′ = F′O′, then LM:QM = IO′:F′O′, and, finally, inverting the proportion, QM:LM = F′O′:IO′.

[112]It has already been established that HI:IU = HD:DT. However, by compounding, HI:IU = (HI:IP):(PI:IU), which is to say that HI:IU = (HI,IP):(IP,IU). Meantime, we know that HI:IP = HD:DP, so, substituting (HD:DP) in the foregoing proportion, we get HI:IU = (HD:DP):(PI:IU). But, by compounding, HD:DT = (HD:DP):(DP:DT), and from earlier conclusions we know that HI:UI (which = [HD:DP]:[PI:IU])= HD:DT. Accordingly, HD:DT = (HD:DP):(PI:IU), but also HD:DT = (HD:DP):(DP:DT), and so (HD:DP):(DP:DT) = (HD:DP):(PI:IU). Dropping the equal term HD:DP from both sides, then, we end up with DP:DT = PI:IU.

[113]The train of logic here is as follows. First, it is clear from figure 5.2.38 that UIH = O′IU + PIO′ + PID + DIH. But since PID = DIH, then UIH = O′IU + PIO′ + 2PID. Now, by previous conclusions, we know that O′IU = O′IH, which = PIO′ + PIH, which = PIO′ + 2PID, so UIH = 2O′IU = 2PIO′ + 4PID. Meantime PIU = UIH – 2PID = 2PIO′ + 4PID – 2PID = 2PIO′ + 2PID, and UIH – 2PID = O′IU + PIO′. But O′IU + PIO′ (which = PIU) = 2PIO′ + 2PID (which = 2DIO′). Thus, PIU = 2O′ID. Furthermore, given that O′ID = CLM (i.e., FLM), by construction, and given that FLM [i.e, FLQ] is half of ADT (i.e., TDP), by construction, then TDP = 2FLM = 2O′ID = PIU.

[114]The method Alhacen lays out in this proposition for determining the point of reflection on the opposite arc from the object-point and center of sight is clearly analogous to the one provided in proposition 25 above for finding the point of reflection on a convex spherical mirror no matter where the center of sight and object-point lie outside its surface. As will become clear in short order, the basic construction for finding any point of reflection in a concave spherical mirror is essentially the same, although the proof is not always equivalent. In the following two sub-theorems Alhacen will prove that, as determined by the method just articulated, the reflection-point on the opposite arc is unique.

[115]If we take HDN as a diameter forming two right angles, angle LDH = 2 right angles – angle NDL, whereas angle MDH = two right angles – angle MDN. But angle LDN < angle FDL, so it is even smaller than angle MDL, of which FDL is a part. Hence, angle LDH > angle MDH.

[116]That is, if the new angle is cut from DHL, then the side of it below HDN would intersect ML between N and L, thus skewing the disproportionality between LH:MH and LN:NM even more.

[117]The reason Alhacen takes the tack he does in this theorem becomes clear at this point, for, by proving that angle DHL, and thus DHT, must be less than angle DHM, he precludes the possibility that reflection between point-pairs, such as T and L that lie unequal distances from the center of the mirror, can occur from point Z. On the one hand, if DHL and DHT could be equal to DHM, then, if point H were to coincide with Z, point T would necessarily coincide with M, so T and L would be equidistant from D. But that contravenes the initial condition of the proposition—i.e., that T and L lie different distances from D. Moreover, if LHD and DHT could be greater than MHD, then, if Z were taken as the point of reflection, point T would lie between M and B. Thus, T and L would lie unequal distances from D and yet would reflect to one another from Z, which is impossible, since such reflection requires that T and L be equidistant from D. Reflection from point Z is thus reserved exclusively for points on diameters BQ and AG that are equidistant from D.

[118]In other words, if a line of reflection is drawn from H in figure 5.2.41a, p. 262, to any point other than T or G on arc TG, it will form an acute angle with the normal dropped to that point. Accordingly, an equal angle can be formed at that point on the other side of the normal, and any point on the resulting line of incidence can serve as an object-point. Line HT cannot, of course, be a line of reflection since angle BTH of tangency is a right angle. Likewise, there can be no line of incidence from point G, because such a line would be tangent.

[119]Here, and in subsequent propositions, I have chosen to translate the expression *angulus reflexionis* as "reflected angle" rather than "angle of reflection," because what Alhacen has in mind is the angle formed at the point of reflection by the line of incidence and the line of reflection rather than the angle formed by the line of reflection and the normal dropped from the point of reflection. Thus, as illustrated in figure 5.2.42, p. 262, the reflected angle (= *angulus reflexionis* in Latin) is OEA, which is twice the angle of reflection BEA understood in its proper sense.

[120]This particular step of adding angle FBA to both ABK and FBT applies to figure 5.2.42 only. In figure 5.2.42a, the common angle ABF is added to either ABK or FBT to yield KBF = TBA.

[121]That FBK + FEA = 2 right angles follows from the fact that angles BFE and BKE in triangles FBE and KBE are right. Therefore, the remaining angles of each triangle—i.e., FEB and FBE, KEB and EBK— sum up to a right angle, so FEB + KEB (= FEA) + FBE + EBK (= FBK) = 2 right angles.

[122]The ostensible intent of this proposition is fairly trivial: to prove that, if a center of sight is chosen on some diameter within a great circle on the mirror, and if a line of reflection is extended from that point to the facing arc, a corresponding line of incidence can be drawn from the selected point of reflection such that the form of any point on that line will reflect to the center of sight from the point selected on the mirror. Thus, no matter what diameter the center of sight lies upon, reflection can occur to it from any point on the facing arc from an infinite number of points on an infinite number of diameters. The ulterior—and more significant—point of the proposition, however, is to establish a relationship between the angle formed by the line of reflection and the line of incidence (i.e., the reflected angle) and the angle adjacent to the angle directly facing the reflected angle. As will become clear in short order, this relationship determines both whether and where reflection may occur in the arc facing the object-point and the center of sight. For instance, in figures 5.2.42, 42a, and 42b, p. 262, it is clear that when the object-point and the center of sight are equidistant from the mirror's center, the adjacent angle is equal to the reflected angle. Thus, in figure 5.2.42, when FB (the distance between object-point F and centerpoint B) is equal to AB (the distance between center of sight A and centerpoint B), angle FBG adjacent to angle ABF is equal to reflected angle AEF. Likewise, in figure 5.2.42a and 42b, where the distance TB between object-point T and centerpoint B is equal to the distance AB between center of sight A and centerpoint B, angle TBG adjacent to TBA is equal to reflected angle AET. From this it is evident that when the adjacent angle and the reflected angle are equal, the object-point and the center of sight must lie equal distances from the centerpoint on their respective diameters. It is also evident that, when those distances are not equal, the adjacent angle will either be greater than or less than the reflected angle.

[123]This proposition bespeaks the rigor of Alhacen's approach to mathematical proof. From the previous proposition (i.e., 42) it is easy to infer that, when the reflected angle ATB and the adjacent angle AGD are equal, the center of sight and the object-point will be equidistant from the center of curvature (see esp. paragraph 2.403). But, as Alhacen recognizes, such an inference is inductive rather than deductive—hence the need to supply a proper proof.

[124]The purpose of the proof to this point is to demonstrate, first, that if the reflected angle—e.g., BTA in figure 5.2.44, p. 264—is less than angle AGD, and if the two points A and B that are mutually reflected lie at different distances from centerpoint G such that BG > AG, then the circle BAT passing through points B, T, and A, will pass above centerpoint G. It follows, therefore, that arc BO on circle BAT is equal to arc AO, O being the point where normal TG intersects the circle. Accordingly, line KO, which intersects line BA at its midpoint, bisects circle BAT and therefore intersects BA orthogonally. Line GN, which bisects angle BGA, passes through point F on AB. Since point F lies to the right of KO, which is perpendicular

to AB, it forms an obtuse angle GFK with AB. Hence, GN and OK will intersect beyond line BA. The crucial point underlying this demonstration is that, for any reflected angle ATB less than angle AGD, the circle passing through the three cardinal points A, B, and T will invariably pass above centerpoint G of the mirror. From this it invariably follows that the relevant lines OK and GN will intersect above line AB. So let us assume that angle BQA is acute, as represented in figure 5.2.44a, p. 265, where it is in fact shown equal to angle ATB so that point Q lies on circle BTA, which passes above centerpoint G. Accordingly, by supposition, angle BQG = angle AQG, so point O′, where normal QG intersects circle BAQT, will bisect arc AB, leaving arc AO′ = arc BO′. According to the procedure described above, then, KO′, which bisects AB, should intersect GF beyond line AB. But instead it intersects it below AB. Hence, point Q cannot be a point of reflection for A and B. Nor, for that matter, can any other point on arc EN, because, no matter what point is chosen on that arc to produce a reflected angle less than angle AGD, the circle passing through that point and points A and B will fall above centerpoint G, and the resulting point O′, where the normal dropped from the chosen point intersects the resulting circle passing through it and points A and B, will lie to the right of G, the result being that O′K will intersect GN below AB. It bears noting that the intersection rule for reflected angle ATB less than angle AGD also holds for reflected angle AQB greater than AGB. In that case, however, the circle passing through A, Q, and B will fall below, rather than above centerpoint G. Thus, as illustrated in figure 5.2.44b, p. 266, when normal GQ is extended to intersect circle ABQ at point O′, and when line O′K is drawn, it will coincide with line OK and will therefore intersect GN above AB at the very same point that OK does.

[125]Under the conditions specified here for reflected angle ACB, the circle passing through A, B, and C must pass through point L, where KO and normal CG intersect if the resulting arcs AL and BL on circle ABT are to be equal. Circle ABC passing through C must therefore be larger than circle ABT, so it can only intersect circle ABT at the two points A and B. Yet, if it is to include C, circle ABC must somehow intersect circle ABT before reaching C from point B, and this is impossible. Furthermore, it is clear that, as C approaches N from T, point L, where normal CG intersects KO, moves upward toward K until it reaches a point where CG actually intersects KO above K. In all such cases, then—i.e., when C lies between T and N—the relevant circle through point L will be larger than ABT, so the proof will hold. On the other hand, if C is chosen between T and Z, CG will intersect the extension of KO below point O. The resulting circle that passes through A, B, and that point will thus be smaller than circle ABT, and so, in order to reach point C outside circle ABT, it must somehow pass outward through arc BT, which is impossible. Altogether, then, given points A and B lying at different distances from centerpoint G of the mirror, there is only one point T on arc ZE at which the angle of reflection ATB is less than angle AGD adjacent to angle AGB subtended by that arc.

[126]Angle TML = angle TNL, because they are subtended by the same arc TL. But angle TNL > angle CTN, which = angle MTO, by construction. Hence, angle TML > angle MTO.

[127]OTF = FTR, because MTF = NTF, since the respective arcs MD and ND that subtend them are equal, by construction. OTF = MTF + MTO, while FTR = NTF +

CTN. But MTO = CTN, by construction, so OTF and FTR are composed of correspondingly equal angles.

[128]As will become clear in proposition 49, case 3, below, there are in fact three possible points of reflection on arc BG. However, only two of them can yield a reflected angle greater than angle ODA. Although not made explicit in the text, this is clearly the limitation intended in this proposition—i.e., that reflection between O and K can occur at only two points when the resulting reflected angle is greater than ODA.

[129]From Euclid, III.22, we know that the opposite angles of any quadrilateral inscribed in a circle sum up to two right angles. Therefore, the point where ND intersects the smaller circle [i.e., *s*] must lie below D, because the complementary angle O*s*K formed by *s*O and *s*K must be smaller than ODK so the resulting angle O*s*K + OTK will sum up to two right angles. That ND extended bisects arc OK follows from the equality of angles OTD and KTD, which are therefore subtended by equal arcs, O*s* and K*s*.

[130]If C were to fall between N and O, and if point *x* were to fall between N and C, i.e., to the right of N and to the left of C, it would necessarily intersect DC below D, because ND and CD intersect at D, and *sx* intersects ND below D. Thus, perpendicular *sx* would form a triangle with CD, one of whose base angles on the left is obtuse, since angle DCK is obtuse, by previous conclusions, and the other of whose base angles, *sx*C, is right.

[131]That is, ND and CD will form a triangle with its vertex at D and its two base angles DNO and DCK obtuse, since we have already determined that the angle at C on the side of K is obtuse. It follows, accordingly, that point *x* must lie between C and K, and C must lie between N and K. That being the case, then, angle KND must be acute, because line TD intersects line KO to the right of angle OCD, which is acute. Accordingly, TD will intersect *sx*, which is a radius of circle OTK, beyond point D at bisection-point *s* of arc OK. Were angle KND not acute, on the other hand, TD would never intersect *sx*. Furthermore, since *sx* intersects KO to the left of DC, and since angle DCK is obtuse, *sx* will intersect DU beyond line KO. Therefore, according to the conditions established in proposition 44, point T is in fact a legitimate point of reflection on arc BG for points K and O on diameters BD and GD.

[132]This passage occurs at the end of paragraph 2.445 in all the manuscripts, but it clearly belongs here at the beginning of paragraph 4.451. The point of the passage is unclear, because it is unclear what is meant by an intersection below TO. If that intersection is taken to occur between D and O on line OD, then clearly TO and DO are unequal, since the one is a radius of the circle and the other is less than a radius. If the intersection is taken to occur below KD (and thus still below TO), then the passage is redundant.

[133]KT:TF is compounded of KT:TL and TL:TF—i.e., KT:TF = (KT:TL):(TL:TF)—but KT:TL = KD:DL, and TL:TF = DL:DO. KD:DO, in turn, is compounded of KD:DL and DL:DO—i.e., (KD:DL):(DL:DO). Therefore, since KD:DL = KT:TL, and DL:DO = TL:TF, then KT:TF = KD:DO. The ulterior point here is to establish that, given KD and DT, for any point T that yields a reflected angle KTD greater than angle ODA, the given proportionality KT:TF = KD:DO will hold. Alhacen's reason for establishing this proportionality becomes clear in the very next proposition, where it

forms the basis for verifying his method for finding a point of reflection on the arc facing points O and K when the reflected angle is greater than the adjacent angle.

[134]In other words, KD and EH are given by construction, and TD is the radius of the circle, so all three are constant. As a result, MI is also constant, and it constitutes the fourth proportional, which can be found by Euclid, VI.12. According to MI's value, therefore, the endpoint F′ of line F′M within circle TEC′ is determinate.

[135]The following explanation should make the force of this proof clear. Let point T in figure 5.2.46f, p. 272, represent one of two legitimate points of reflection on arc BG of the mirror such that the reflected angle—OTK in this case—is greater than angle ODA. The object is to find point F′ and thus to define MI. To do this, we cut angle OTF = angle ODA from reflected angle OTK. We then bisect the remaining angle KTF with line TZ to find point E on diameter AG. From K we drop KH perpendicular to TZ so as to meet the extension of normal TD at point H. We then draw EC′ parallel to KH, point C′ being where that parallel intersects TH. Through points T, E, and C′ we produce circle TEC′ to intersect diameter AG at point M and then cut angle F′ME = angle EHD from angle TME. Line F′M will thus cut normal TD at point I to yield segment IM such that the proportionality KT:DT = EH:IM holds. We know from proposition 20, lemma 2, that two lines can be dropped from F′ through diameter TD of circle TEC′ to yield a segment equal to MI. This second line is F′N, which cuts normal TD at point K′, so K′N = IM.

Now, let T′ be another legitimate point of reflection on arc GB such that the reflected angle OT′K formed on it is also greater than ODA. This case is illustrated in figure 5.2.46g, p. 273, which is superimposed on the previous figure so that all but lines F′M and F′N of that previous figure are backgrounded in pale gray. Just as before, we cut angle OT′F from angle of reflection OT′K and bisect angle KT′F with T′Z′ to locate point E′, where line T′Z′ intersects diameter AG. We drop KH′ perpendicular to line T′Z′ so that it intersects the extension of normal T′D at H′. Then we draw E′C″ parallel to KH′ so as to intersect T′H′ at point C″. Through the resulting points T′, E′, and C″ we produce circle T′E′C″, which intersects diameter AG at point M′. Taking angle F″M′E′ = angle E′H′D, we cut it from angle T′M′E′ to end up with line F″M′, which intersects normal T′D at point I′. The resulting segment I′M′ will therefore be such that KD:TD = E′H:I′M′. Since KD:TD = EH:IM, then E′H:I′M′ = EH:IM. As in the previous case, so in this one, there is a second line F″P that intersects normal T″D at point L such that LP = I′M′.

If we turn, finally, to figure 5.2.46h, p. 274, which is abstracted from the previous one and somewhat enlarged, the implications of this analysis become clear. Take point T and its associated point F′. As we just established, F′N is the second of the two lines extending from point F′ which, on passing through normal TD, produce segments equal to IM, K′N being the segment in this case. If from the second point of reflection T′ we extend a line T′X parallel and equal to F′N, that line will intersect diameter ADG at point R such that T′R = K′N. That this is indeed the case can be easily understood if the production of line T′X is described in mechanical terms. Accordingly, as represented in figure 5.2.46k, p. 274, line F′N is shifted toward line T′X in the direction of the dotted arrows until point K′ on F′N coincides with point T′ and point N on F′N coincides with R. In this position, the line will protrude above T′ by the distance ZT′, which = F′K′. That line is then slid downward along T′X until its upper endpoint Z coincides with T′ and its lower endpoint N coincides with X. The corresponding distances ZT′ and RX it will have moved in the process are equal to F′K′, so RX = F′K′. The remainder T′R of the resulting line T′X will thus be equal to K′N. As is evident from figure 5.2.46h, there is no point other than T′ on arc BG at which a line extended equal and parallel to F′N will intersect diameter AG in such a way that the segment between that point of intersection and the chosen point on arc BG will be equal to K′N, which = MI. Thus, from the perspective of T, T′ is the only possible alternate point of reflection on arc GB that yields a reflected angle greater than ODA and fulfills the requisite proportionality KD:DT = EH:IM.

Now, take point T′ and its associated point F″ in figure 5.2.46h. As established earlier, F″P is the second of the two lines extending from point F″ which, on passing through normal T′D, produce segments equal to I′M′, LP being the segment in this case. If from the first point of reflection T we extend a line TY parallel and equal to F″P, that line will intersect circle TEC′ at point S, the result being that TS = LP. Again, this can be easily understood in mechanical terms, line F″P being shifted parallel to TY so that endpoints F″ and P coincide with T and Y, respectively. Thus, excess SY on TY that extends beyond circle TEC′ = F″L, leaving TS = LP. As is clear from the diagram, if we choose any point on arc BG between B and *x* and drop the equivalent of line TY from it, TY will not intersect circle TEC′ to yield a segment equal to LP. But at point *x*, which lies just above G, line *xz* extended parallel and equal to F″P will cut circle TEC′ at point *y* such that *xy* = LP. As Alhacen shows somewhat later, however, angle K*x*O formed at this point will be less than ODA. So, from the perspective of T′, T is the only possible alternate point of reflection on arc GB that yields a reflected angle greater than ODA and fulfills the requisite proportionality KD:DT = EH:IM. Altogether, then, T and T′ are the only possible points of reflection on arc GB that yield reflected angles greater than ODA, which is what Alhacen set out to prove in the first place.

[136]The requisite proof is based on cutting from angle OTK an angle OTF equal to angle ODA and extending TF until it meets KD at F. From there it is a matter of drawing on the previous theorem to demonstrate that KT:TF = KD:DO, which was shown to apply to any legitimate point of reflection on arc GB. Although Alhacen does not make it explicit, this same method will yield the second point of reflection. As was pointed out in proposition 25 above, a second line, K″D″ in figure 5.2.47a, p. 275, can be produced from line QH to line HT′ such that K″D″:D″T′ = KD:DT, the original K′D′ represented by the lighter line passing through E. Then, according to the procedure outlined in the proposition, we need only form angle KDT″ equal to angle K″D″T′ to find point T″ of reflection.

[137]Given the construction in figure 5.2.49b, p. 277, it is possible for the form of B to reflect to A from part of the arc facing centerpoint G, and, since the reflected angle formed by them at any point on that arc is greater than adjacent angle BGD, it is possible for there to be as many as two possible points of reflection. Overall, then, in this case there can be as many as three reflections, two from the arc facing G and one from arc KD.

[138]This point is illustrated in figure 5.2.49e, p. 278, which is based on the construction in proposition 47 above. Accordingly, O and K are the relevant points on diameters BD and GK, and KD > OD. As determined in proposition 47, T and T_1 are the two points of reflection on arc BG that form reflected angles OTK and OT_1K greater than angle ODA. If we pass a circle through points D, O, and K, it will cut arc BG at points E and F so that all the angles formed by points within arc EF will be greater than angle ODA, whereas all the angles formed within remaining arc-segments FB and EG will be less than angle ODA. Point T_2, which is the third legitimate point of reflection in arc BG, lies just below point E within arc EG, so it yields a reflected angle OT_2K less than angle ODA. That it cannot have a counterpart on arc FB follows from proposition 44 above, where it is demonstrated that within arc BG there cannot be two legitimate reflected angles smaller than angle ODA. Finally, in arc AQ′ opposite BG, T_3 constitutes the only legitimate point of reflection. Altogether, then, O and K are situated within the mirror in such a way as to yield four, but no more than four, points of reflection.

To this point Alhacen has provided the method for locating points T and T_1 on arc BG (proposition 47) and point T_3 on arc AQ′ (proposition 38), but he has not yet shown, nor will he show, how to locate point T_2 on arc BG. The method for determining this point is essentially the same as in proposition 38 above, albeit with a few adjustments. I will follow the format of that proposition as closely as possible for the sake of clarity and consistency.

Accordingly, let center of sight T on diameter BDQ′ in figure 5.2.49f, p. 279, replace point O in figure 5.2.49e, let object-point H on diameter ADG replace K, let I replace T_2, and let HD > TD. Take some arbitrary line LQ and cut it at point K so that QK:KL = TD:HD, by Euclid, VI.10. Bisect LQ at point N, from point N draw perpendicular NF, and at point L form angle FLQ equal to half of angle BDG. Let FL intersect FN at point F, and from point K draw a line to side FL, intersecting it at point C such that KC:CL = HD:DB, by proposition 24, lemma 6 above. At centerpoint D of the mirror form angle IDB equal to angle LCK. That I is the sought-after point of reflection is demonstrated as follows.

At point I form angle O′ID equal to angle CLK. To the resulting line O′I drop perpendicular TC′ from point T, extend line IC′ to point F′ so that C′F′ = C′I, and draw lines TF′ and TI.

Now, since angle O′DI (i.e., IDB) = angle LCK, by construction, and since angle O′ID = angle CLK, by construction, triangle CLK will be similar to triangle IO′D. Therefore angle IO′D = corresponding angle LKC. But angle C′O′T (i.e., IO′D) = angle LKC (i.e., NKM), and right angle TC′O′ = right angle KNM, both being right by construction, so it follows that triangles NKM and C′TO′ are similar, by Euclid, VI.4.

Now, if line ID is extended beyond D until it intersects the extension of C′T at point R, angle RDT = angle MCF, because angle MCF = two right angles – LCK, whereas angle RDT = two right angles – TDI, and LCK = TDI by construction. Triangle RDT will thus be similar to triangle CMF, since angles CMF and RTD = vertical angles KMN and O′TC′, respectively, which are corresponding angles in similar triangles NMK and C′TO′. Therefore, RD:DT = FC:MC, so, by Euclid, V.16, DT:MC = RD:FC.

But, since triangle O′ID is similar to triangle CKL, by previous conclusions, KC:CL = O′D:DI, so KC:O′D = CL:DI. Moreover, KC = MC + MK, and O′D = DT + O′T, so (MC + MK):(DT + O′T) = CL:DI, and so, by Euclid, V.17, MC:DT = CL:DI. But MC:DT = FC:RD (given that DT:MC = RD:FC by previous conclusions), so FC:RD = CL:DI, and so RD:DI = FC:CL. Accordingly, RI:DI = FL:CL, by Euclid, V.18. But DI:IO′ = CL:LK, since triangle DIO′ is similar to triangle CLK, by previous conclusions. Therefore, RI:IO′ = FL:LK, by Euclid, V.22, so IO′:RI = LK:FL. But RI:IC′ = FL:LN, because triangle RIC′ is similar to triangle FLN, since right angle RC′I = right angle LNF, and angle RIC′ = corresponding angle FLN, by construction. Hence, by Euclid, V.22, IO′:IC′ = LK:LN, which reversed is LN:LK = IC′:IO′. But, by construction, IC′ = F′C′, and QN = LN, so QN:LK = F′C′:IO′, and so, by Euclid, V.17, QK:LK = F′O′:IO′.

Now, if line UI is drawn from point I parallel to TF′, and if line DB is extended until it intersects UI at point U, triangle O′UI will be similar to triangle TO′F′, so angle O′IU = angle TF′O′. Therefore, TO′:O′U = QK:KL, because TO′:O′U = F′O′:IO′ = QK:LK, by previous conclusions, and so TO′:O′U = TD:HD, because QK:KL = TD:HD, by construction. But since triangle TC′I = triangle TC′F′, because common side TC′ is perpendicular to equal bases F′C′ and C′I, then angle TF′C′ = angle TIC′, so angle TIC′ = angle UIO′, which = angle TF′C′, by previous conclusions. By Euclid, VI.3, therefore, TO′:O′U = TI:UI, because angle UIT in triangle UIT is bisected by IO′, and so TI:UI = TD:HD, since both are as TO′:O′U, by previous conclusions. At point I, finally, form angle PID = angle DIT, and let P be a point on diameter DB.

It has just been established that TI:IU = TD:HD, and, by compounding, TI:IU = (TI:IP):(IP:IU). By Euclid, VI.3, however, TI:IP = TD:DP, since angle TIP is bisected by DI, so, with TD:DP substituted in the foregoing proportion, we get TI:IU = (TD:DP):(IP:IU). Also, by compounding, TD:HD = (TD:DP):(DP:HD), but we know from previous conclusions that TI:UI (which = [TD:DP]:[IP:IU]) = TD:HD. Accordingly, TD:HD = (TD:DP):(IP:IU), but also TD:HD = (TD:DP):(DP:HD), and so (TD:DP):(DP:HD) = (TD:DP):(IP:IU). With the equal term TD:DP dropped from both sides, then, DP:DH = IP:IU.

Now angle O′IT is half of angle UIT, by previous conclusions, whereas angle DIT is half of angle PIT, since angle PID = angle DIT by construction. It follows that angle DIO′ is half of angle PIU. But angle DIO′ is half of angle HDP, since it is equal to angle FLQ, which, by construction, is equal to half of angle BDG, the vertical angle of HDP. Therefore, angle PIU = angle HDP, and DP:HD = PI:IU, by previous conclusions. Hence, triangle UIP is similar to triangle HDP, by Euclid, VI.5, and so angle UPI = angle HDP. Point H will thus lie on line PI, because angle HPD + angle HPQ′ = two right angles, and angle O′PI + angle HPQ′ = two right angles, HPQ′ being adjacent to both HPD and O′PI. Therefore, since angle of reflection TID = angle PID = angle of incidence HID, the form of point H is reflected to point T from point I.

[139]This last claim is based on proposition 37, cases 2 and 3, above. In case 2 it was demonstrated that, if the perpendiculars drawn from the two points H and T (in figure 5.2.37b, p. 258) converge at point E on facing arc BG, then E is the only

point of reflection within that arc. Likewise, H and T can reflect to one another from point Z, and only point Z, on the opposite arc QA. Altogether, then, only two reflections will occur. In case 3 it was demonstrated that, if EH and ET are not perpendicular to the diameters (in figure 5.2.37c, p. 259), reflection will occur from points M, E, and L, but from no others, on facing arc GB. Likewise, reflection will occur from point Z, and only point Z, on opposite arc QA. Altogether, then, four reflections will occur in this case. Since there is no intermediate case, it follows necessarily that, when the two points are equidistant from the center, it is impossible for there to be three, and only three, reflections.

[140]In the cylinder at the top of figure 5.2.50a, p. 281, GA represents the normal common to both the circle GLA parallel to the bases and the ellipse FGK formed by an oblique cut along major axis FDK, GA being its minor axis. Accordingly, line X'GY' tangent to the ellipse at point G and line XGY tangent to the circle at the same point define the plane tangent to the cylinder along the line of longitude passing through G. It is impossible to represent this situation meaningfully from a perspective directly above the axis of the cylinder, because, in that case, the circle and the ellipse will coincide. Accordingly, I have represented the situation in figure 5.2.50 by inscribing the circle inside the ellipse, although in the manuscripts only the circle is given. Oddly enough, Risner represents the situation by having the circle circumscribe the ellipse so that the normal coincides with the major rather than the minor axis.

[141]Thus, as illustrated in figure 5.2.50a, p. 281, the form of point C will reflect from points H and L on circle ALGH, and, as shown in proposition 34, the form of any point on AD between A and D will reflect to E from two points on the circle on either side of AG. That there can only be two such points is demonstrated in proposition 36.

[142]There is obviously some confusion in the text at this point. Point F will not be the image-location for a center of sight located at H and an object-point located at T, as specified in the previous paragraph. It will be the image-location for a center of sight lying at T if the object-point is at H. Although none of the manuscripts seems to recognize the problem, Risner makes the necessary adjustment, putting the center of sight at T where it belongs.

[143]This point is easier to see if we abstract the case of reflection-point Q from figure 5.2.52, p. 283. Accordingly, in the segment of the cylinder provided in the top diagram of figure 5.2.52a, p. 284, points A and B, respectively, represent the object-point and the center of sight in the plane of the base circle, whose centerpoint is H. Within that plane E is a legitimate point of reflection, so normal EH bisects angle AEB, leaving angle of incidence AEH = angle of reflection BEH. If we drop a line orthogonal to the plane of the base circle at point B and choose some point T on it, that point will represent a corresponding center of sight, with Q the point on the mirror from which the form of point A will reflect to center of sight T. The circle of the plane within which Q lies is represented by the gray section passing through Q. Now, as this case is actually constructed, T lies above Q within a higher plane represented by the topmost gray section. Thus, the form of point A moves upward along AQ and, on reflecting from Q, moves yet upward along QT to the center of

sight. The figure to the right of the cylindric section represents a head-on view of the situation along a line of sight parallel to AH and just above the plane of the base-circle so that A lies directly behind H. Normal QU, which passes through AT at point I, bisects angle AQT so that angle of incidence AQU = angle of reflection TQU. In other words, quadrilateral AQTU with its diagonals TA and QU in the plane of reflection is simply a projection of quadrilateral AEBH with its diagonals BA and EH in the plane of the base-circle. Moreover, normal QU lies in the plane of the circle passing through Q, so it follows that the plane of reflection forms an elliptical section on the cylinder's surface with normal QU as its minor axis.

[144]For example, the case of reflection-point L is represented in the lower set of diagrams in figure 5.2.52a, p. 284, both representing the same views as in the upper set. Point of reflection L lies in the plane of the circular section represented by the lower segment in gray, and center of sight T lies in the plane of the circular section above it. The normal in this case is LX, and the resulting quadrilateral ALTX with its diagonals LX and AT in the plane of reflection is a projection of quadrilateral ADBH with its diagonals DH and AB in the plane of the base-circle. Thus, angle of incidence ALX = angle of reflection TLX, normal LX lies in the plane of the circle passing through point L, and the plane of reflection forms an elliptical section on the cylinder's surface with normal LX as its minor axis. The same sort of analysis can be applied to M, the remaining point of reflection.

[145]As before, the proof depends on extending a line from O to axis HU parallel to AC, which is parallel to HZ and lies in the same plane as OZ, which is parallel to HU, so that line will intersect the axis. Accordingly, the angle formed by that line and line of reflection OT will be equal to the angle formed by that line and line of incidence AO, so that line will be normal to point of reflection O on the mirror's surface within plane of reflection AOT.

[146]That each point of reflection is unique to a given elliptical section on the cylinder's surface—and vice-versa—follows from proposition 50 above, where it is shown that reflection can occur from that point, and that point only, where the ellipse intersects the circle whose diameter forms its minor axis.

[147]The various points in this paragraph are illustrated in figure 5.2.52b, p. 285. As we just established, each of the four points of reflection M, L, Q, and O lies in a unique plane of reflection that forms a particular elliptical section on the cylinder's surface. We also established that each of those points lies in a plane that forms a particular circular section on the cylinder's surface. These circles are stacked one above the other in ascending order, starting with the base circle and moving upward through the circles passing through points M, L, O, and Q. There are thus five circles in all. Furthermore, as we established, the diameter of each circle coincides with the minor axis of the ellipse particular to the given point of reflection. Finally, since A and T are common to all four planes of reflection, all four elliptical sections particular to their points of reflection must intersect along line AT, which is extended along RP.

Now, imagine the circle in figure 5.2.52b to represent the aforementioned stack of five circles viewed from directly above so that point M on its circle lies directly above point G on the base circle, point L directly above point D, point Q directly above point E, and point O directly above point Z. Imagine, as well, that each elliptical section associated with its particular circle is rotated about its minor axis—which is the diameter of its associated circle—until it lies in the same plane as that circle. In the course of this rotation, the line coincident with RP on each of the elliptical sections will be moved until it comes to rest in a position slightly displaced from its original position coincident with RP, but for the sake of convenience we can ignore this displacement because it is so slight.

Accordingly, in figure 5.2.52b, each of the ellipses must be understood to lie in the plane of its associated circle in a stack of four. The lowermost ellipse *mn* is therefore associated with the circle passing through M, diameter MN of that circle being the minor axis of the ellipse, the next in line, *lk*, associated with the circle passing through L, its diameter LK being the minor axis, and so forth up the line from O (*op*) to Q (*qs*), the respective diameters/minor axes being OP and QS. Taking RP as common to all these ellipses, we can see that RP is normal to ellipse *qs* associated with point of reflection Q on the side of R and, furthermore, that it cannot be normal to any of the remaining three ellipses on the side of either R or P. Therefore, RP can be normal to only one of the four ellipses. Moreover, the only way it could be normal to more than one would be if it coincided with the minor or major axes of the relevant ellipses. That this cannot be the case is clear from the fact that RP does not pass through centerpoint H of the circle through which the minor and major axes of all the ellipses must pass. Or, as Alhacen observes, in order to be a normal common to more than one of the ellipses, RP would have to meet the axis, which passes through centerpoint H. Therefore, each ellipse has a distinct normal dropped to it from object-point A. The reason for establishing this point, of course, is to establish that for each ellipse—and thus for each plane of reflection—there is a distinct image, each image lying at the intersection of the particular normal and the particular line of reflection. Thus, in the case illustrated, the image of object-point A will lie at center of sight T, where normal RATP, dropped from object-point A to the reflecting surface, and line of reflection QT intersect.

[148]In other words, if the center of sight were to lie at T, then we would reverse the process in the theorem to locate point B and then find the points of reflection in the base circle and then extrapolate from that. The method prescribed here is essentially the same as that prescribed for convex cylindrical mirrors.

[149]That in the case of any conical mirror the common section of the plane of reflection and the mirror's surface cannot be a circle was made clear in book 4. If that section is a line of longitude, of course, then the rules governing both reflection and image-location will be precisely the same as those governing them in plane mirrors, so the image will always appear behind the reflecting surface.

[150]Thus, as illustrated in figure 4.5.13, p. 212, if ARFB represents the concave conical mirror, and if the center of sight lies at vertex-point K of the lower cone, then all the lines of sight extended from K normal to the mirror's surface will strike it along circle RFB, so reflection will occur back to K along those same lines. Accordingly, all that is visible when the eye is so placed is that segment of the eye concentric with K from which the rays can reach the mirror and reflect back to K—i.e., the slice of the cornea bounded by the pupil through which the rays normal to

circle RFB can reach to the center of sight. On the other hand, if the center of sight is displaced from the axis, then only one line of sight will be normal to the mirror, so all that will be visible is the single point on the cornea through which that line of sight passes.

[151]Presumably, this latter qualification is meant to take into account the lone reflection from the arc on the opposite side of the axis, in which case, of course, the maximum number of possible reflections will be four.

[152]The claim that TP is parallel to NZ is true if, and only if, points H and Z are equidistant from point Q, in which case QT will be perpendicular to HZ at point N and will bisect it at that point. As far as the proof is concerned, though, whether TP is parallel to NZ is irrelevant; all that matters is that TP be tangent to the circle. That it is in fact tangent follows from the fact that NT is parallel to ZF, by construction. Being perpendicular to ZF, then, TP is also perpendicular to normal QT and is thus tangent to the circle at point T.

FIGURES FOR INTRODUCTION AND LATIN TEXT

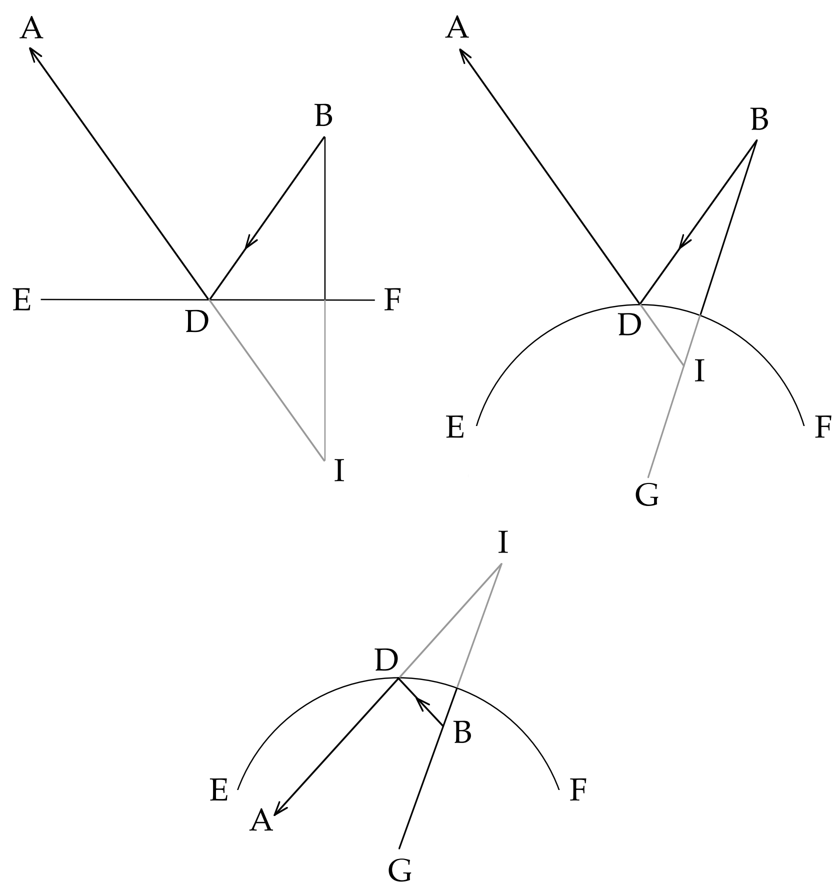


figure 1

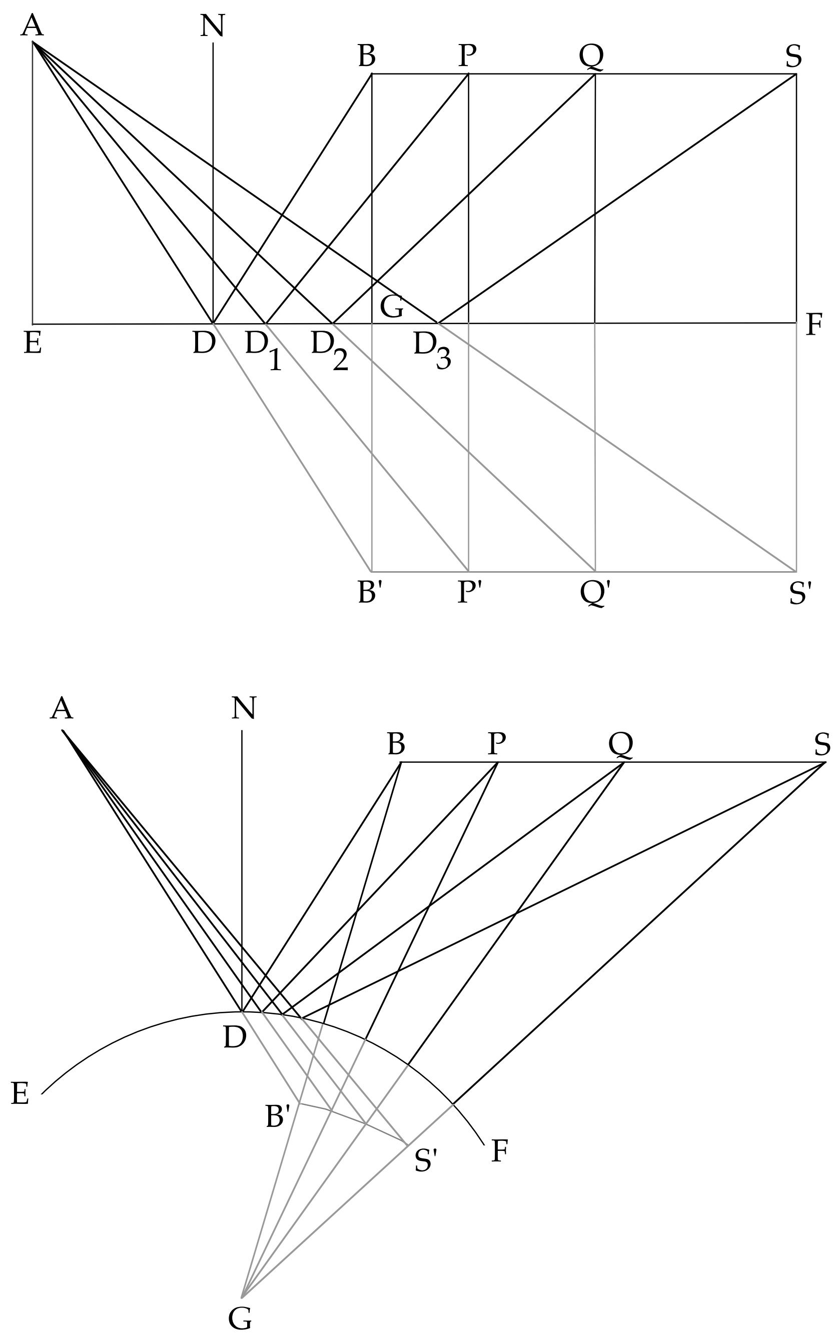


figure 2

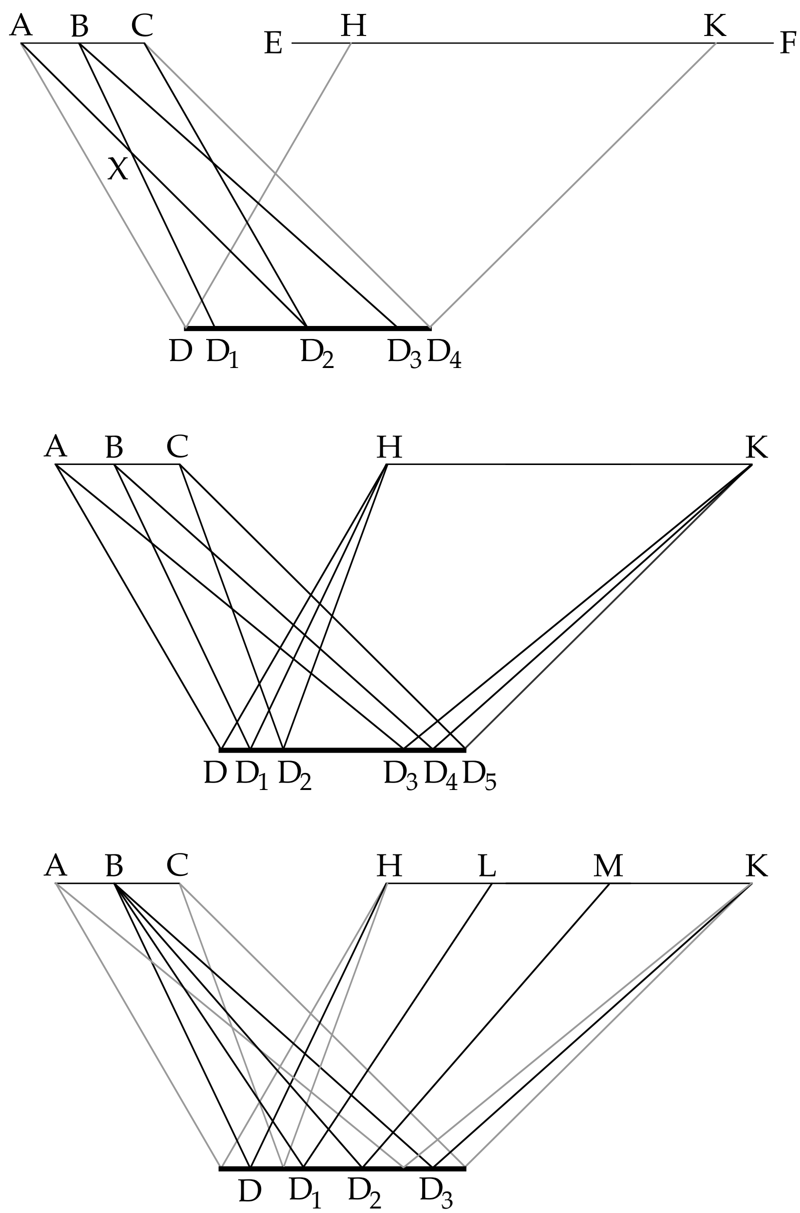


figure 3

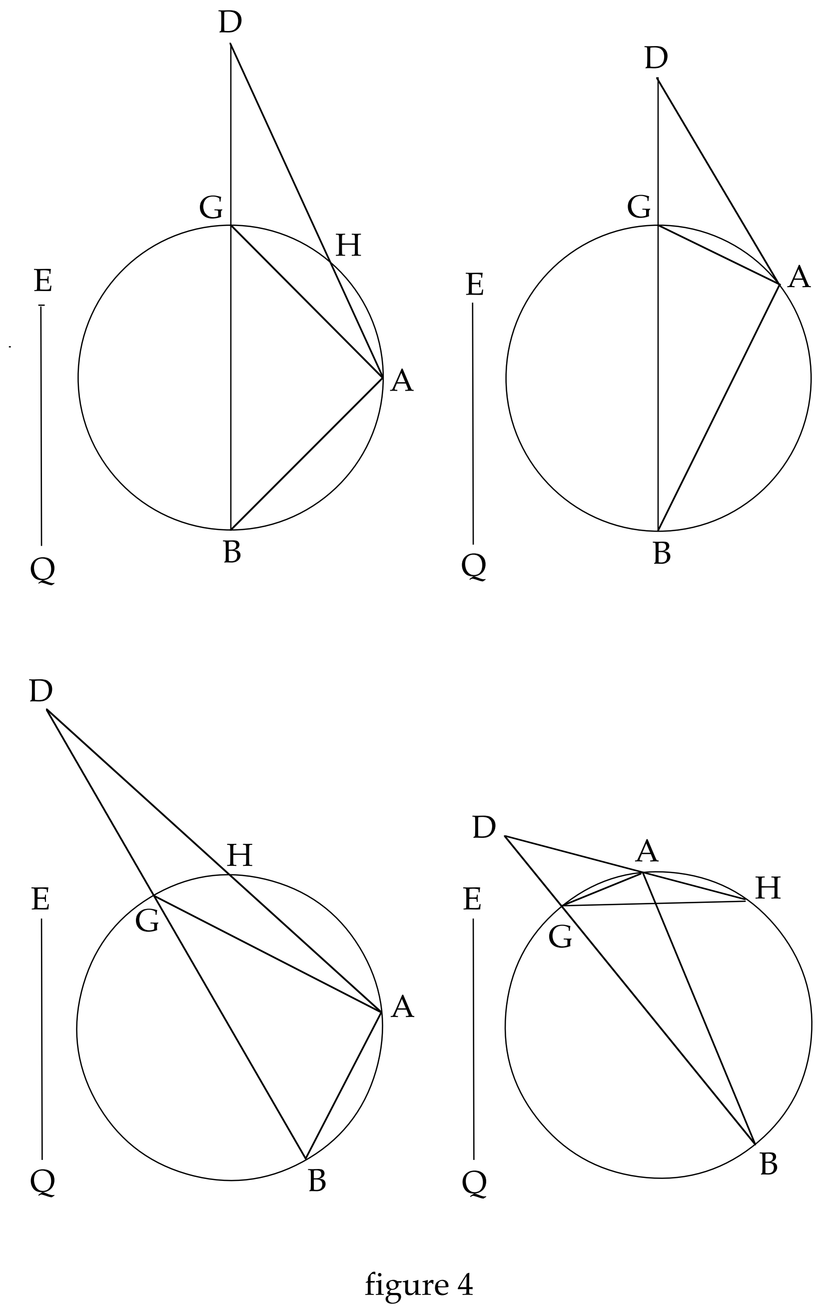


figure 4

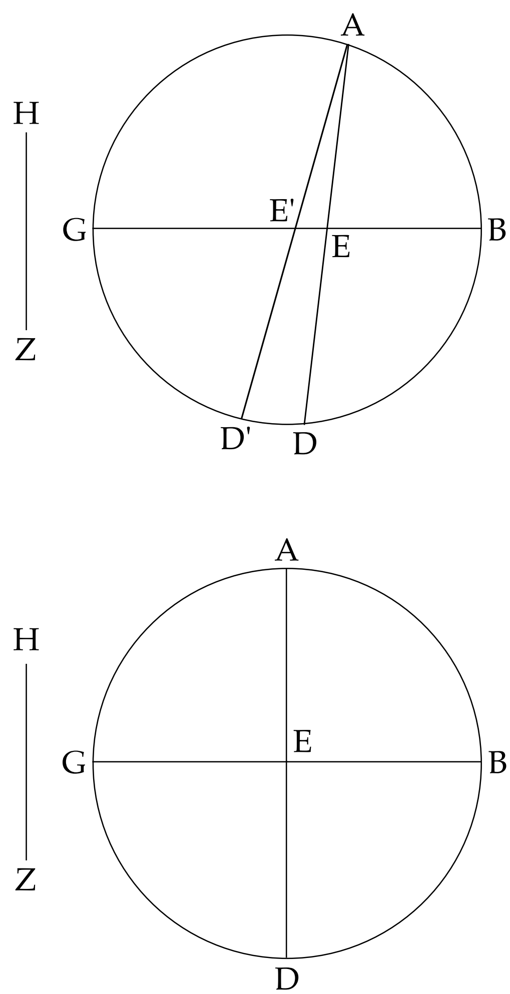


figure 5

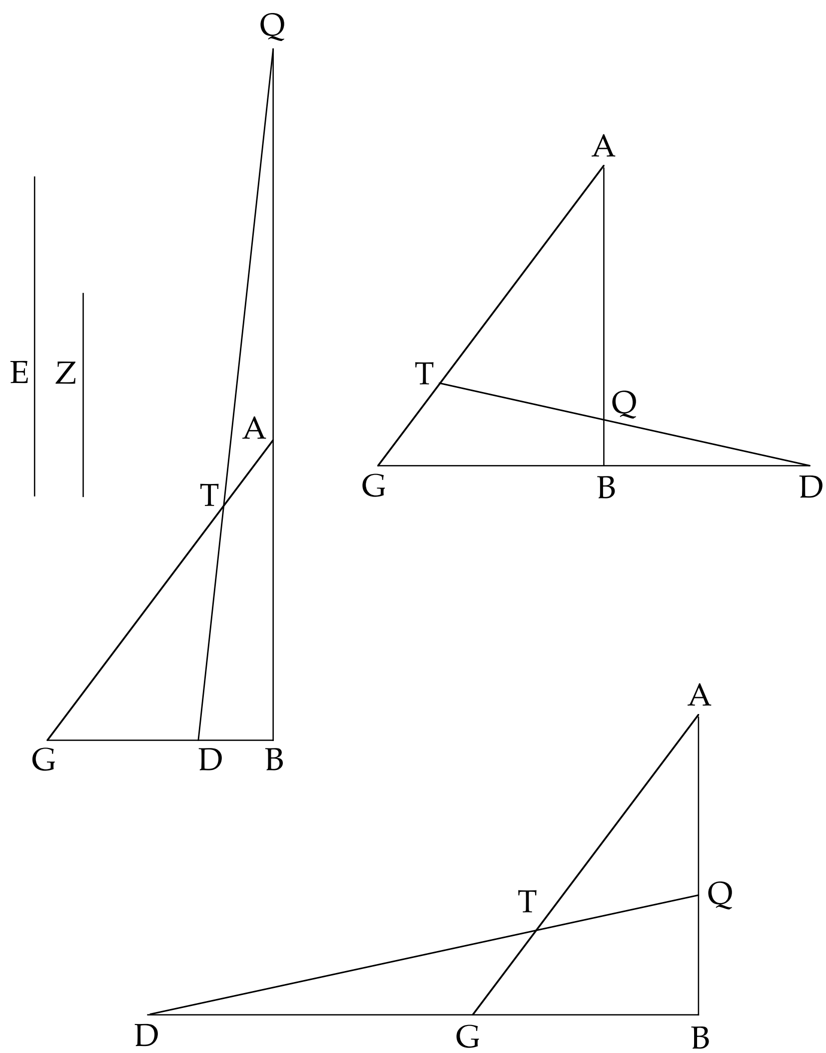


figure 6

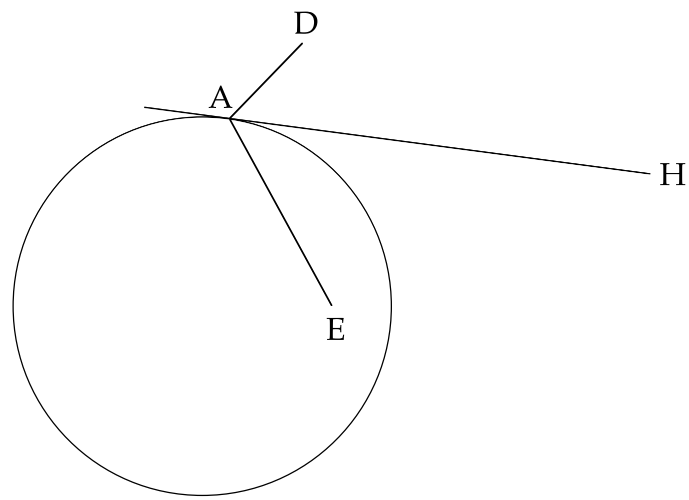


figure 7

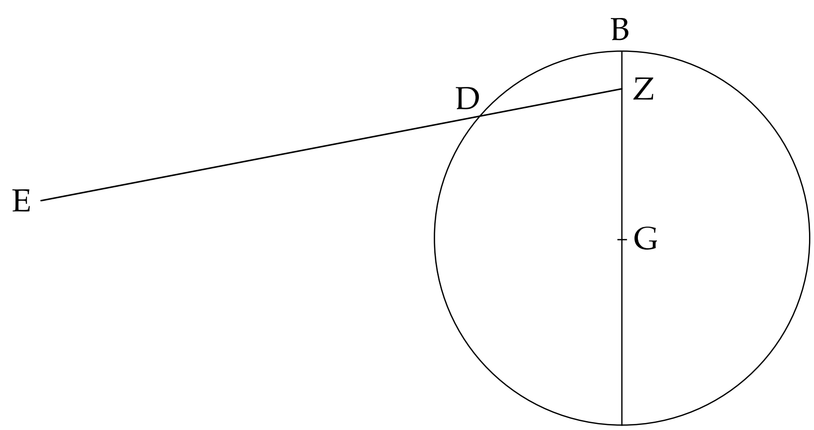


figure 8

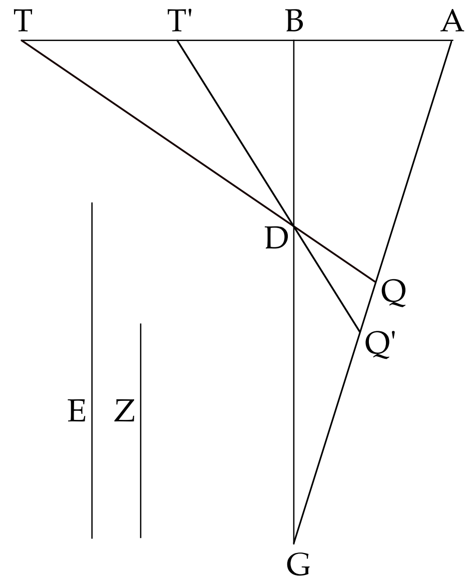


figure 9

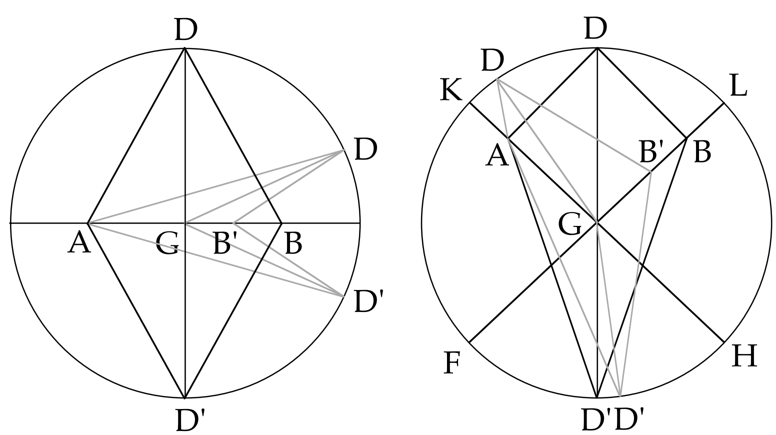


figure 10

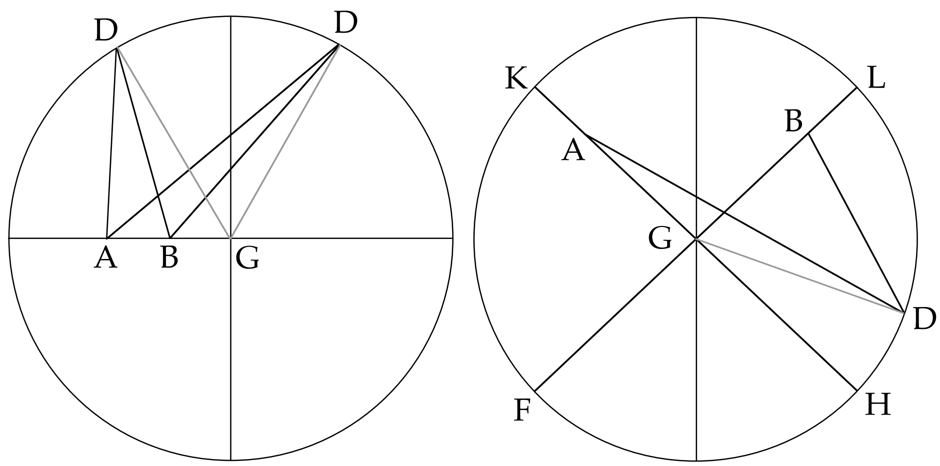


figure 11

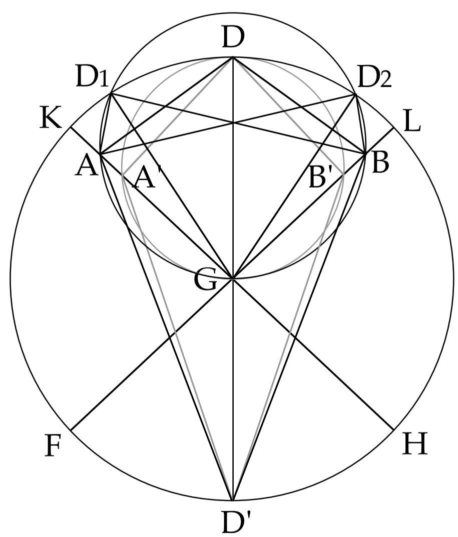


figure 12

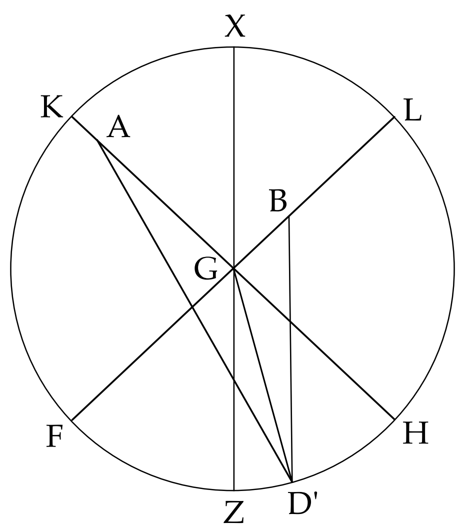


figure 13

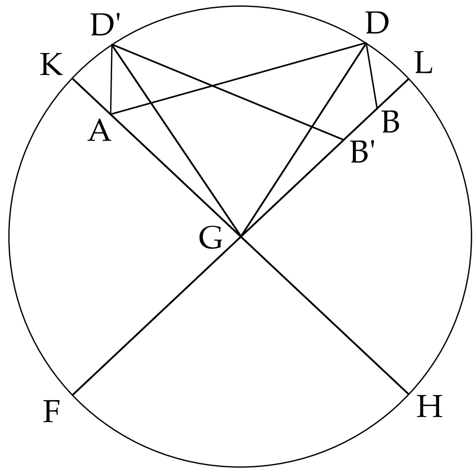


figure 14

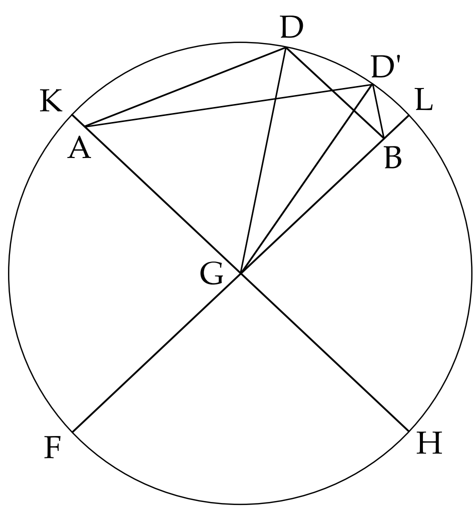


figure 15

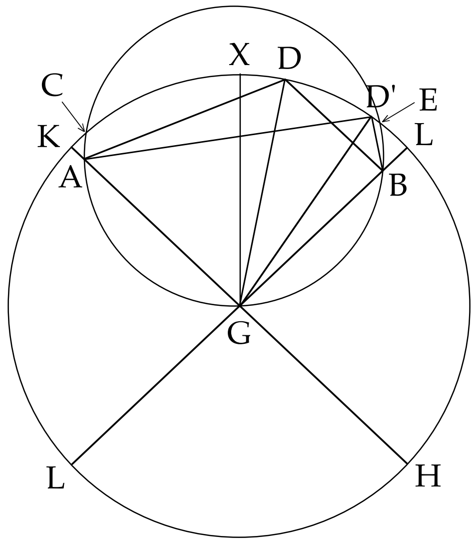


figure 16

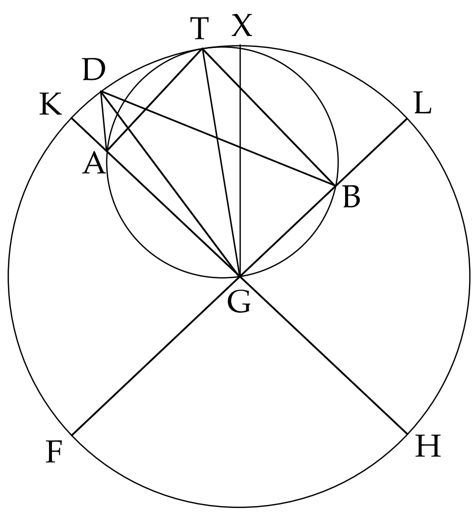


figure 16a

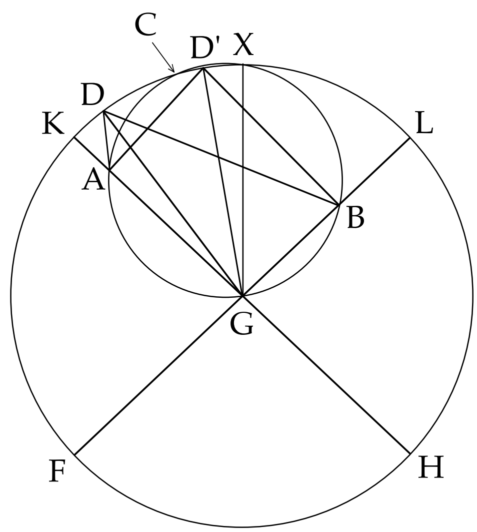


figure 16b

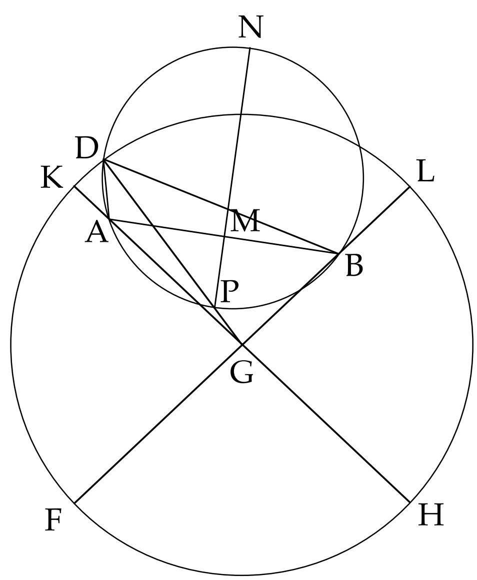


figure 16c

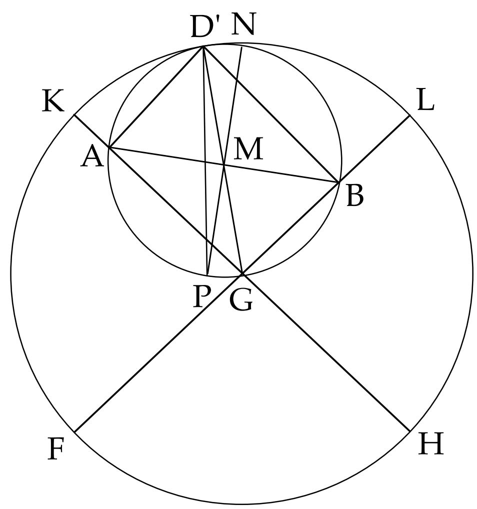


figure 16d

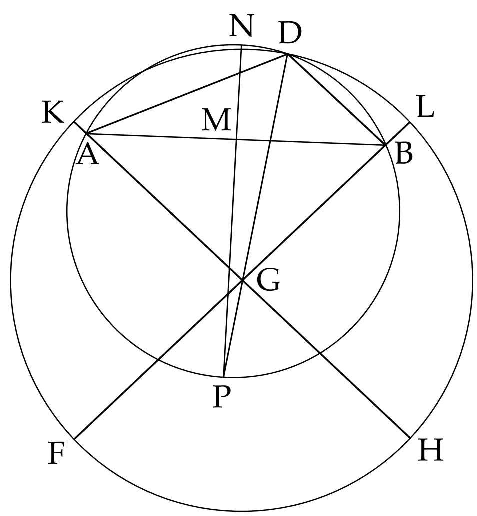


figure 16e

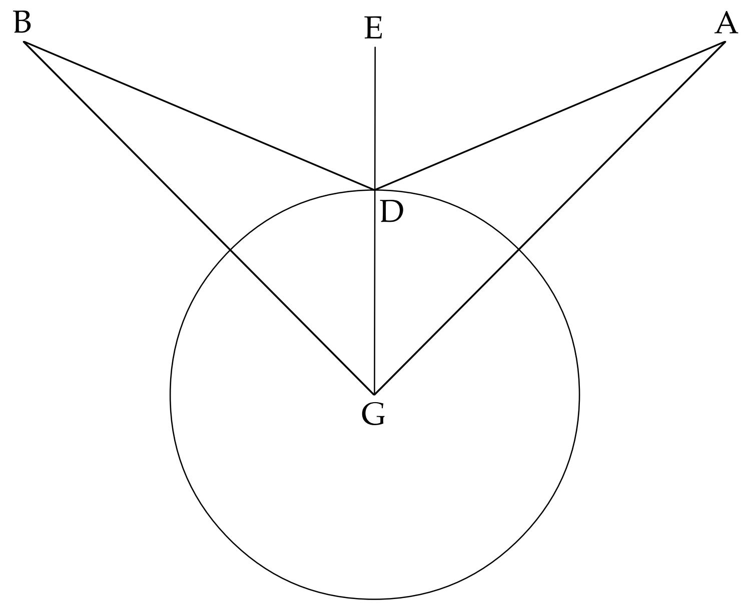


figure 17

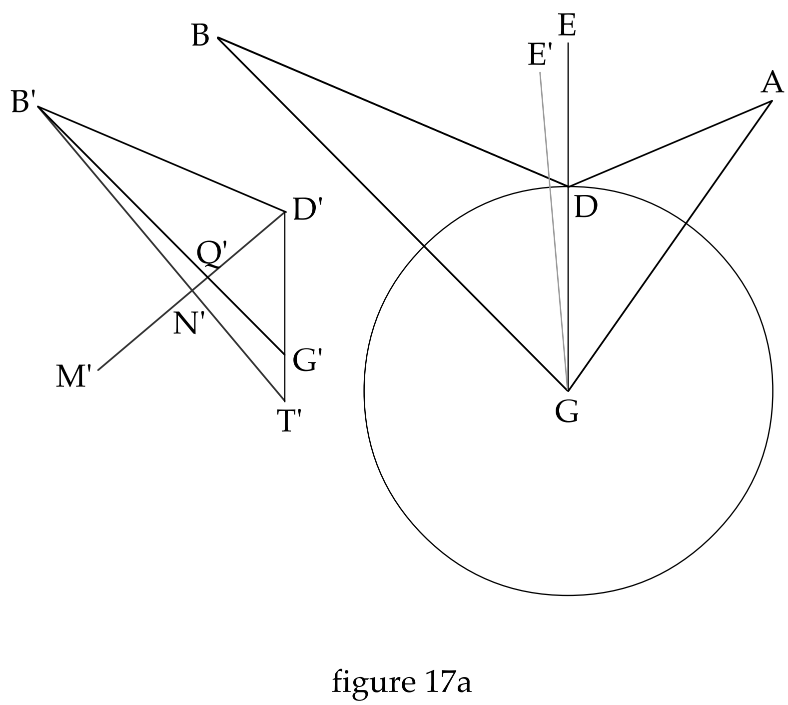


figure 17a

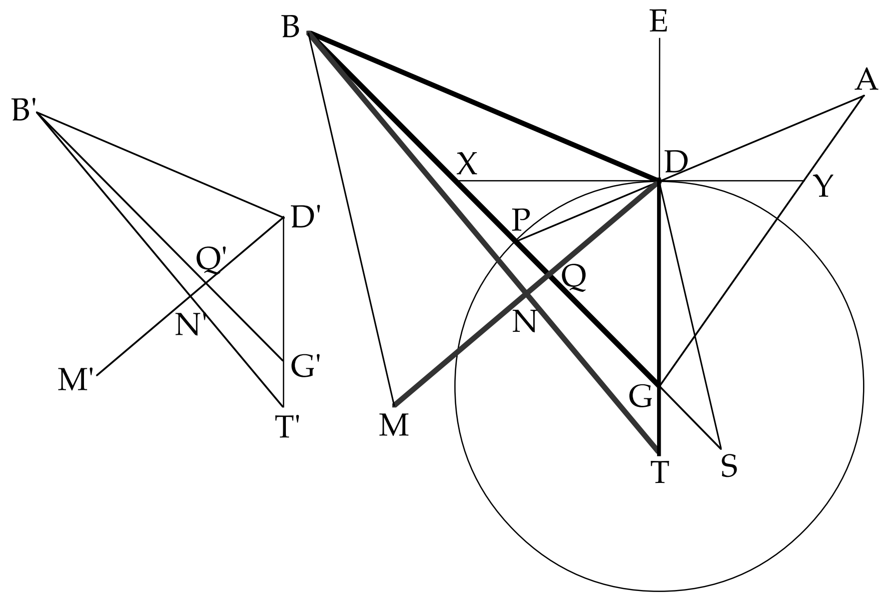


figure 17b

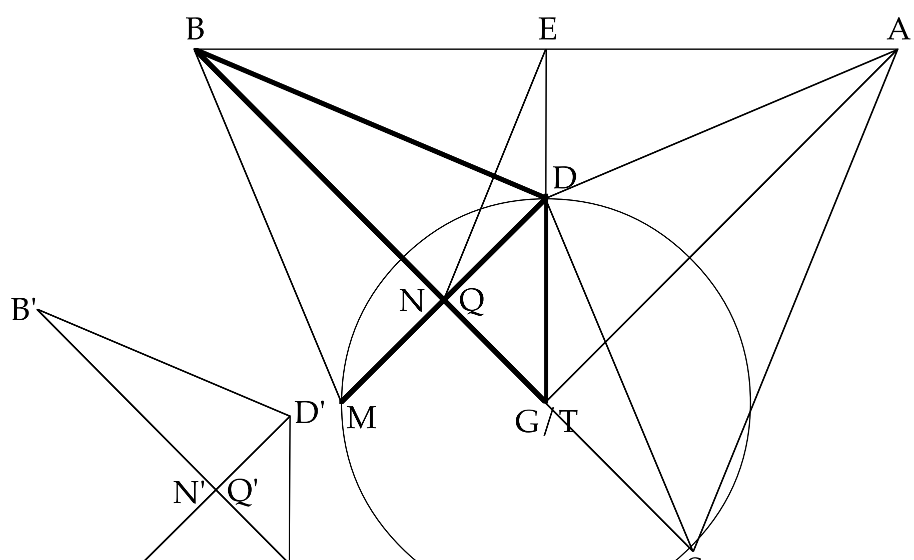


figure 17c

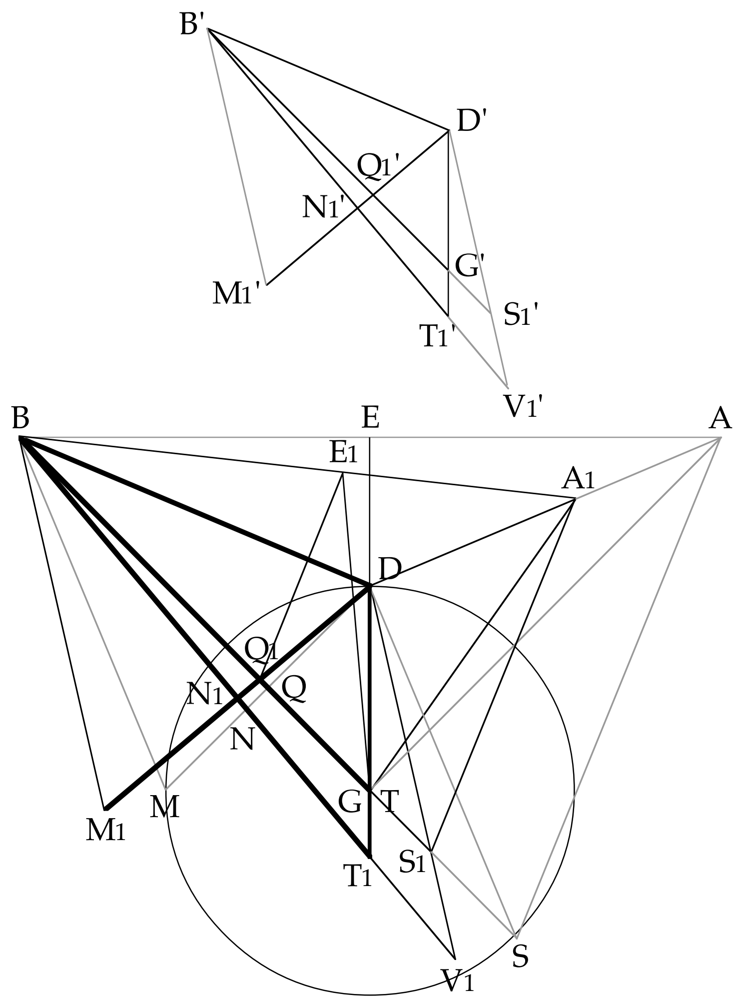


figure 17d

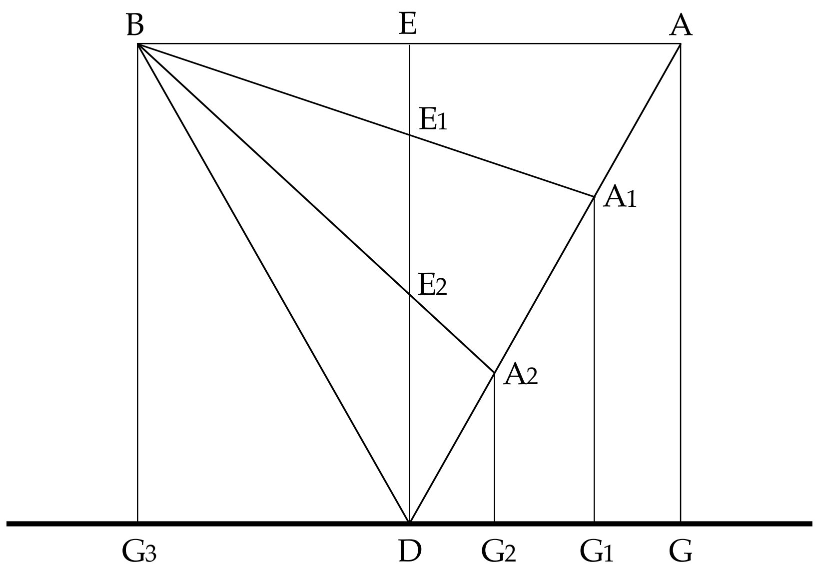


figure 17e

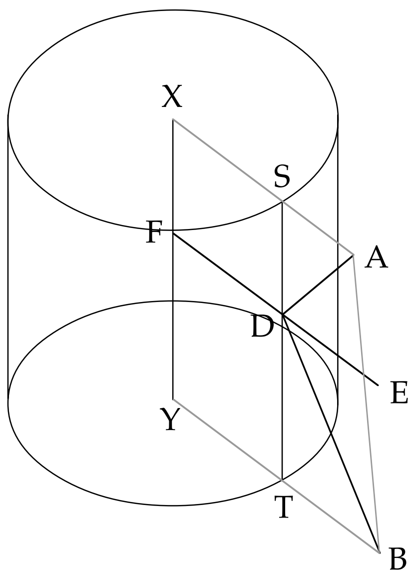


figure 18

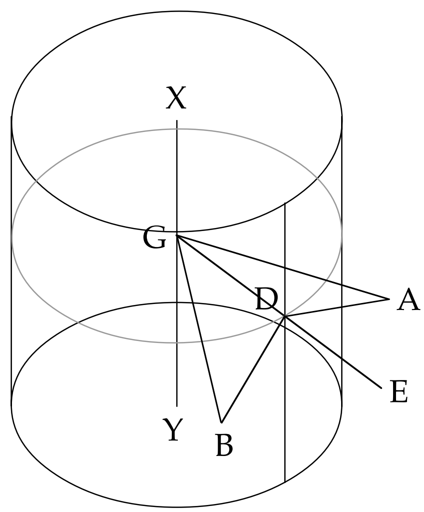


figure 18a

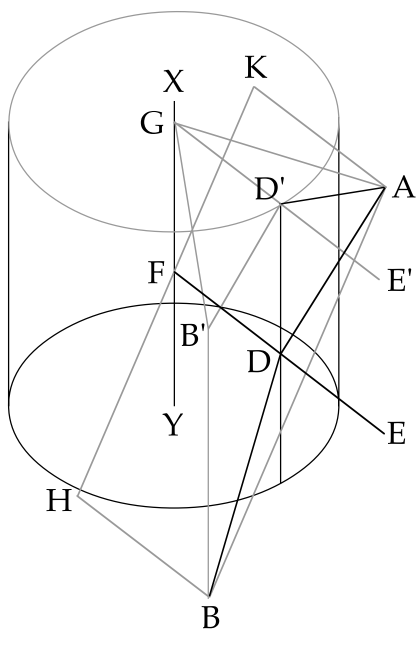


figure 18b

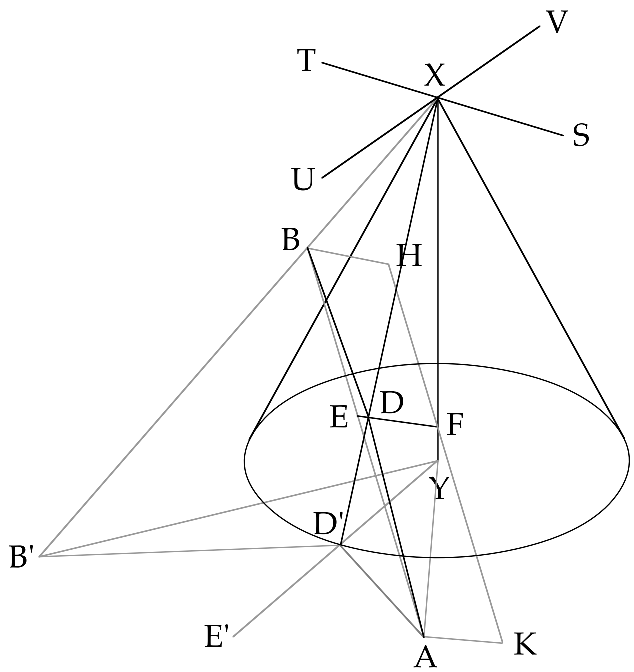


figure 19

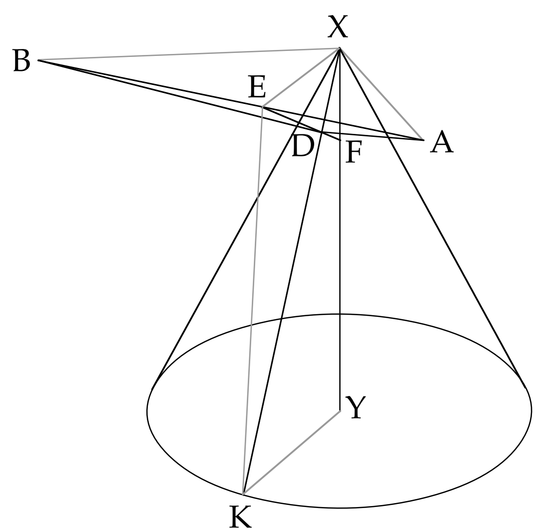


figure 19a

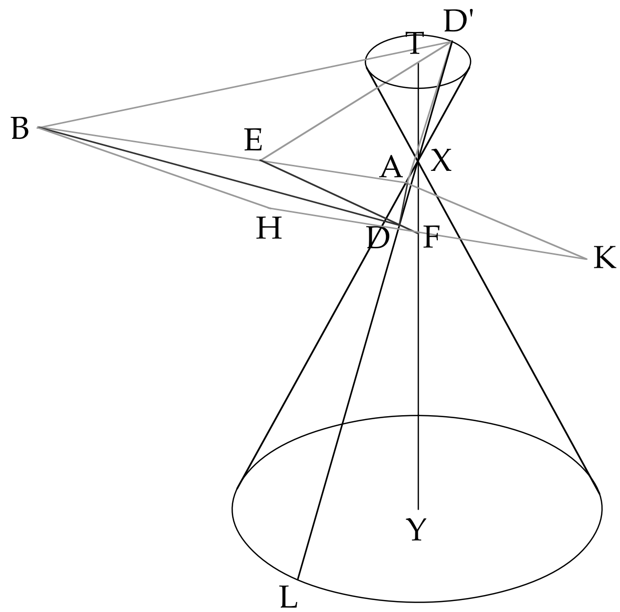


figure 19b

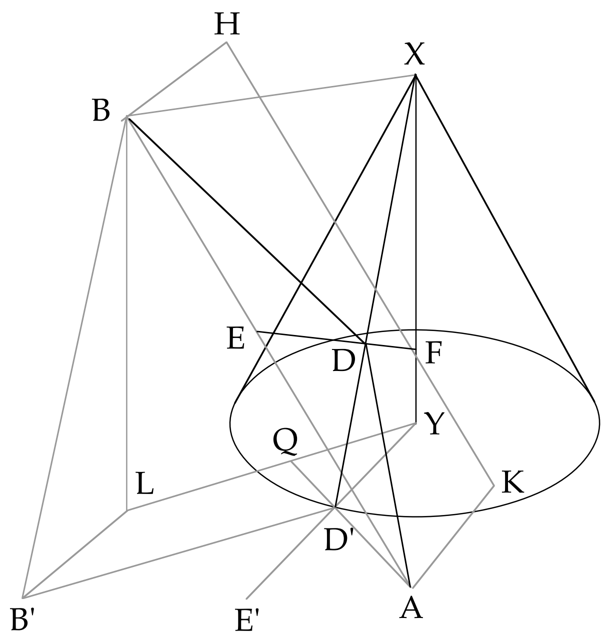


figure 19c

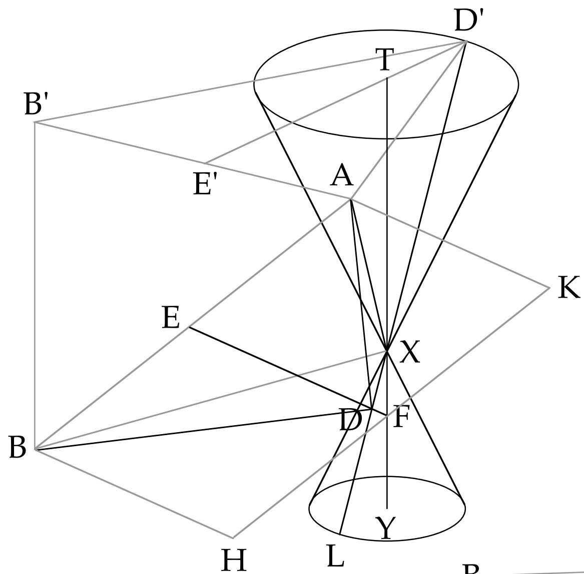


figure 19d

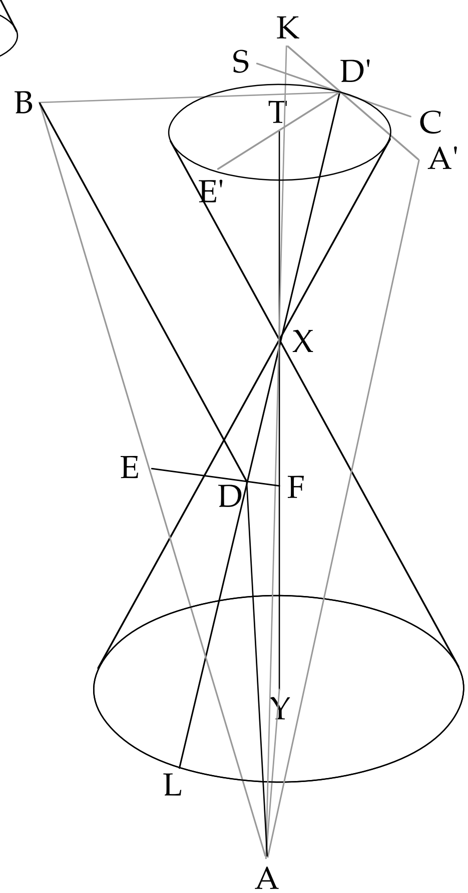


figure 19e

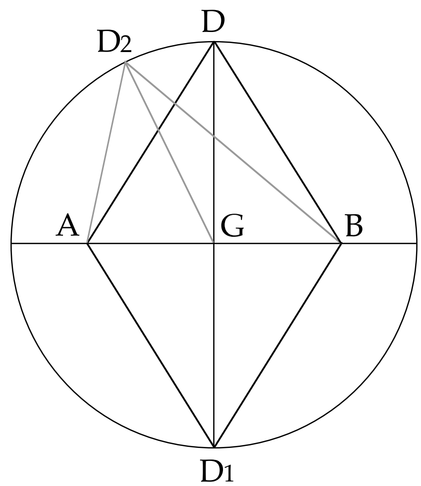


figure 20

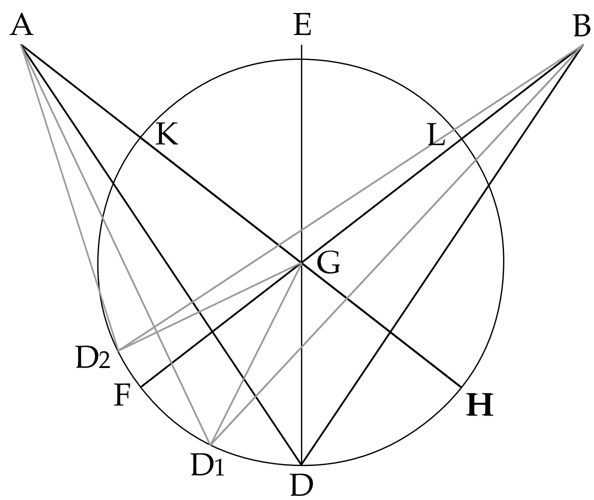


figure 20a

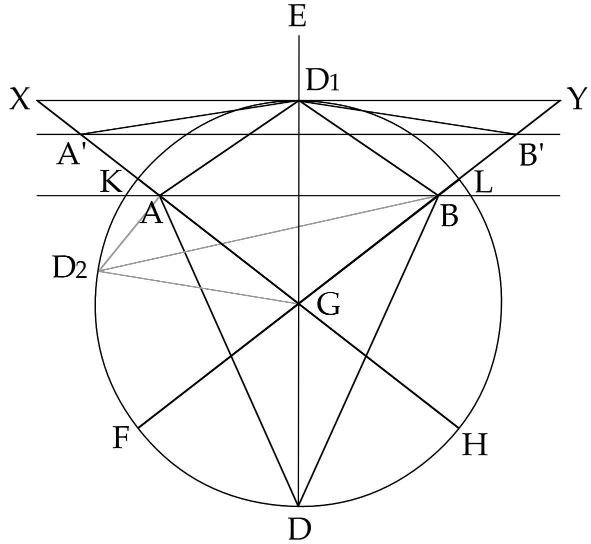


figure 20b

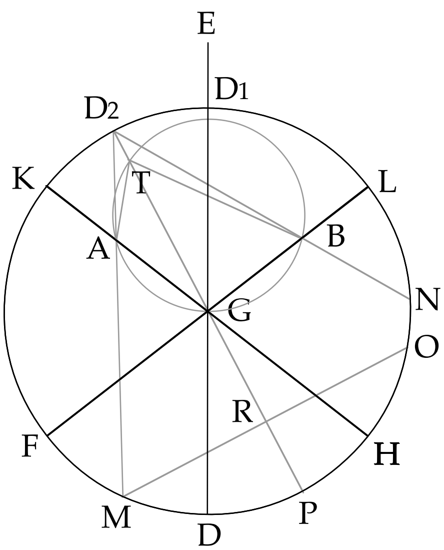
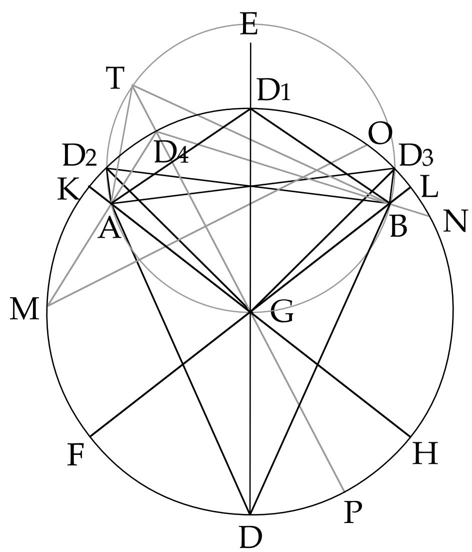


figure 20c



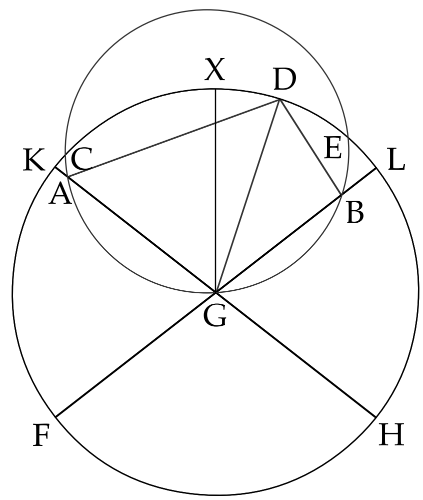


figure 20e

figure 20d

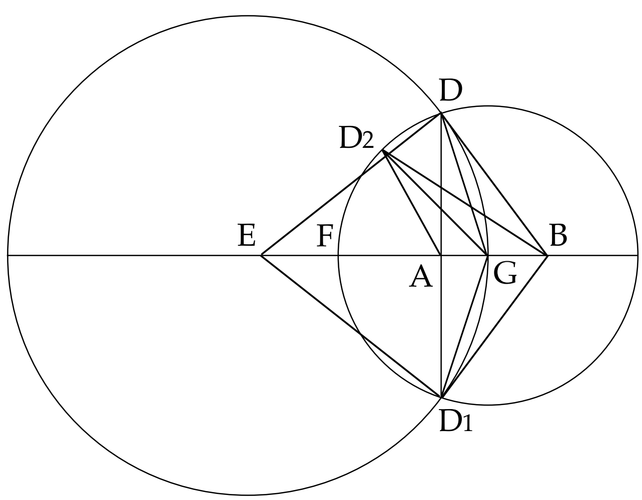


figure 20f

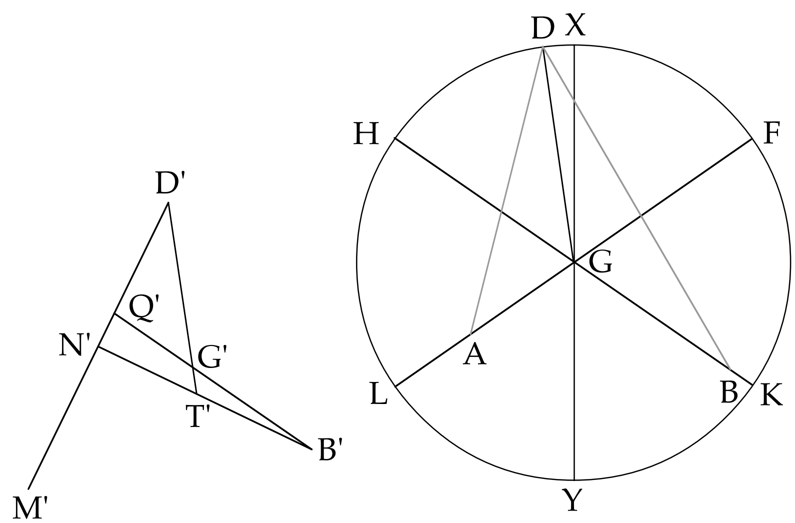


figure 20g

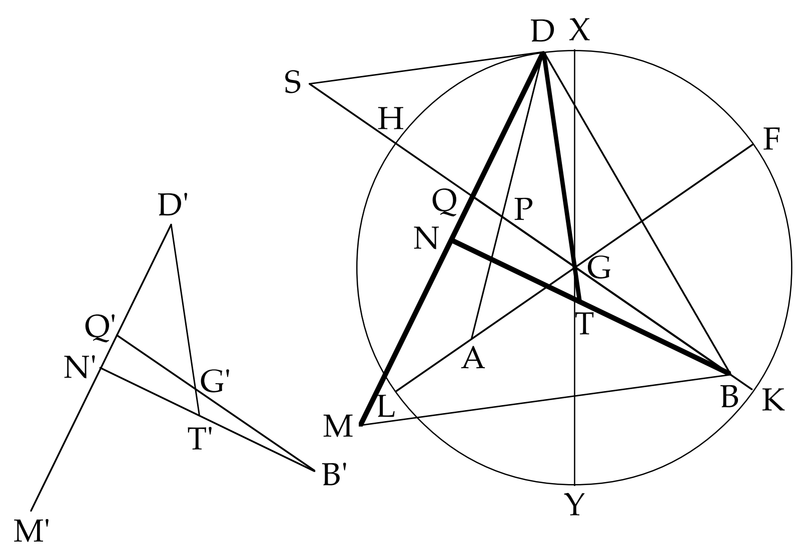


figure 20h

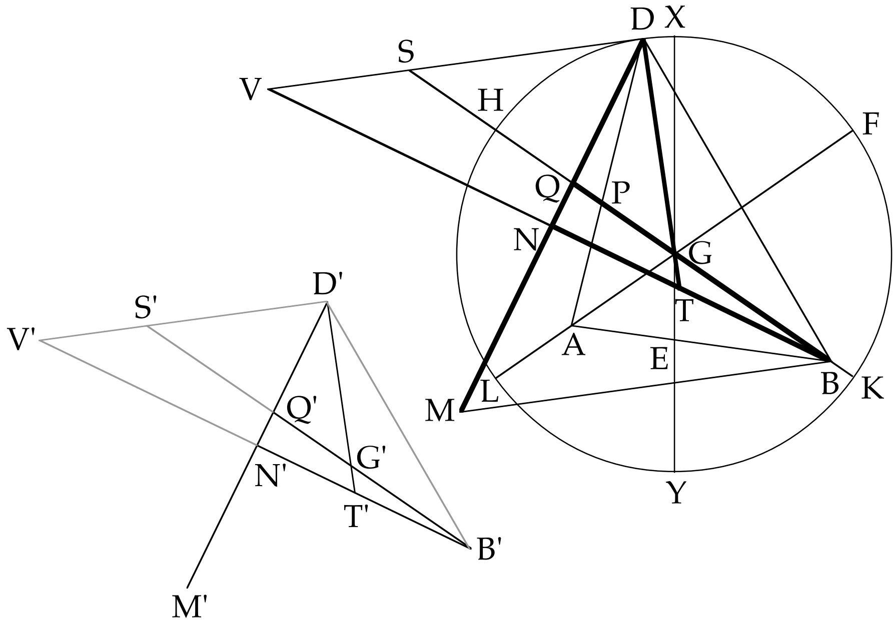


figure 20k

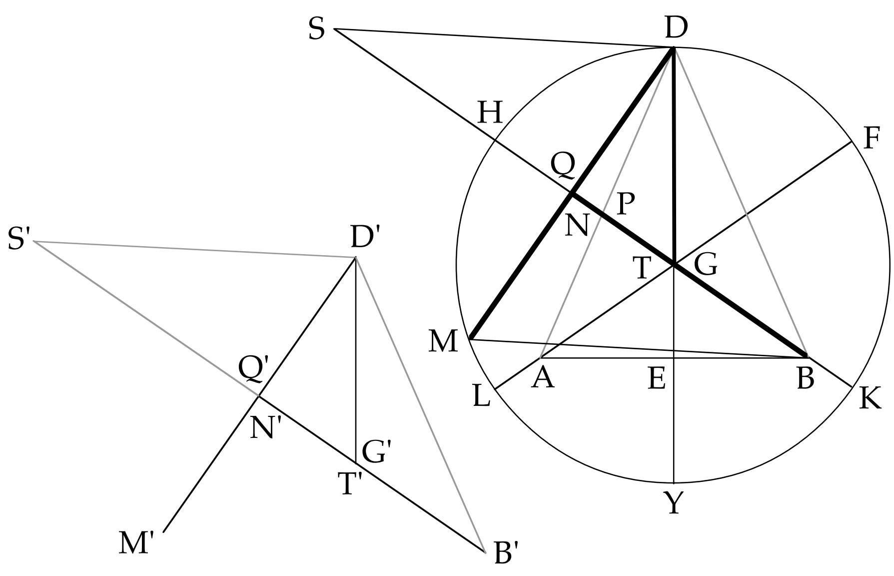


figure 20m

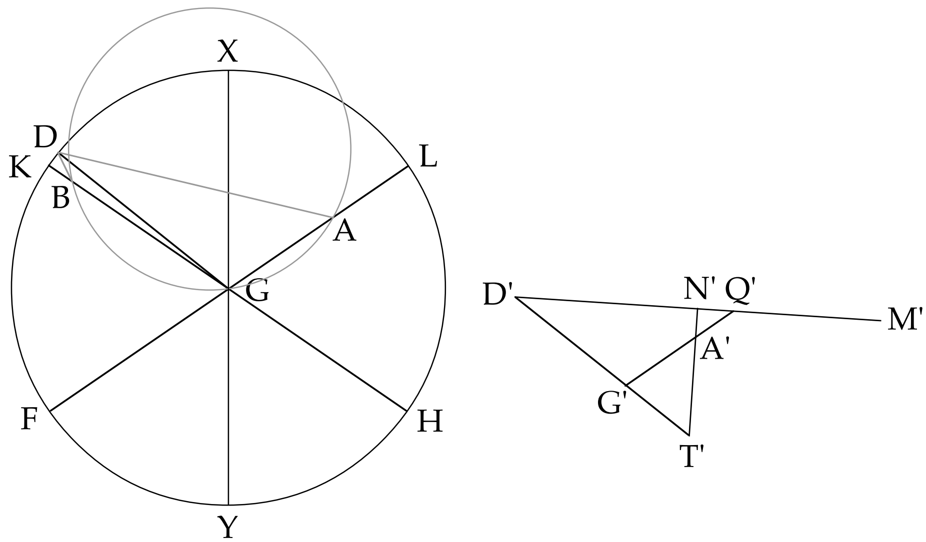


figure 20n

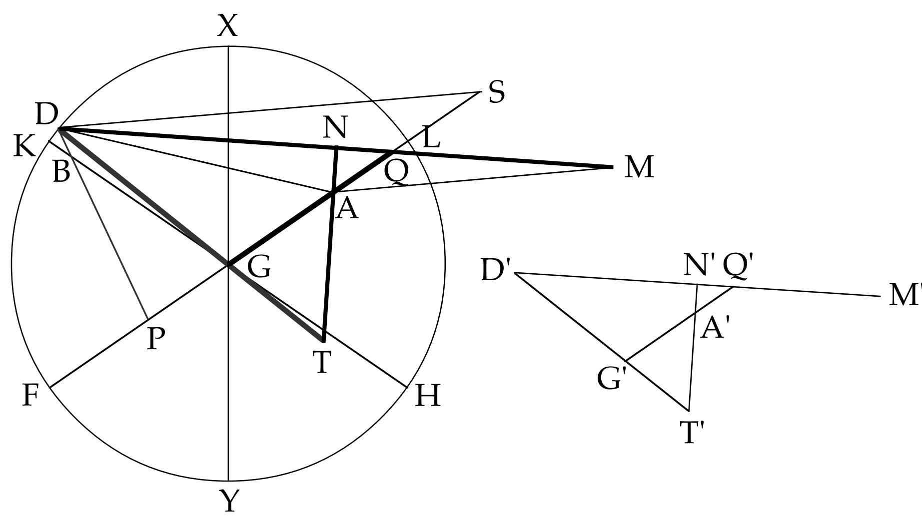


figure 20p

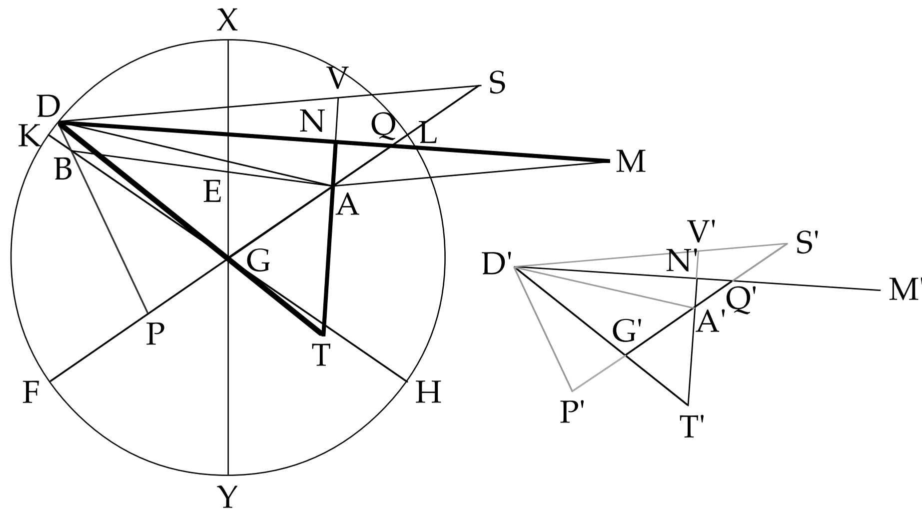


figure 20q

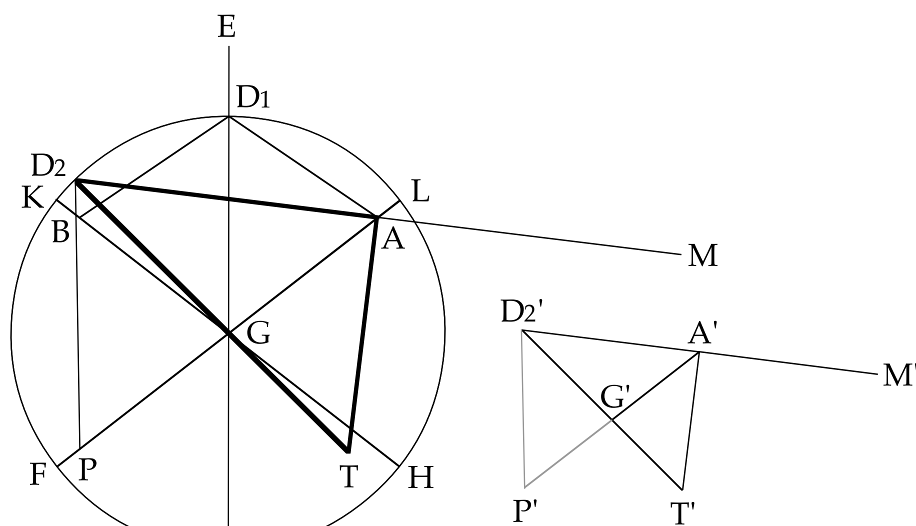


figure 20r

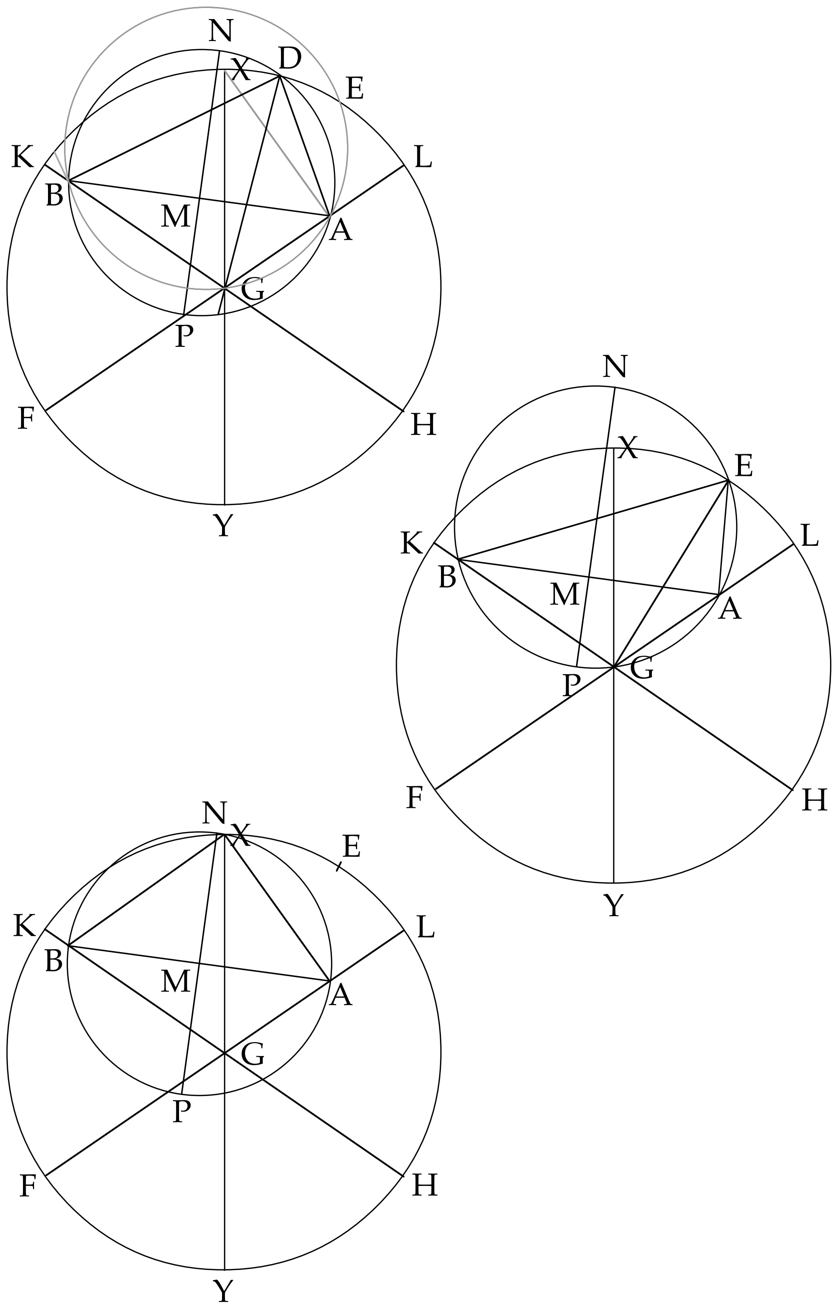


figure 20s

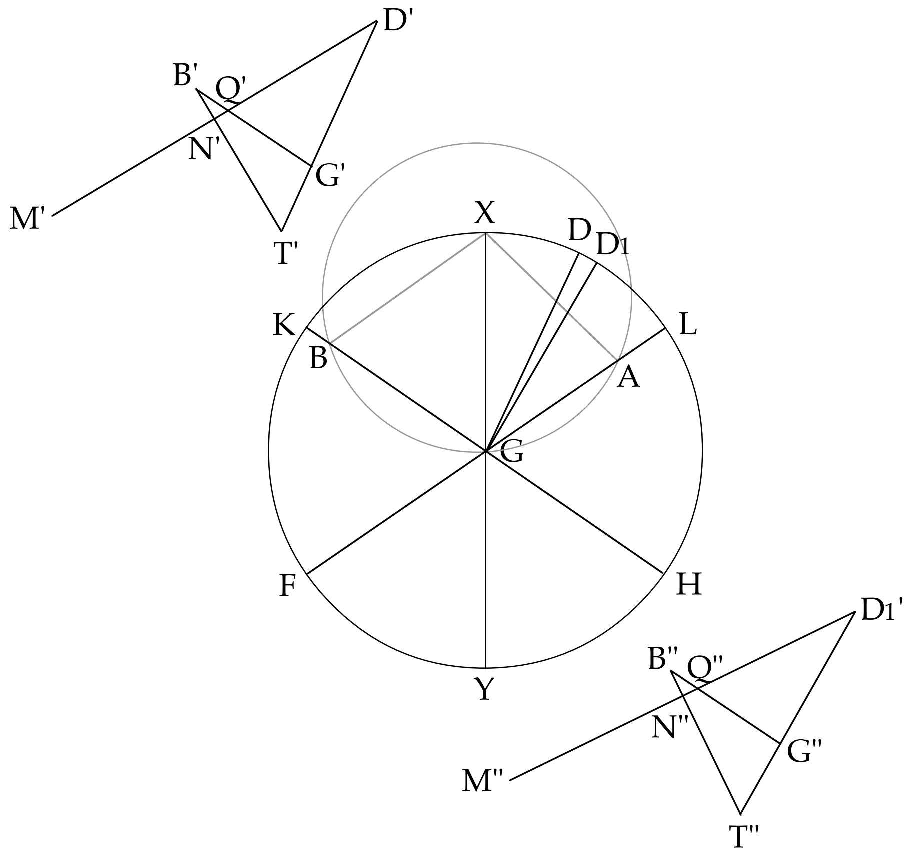


figure 20t

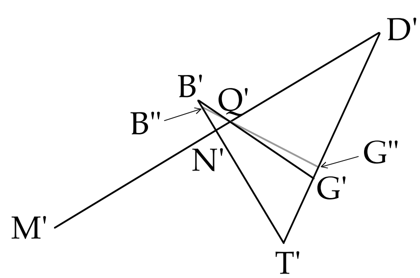


figure 20v

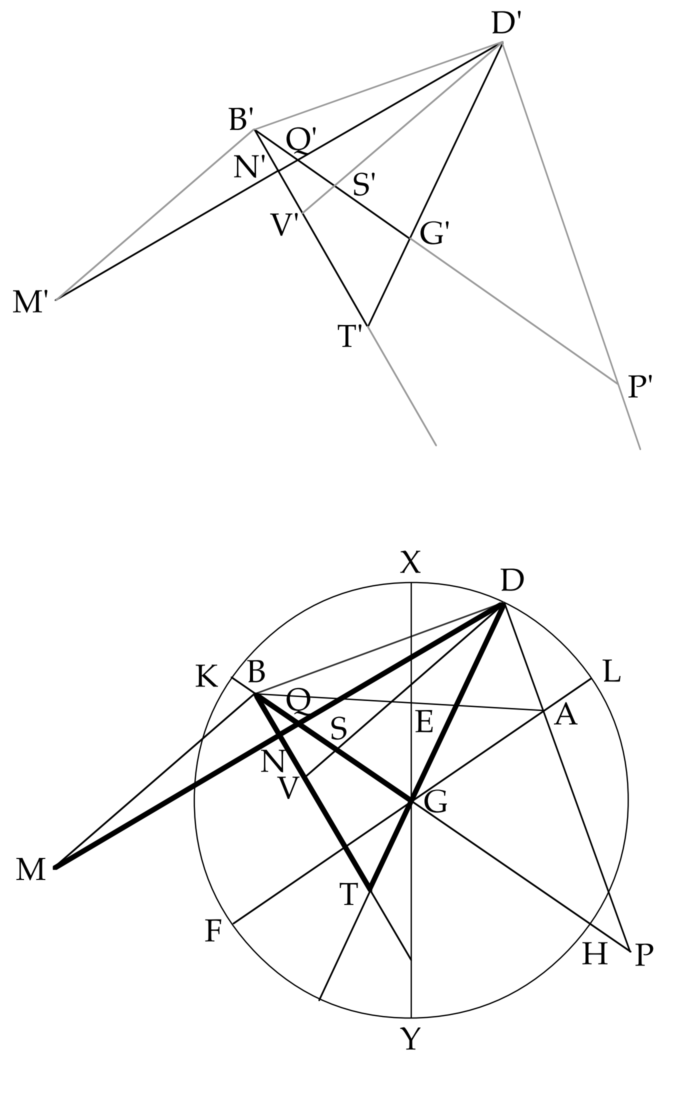


figure 20x

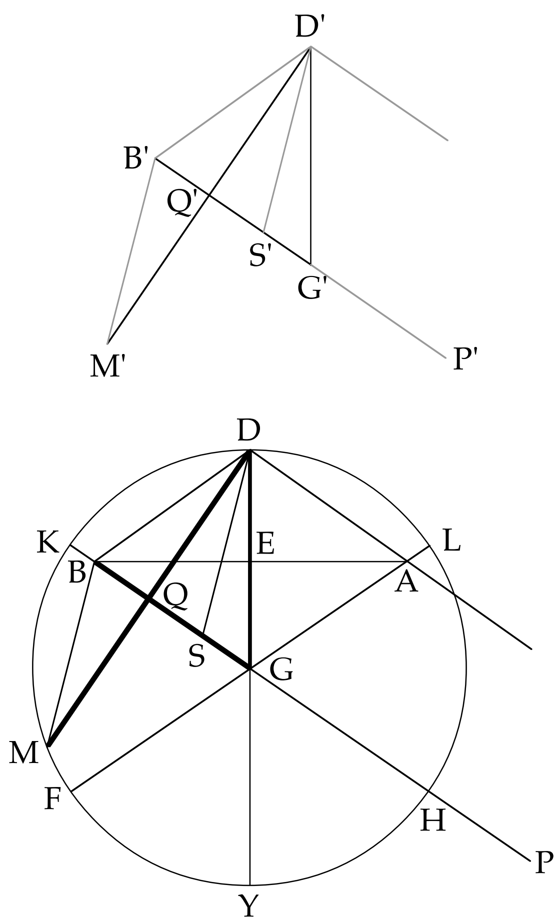


figure 20y

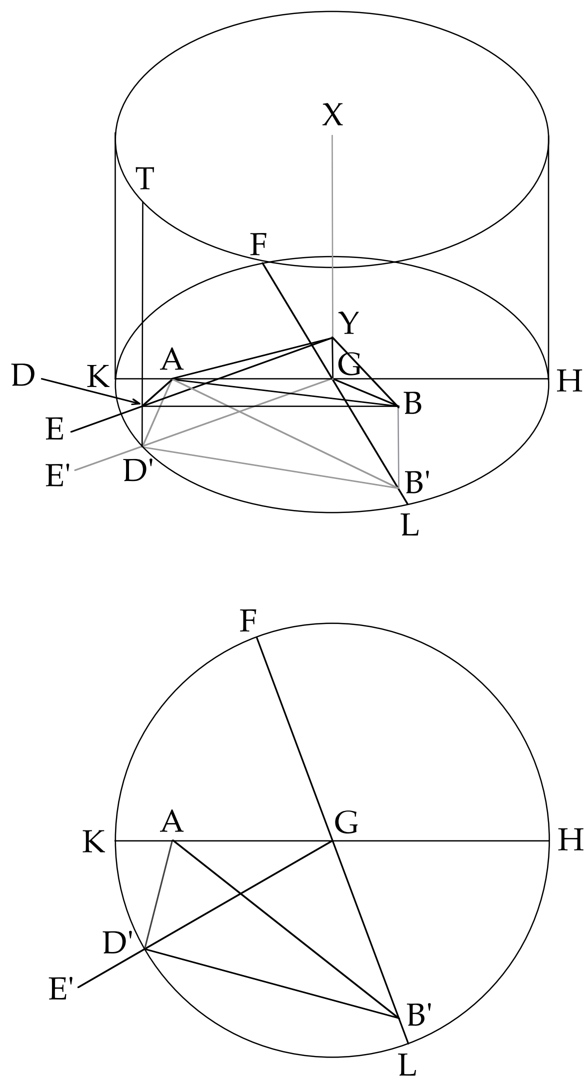


figure 21

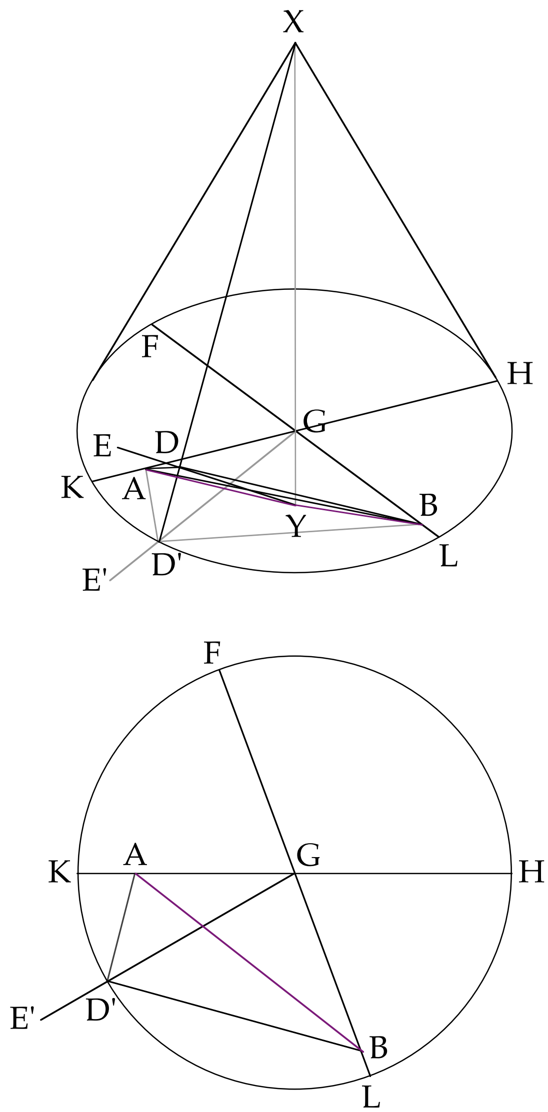


figure 22

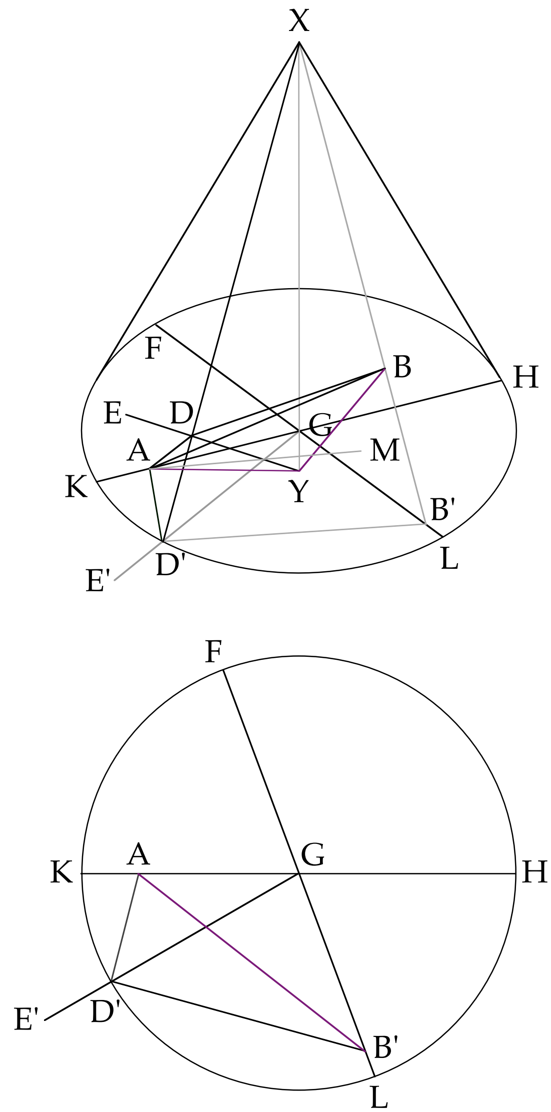


figure 22a

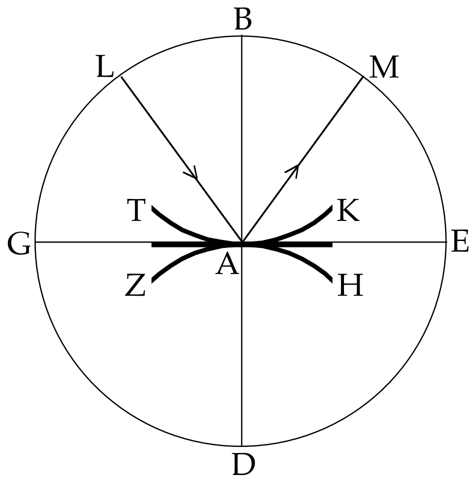


figure 23

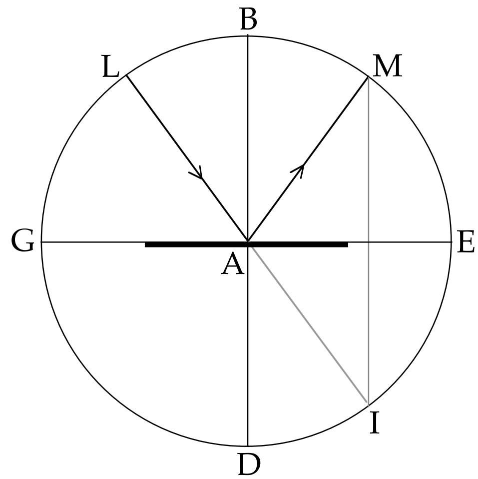


figure 23a

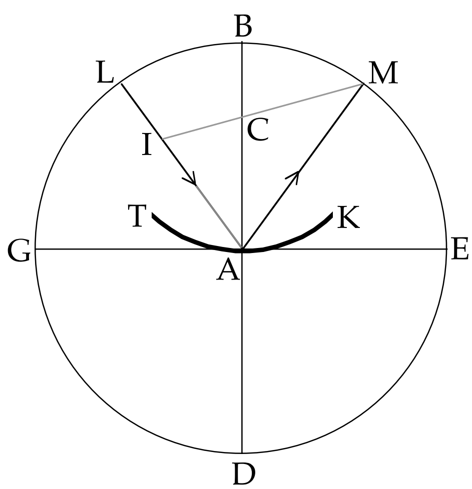


figure 23b

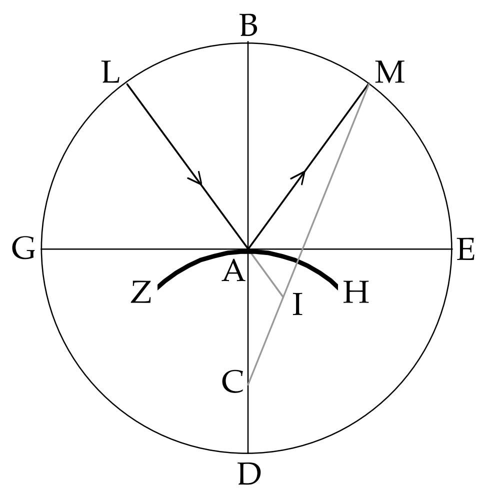


figure 23c

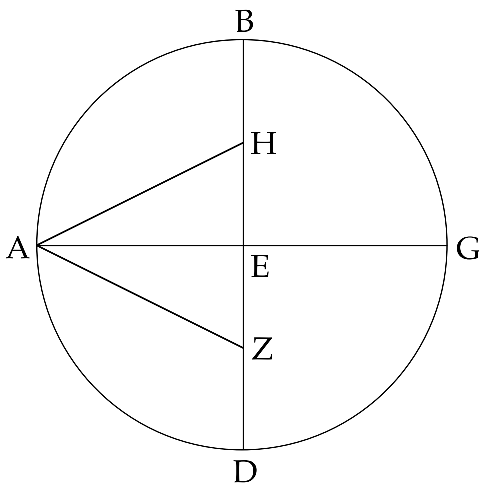


figure 24

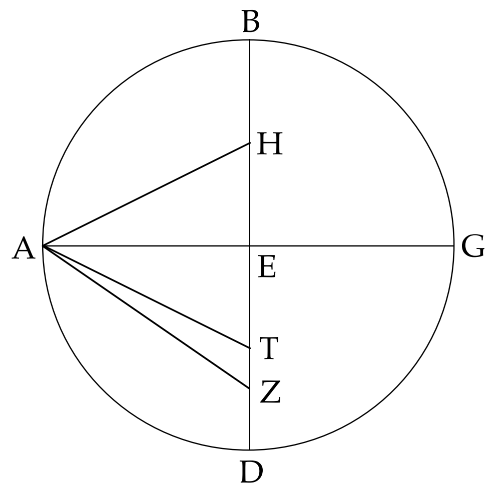


figure 24a

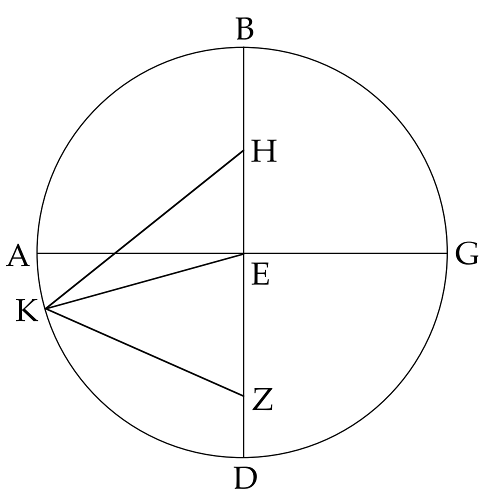


figure 24b

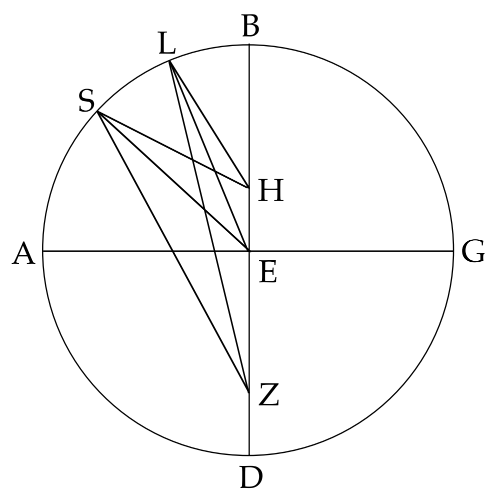


figure 24c

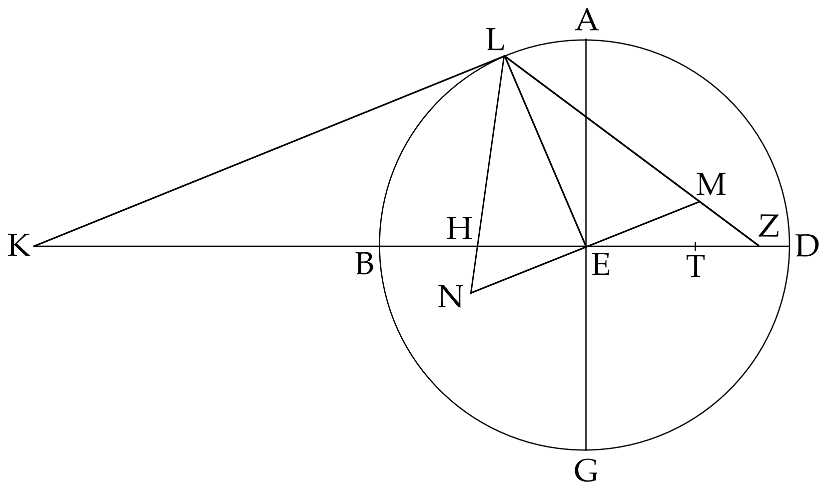


figure 25

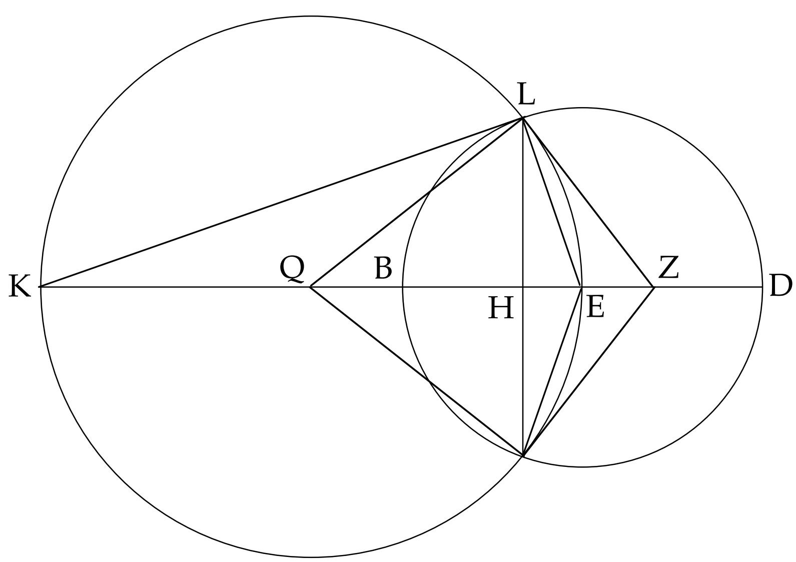


figure 25a

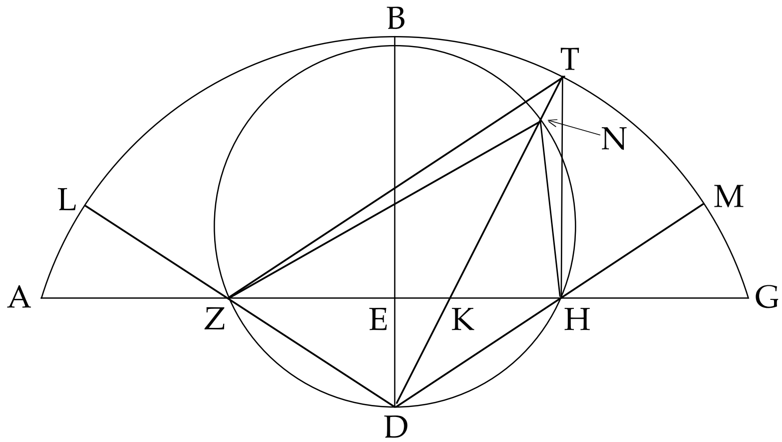


figure 26

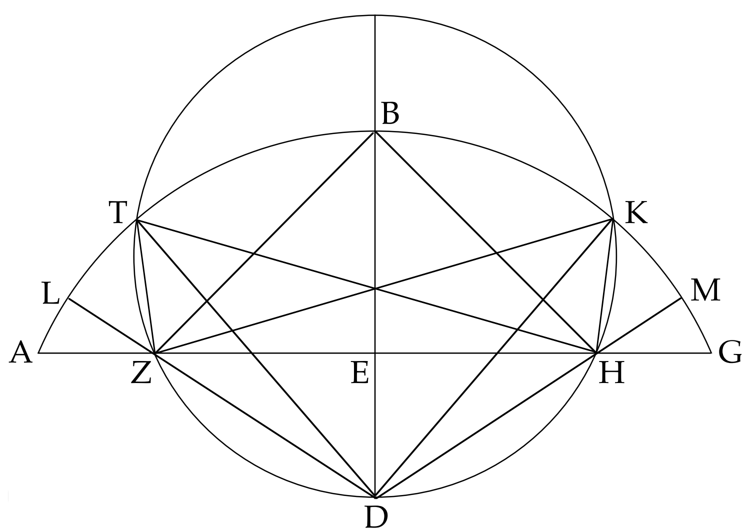


figure 26a

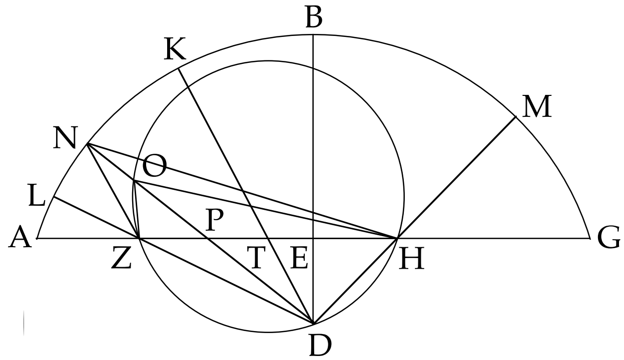


figure 27

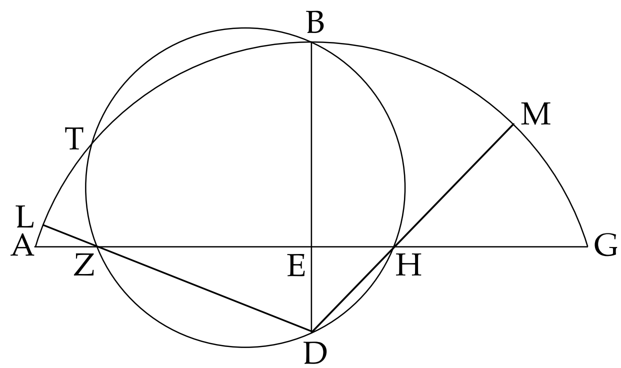


figure 27a

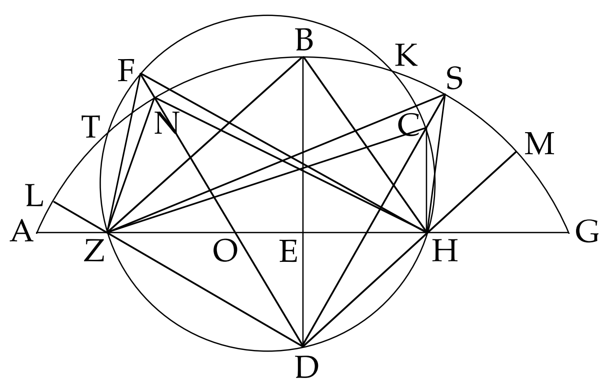


figure 27b

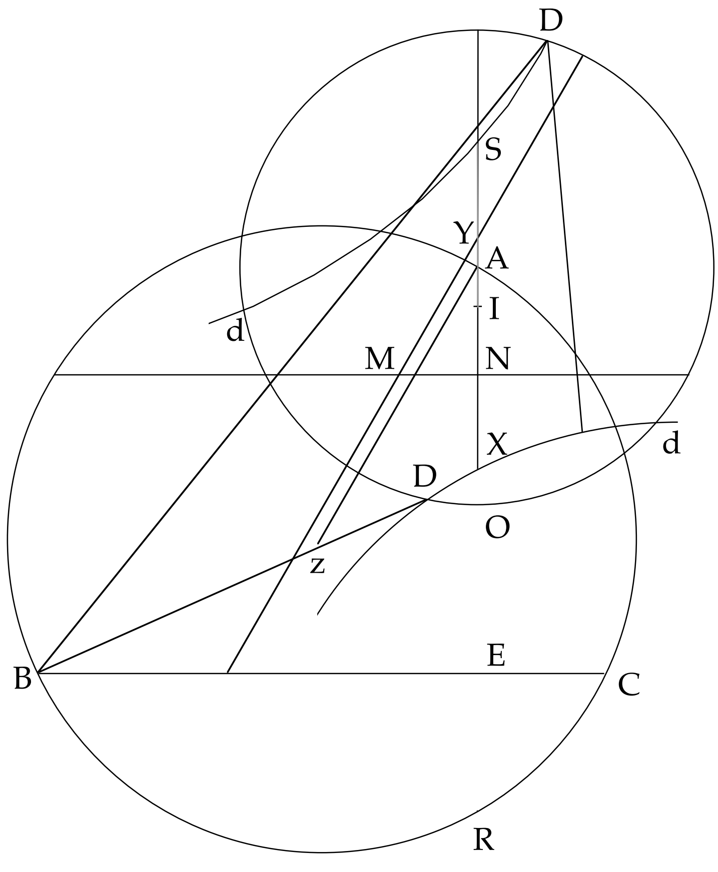


figure 28

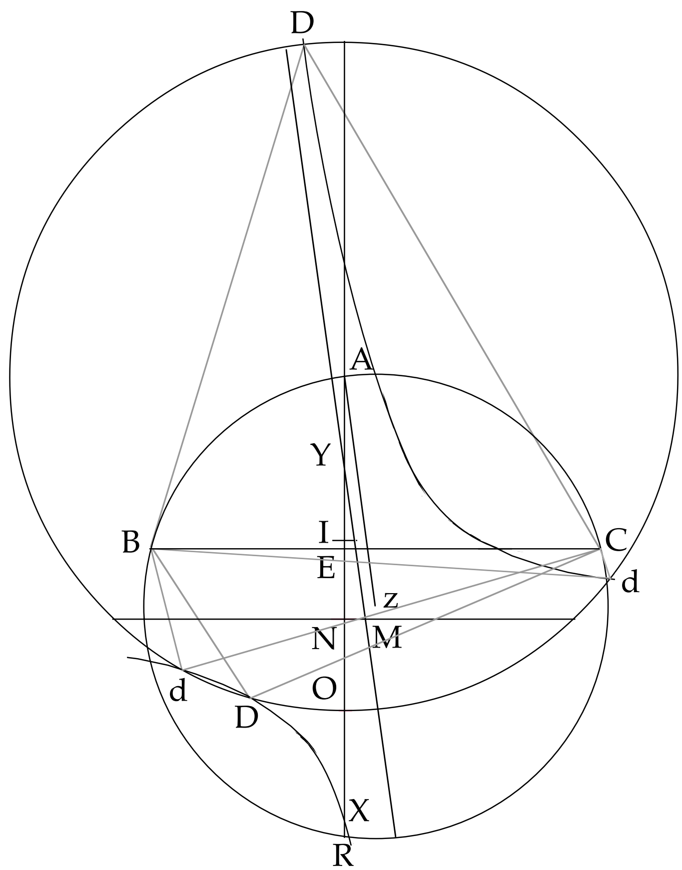


figure 28a

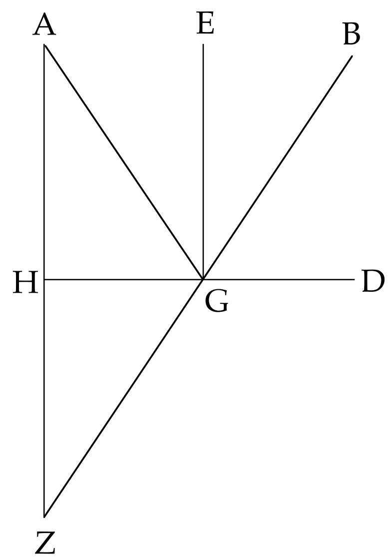


FIGURE 5.2.1

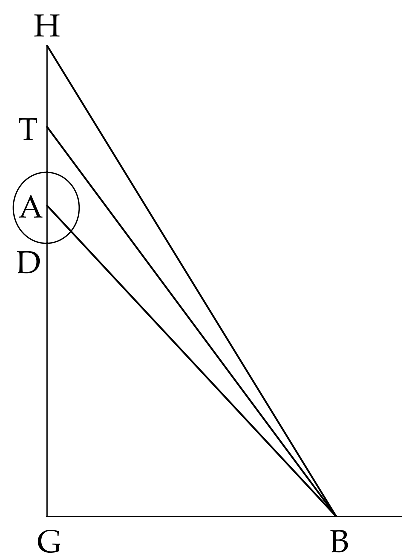


FIGURE 5.2.2

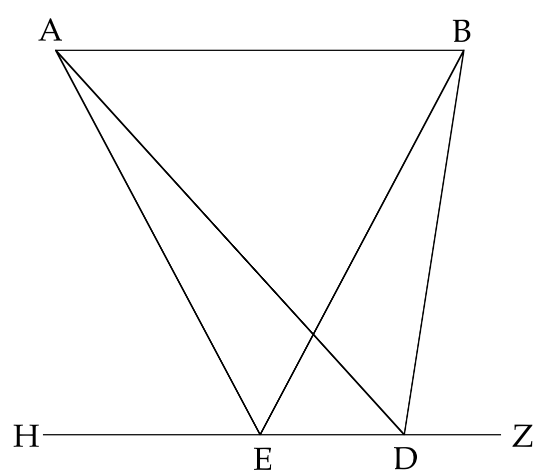


FIGURE 5.2.3

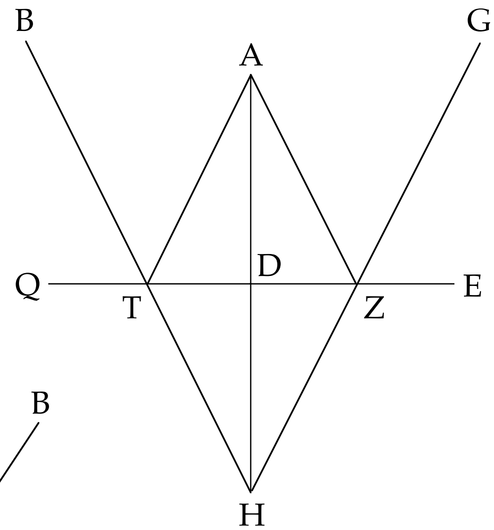


FIGURE 5.2.4

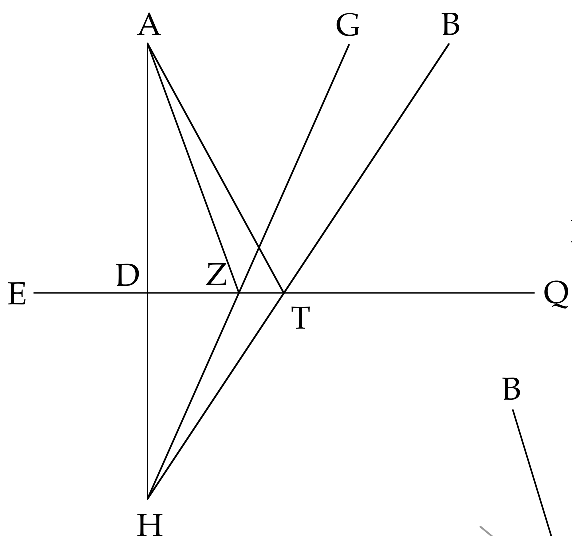


FIGURE 5.2.4a

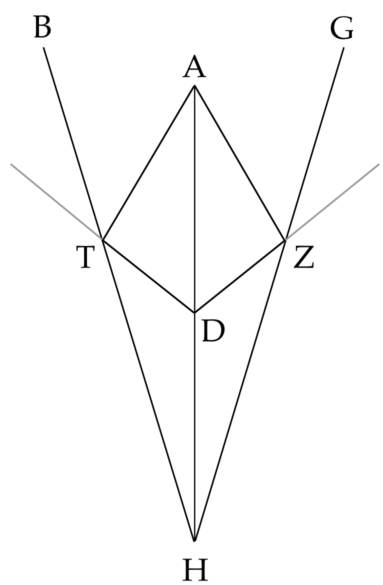


FIGURE 5.2.4b

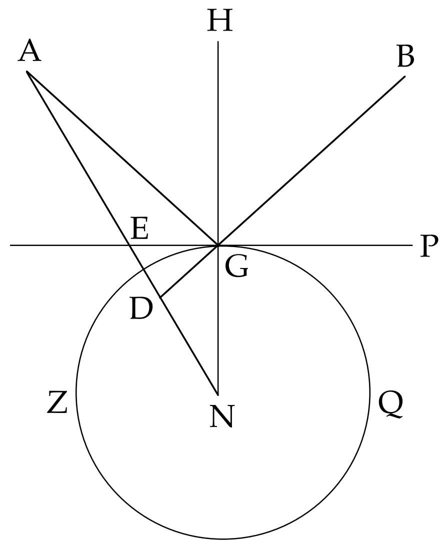


FIGURE 5.2.5

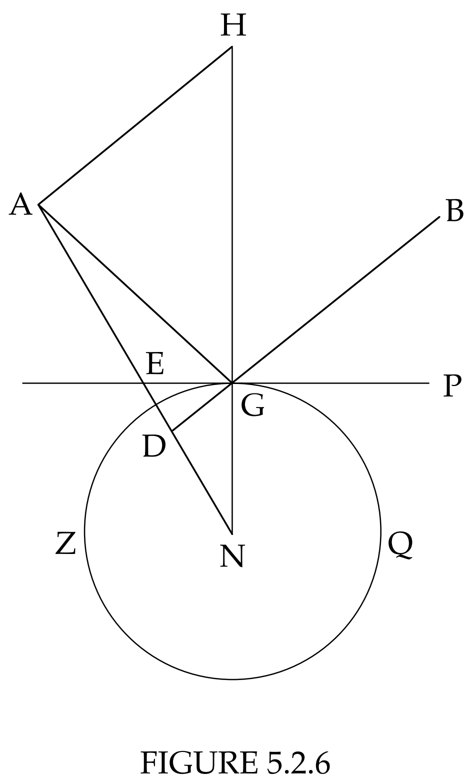


FIGURE 5.2.6

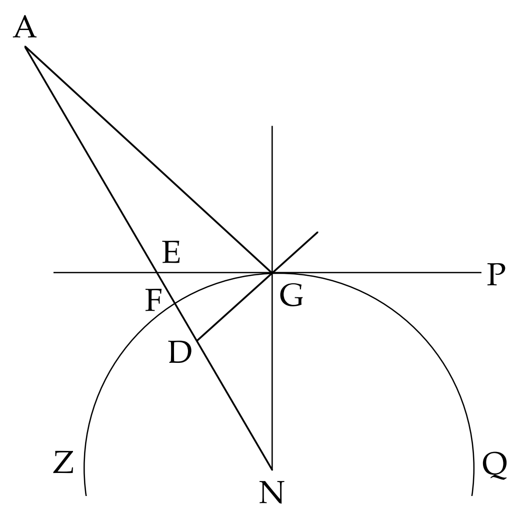


FIGURE 5.2.7

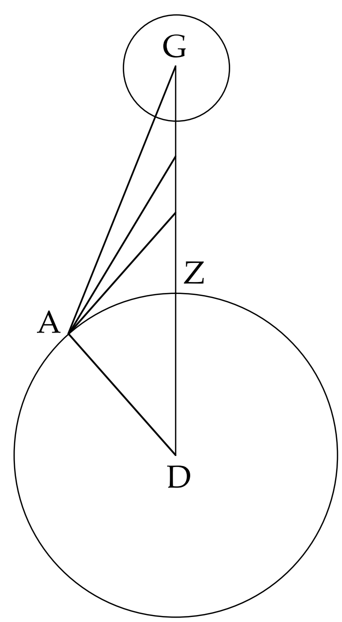


FIGURE 5.2.8

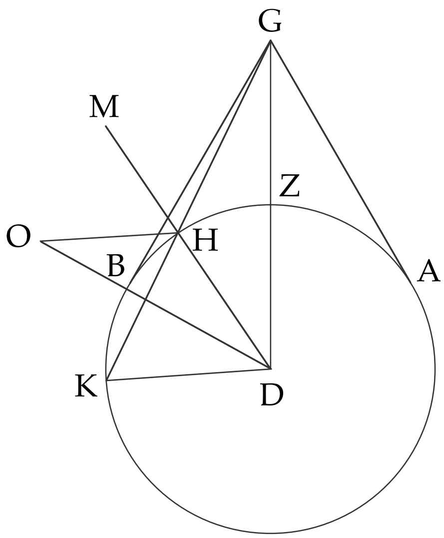


FIGURE 5.2.9

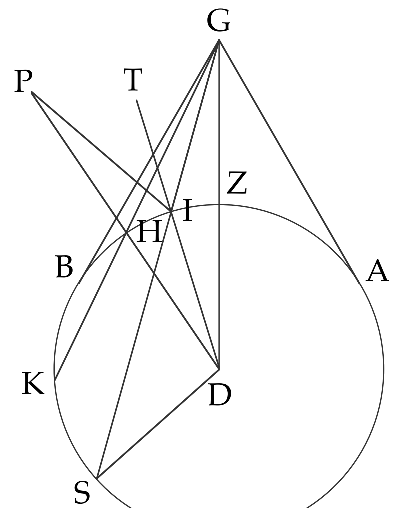


FIGURE 5.2.9a

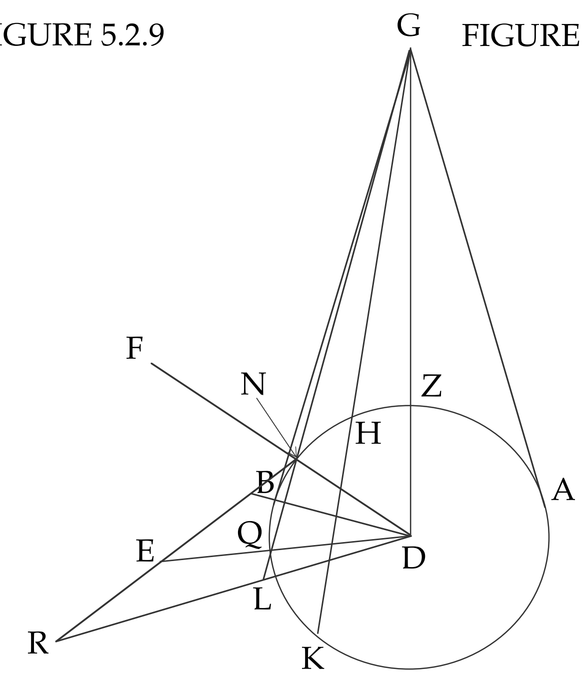


FIGURE 5.2.9b

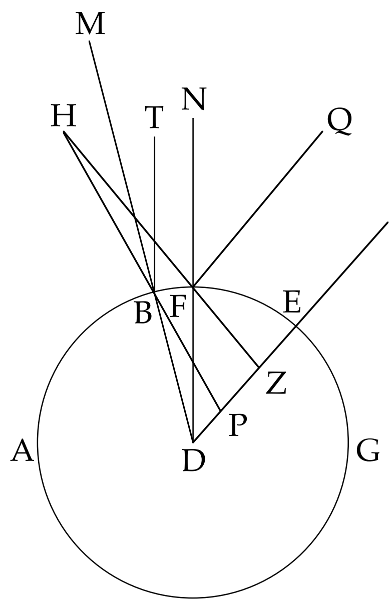


FIGURE 5.2.10

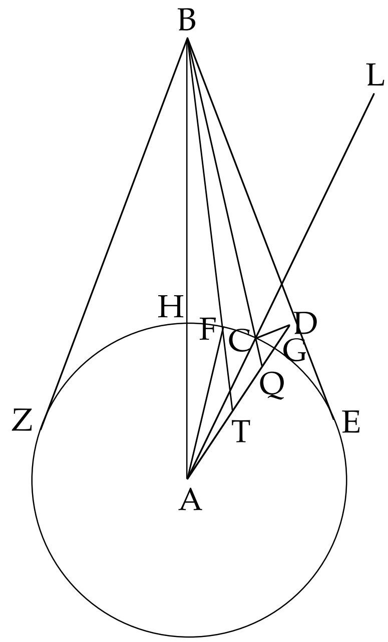


FIGURE 5.2.11

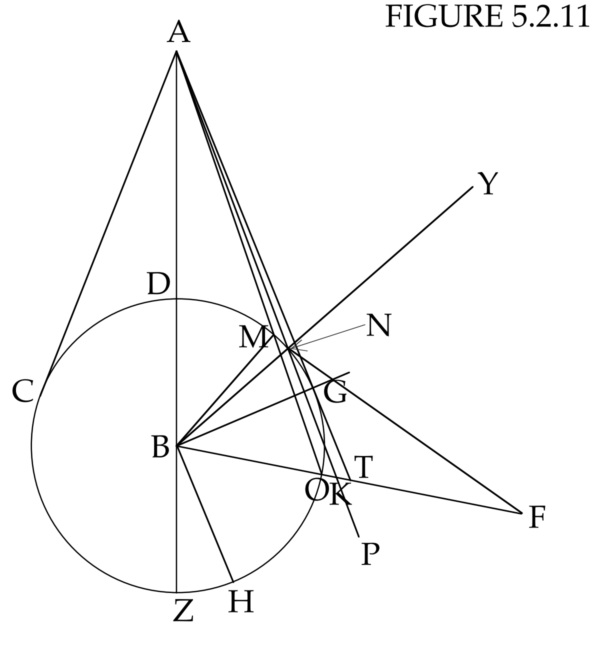


FIGURE 5.2.12

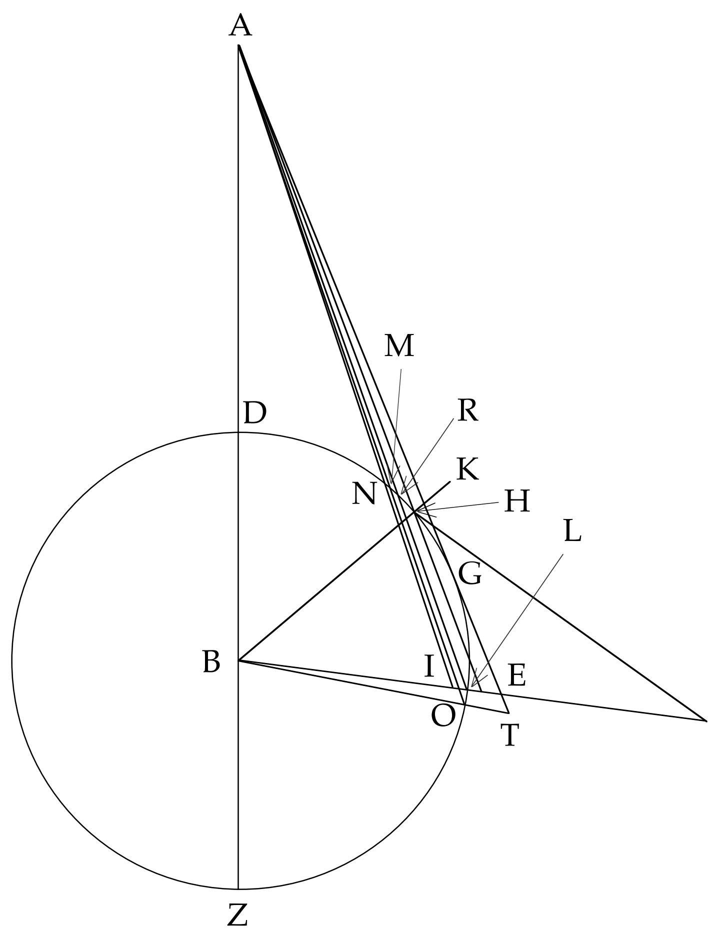


FIGURE 5.2.13

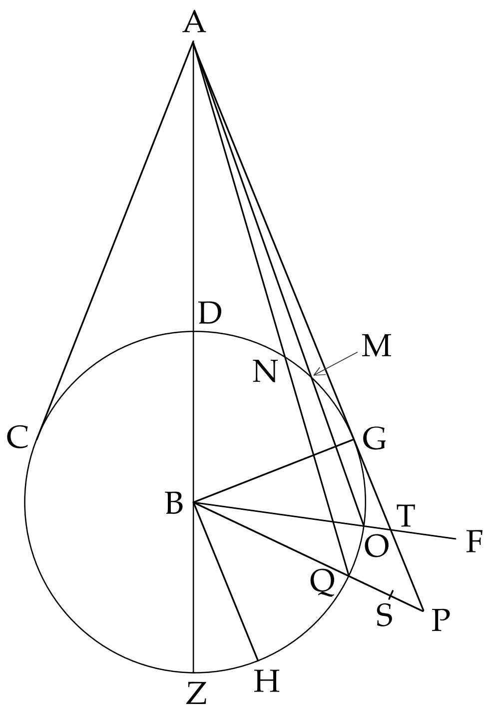


FIGURE 5.2.14

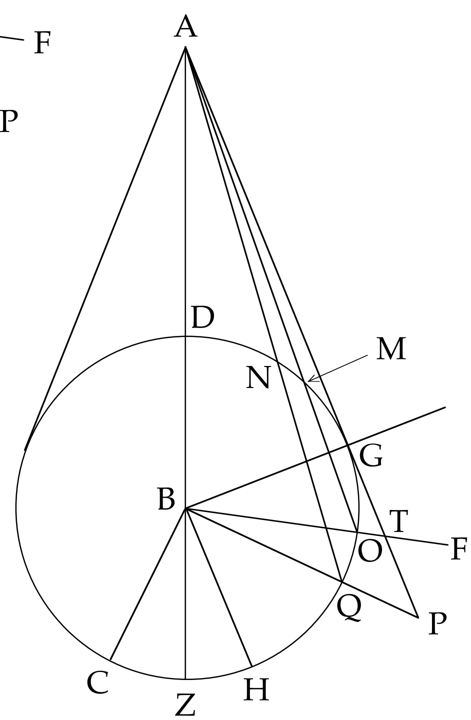


FIGURE 5.2.15

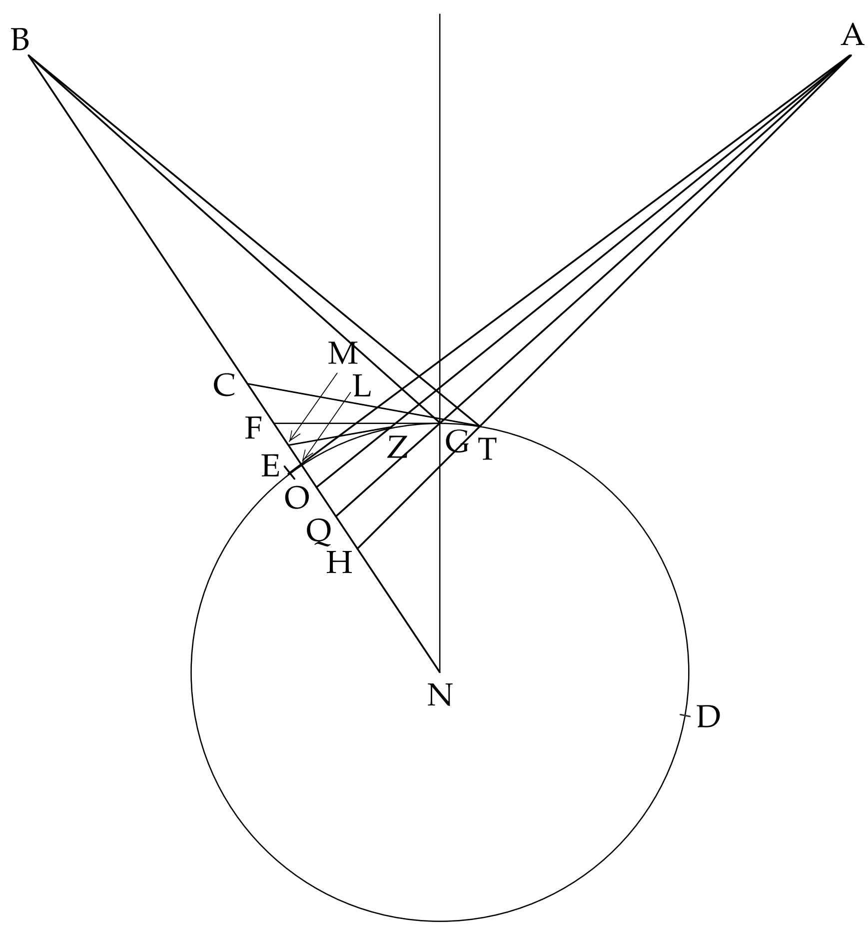


FIGURE 5.2.16

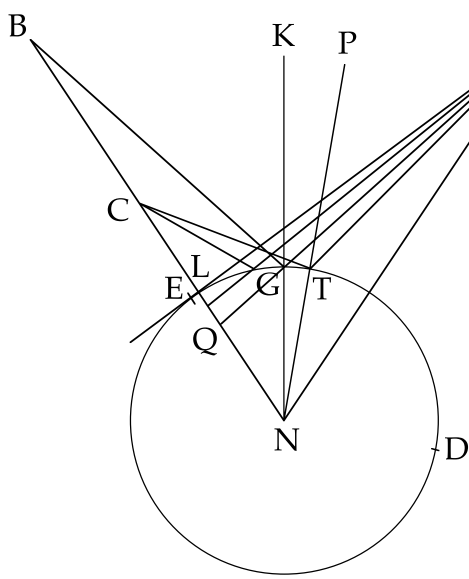


FIGURE 5.2.17

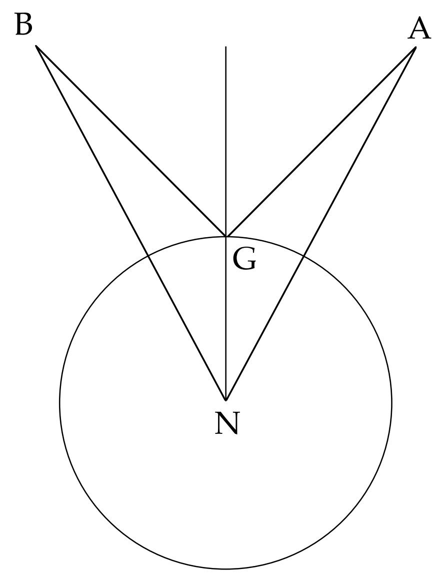


FIGURE 5.2.18

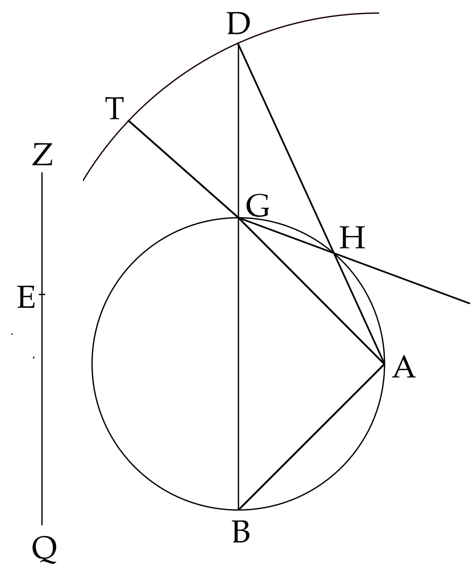


FIGURE 5.2.19a

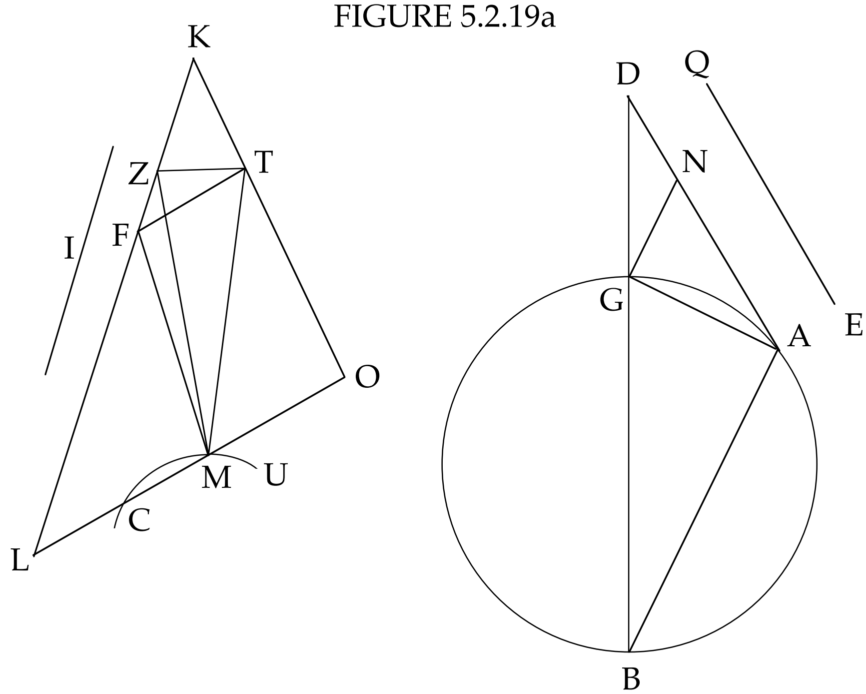


FIGURE 5.2.19c

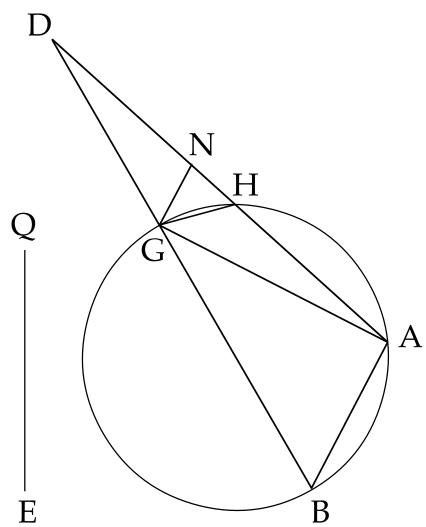


FIGURE 5.2.19d

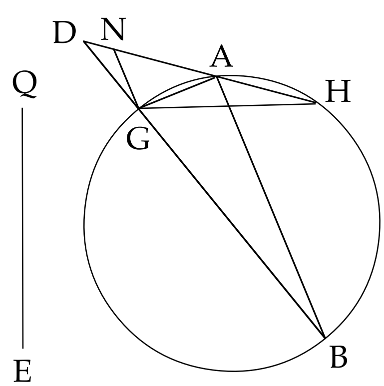


FIGURE 5.2.19e

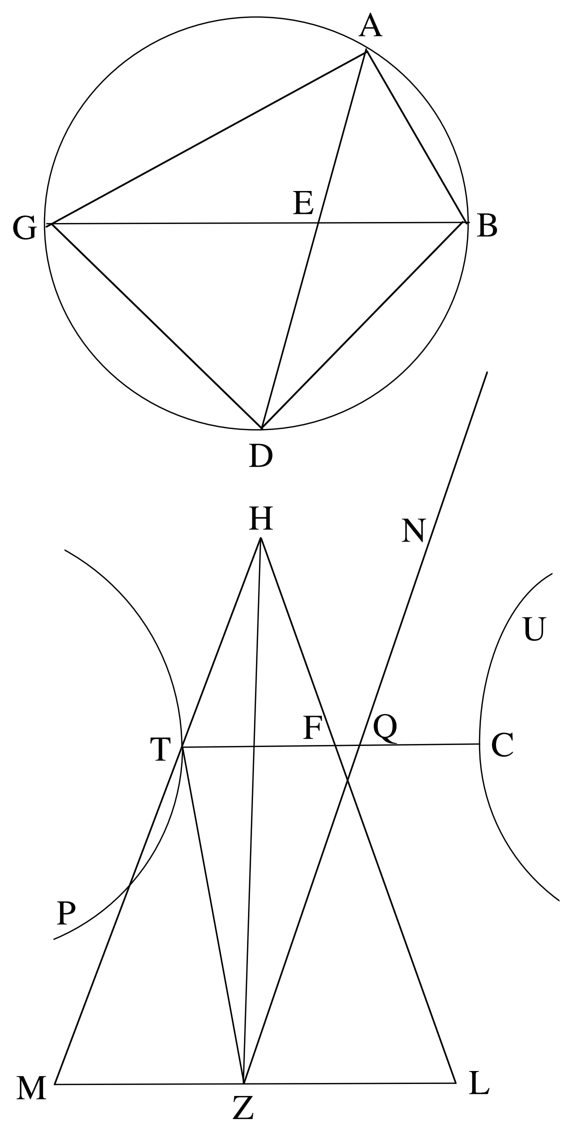


FIGURE 5.2.20

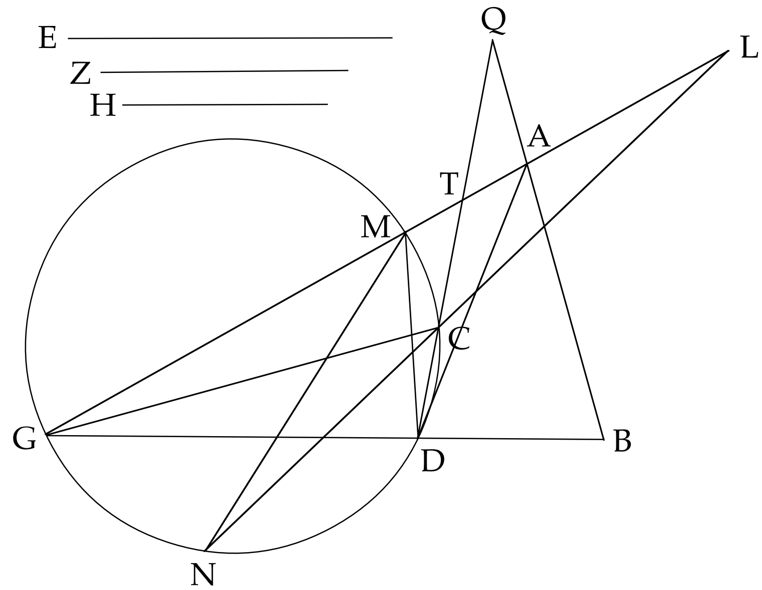


FIGURE 5.2.21

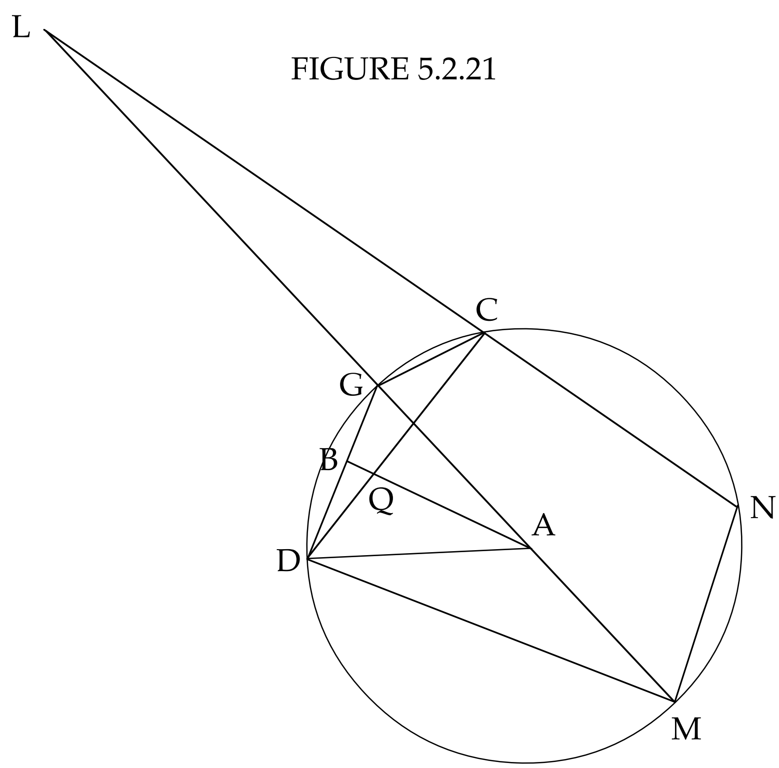


FIGURE 5.2.21a

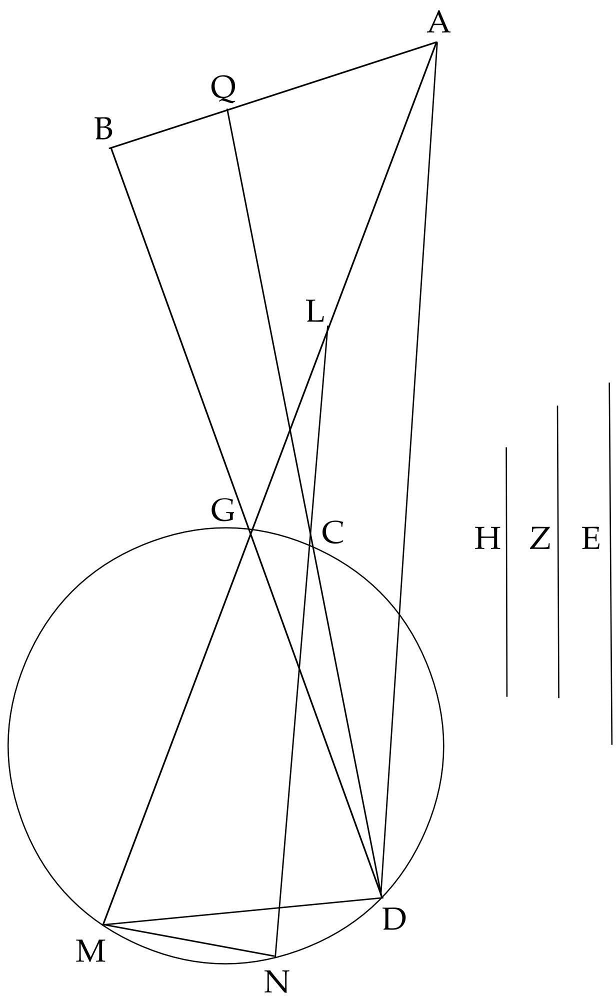


FIGURE 5.2.21b

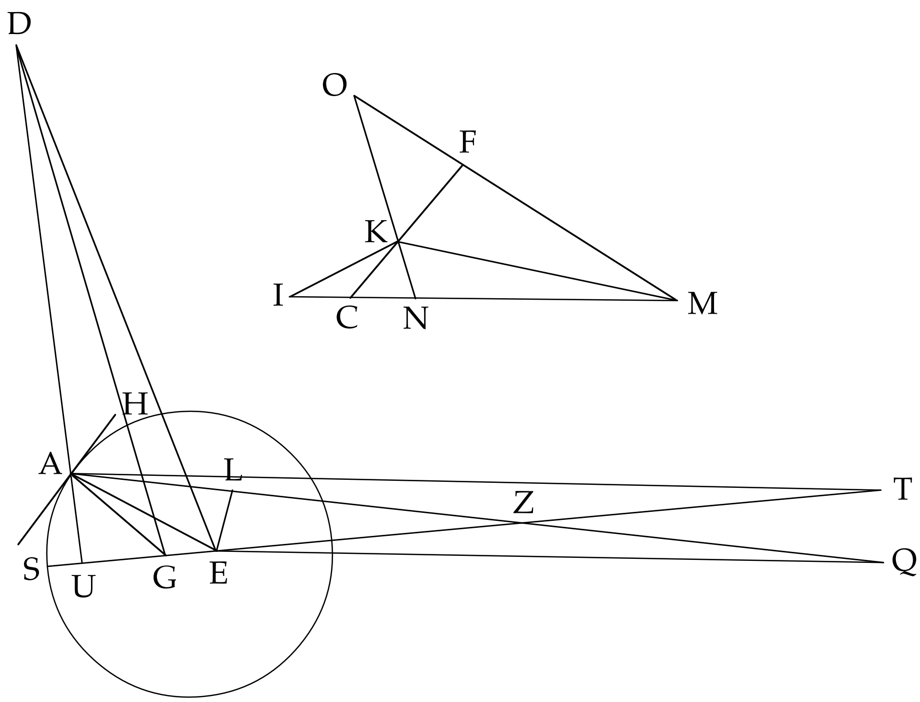


FIGURE 5.2.22

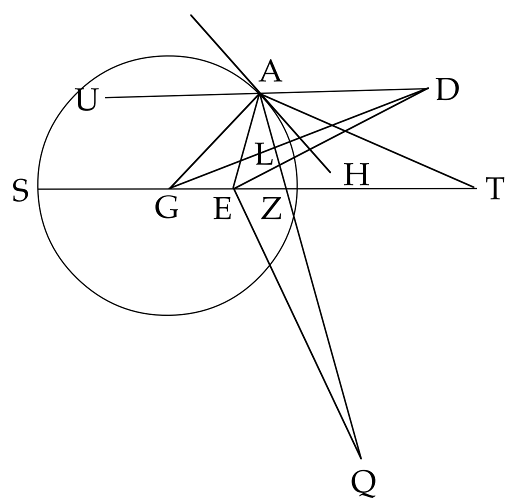


FIGURE 5.2.22a

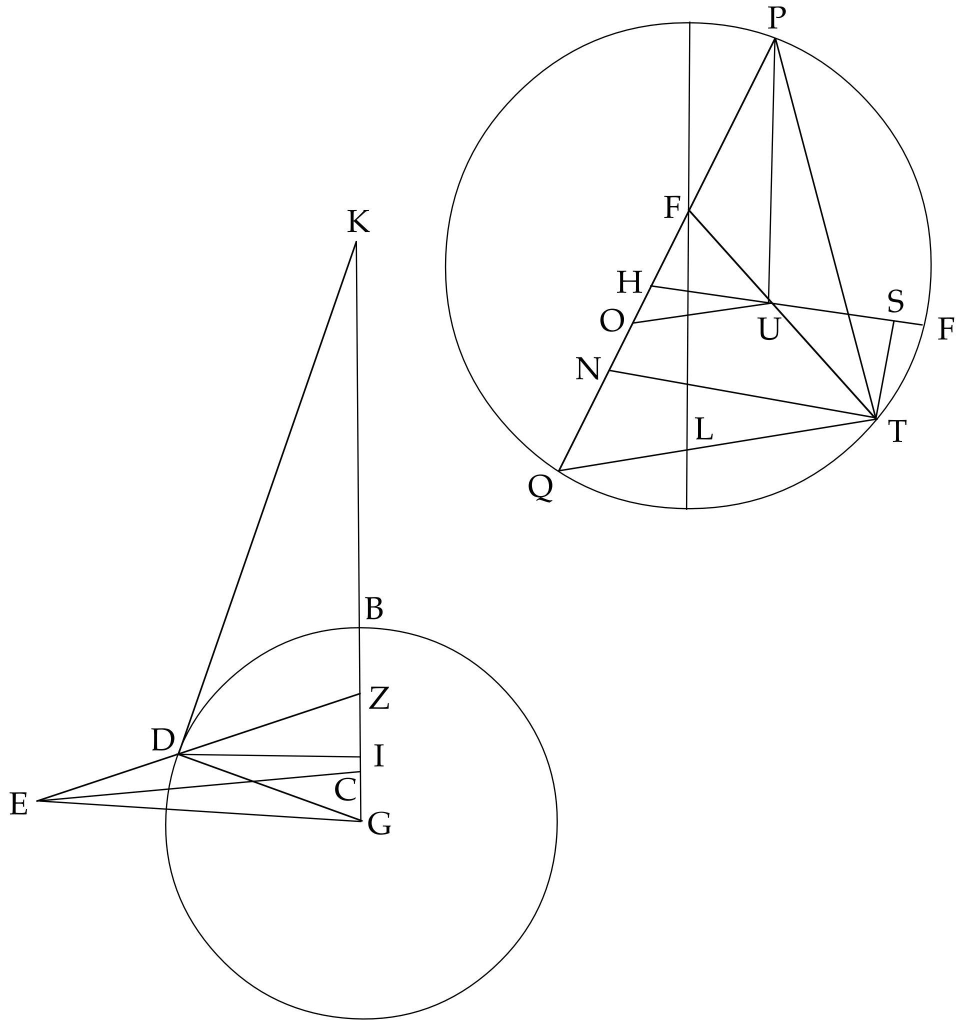


FIGURE 5.2.23

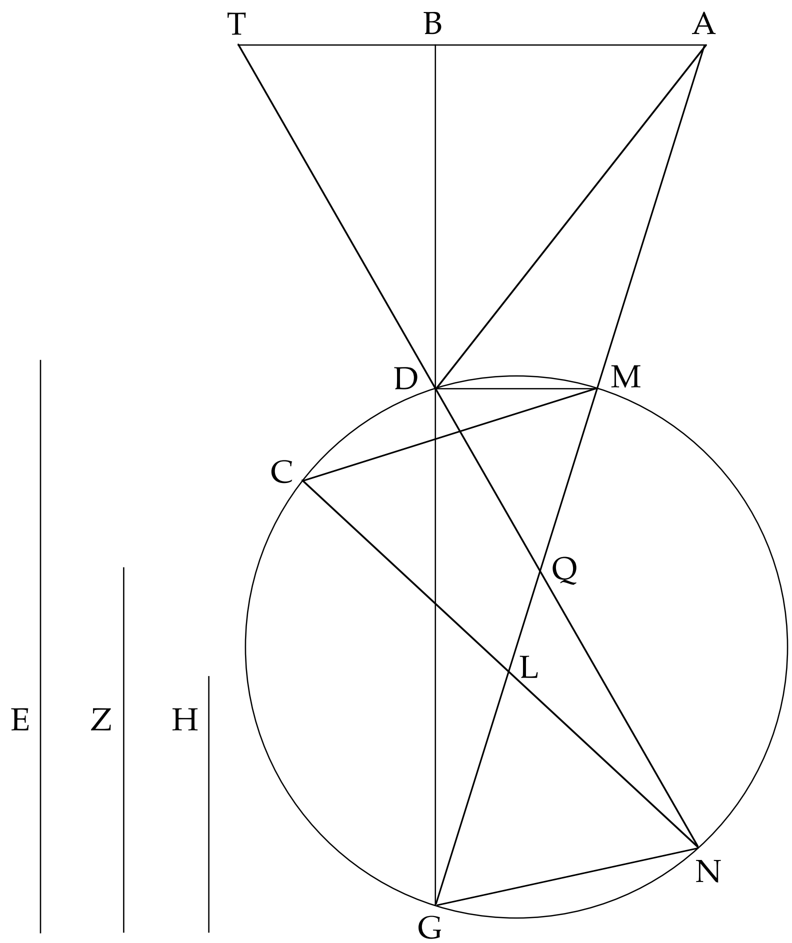


FIGURE 5.2.24

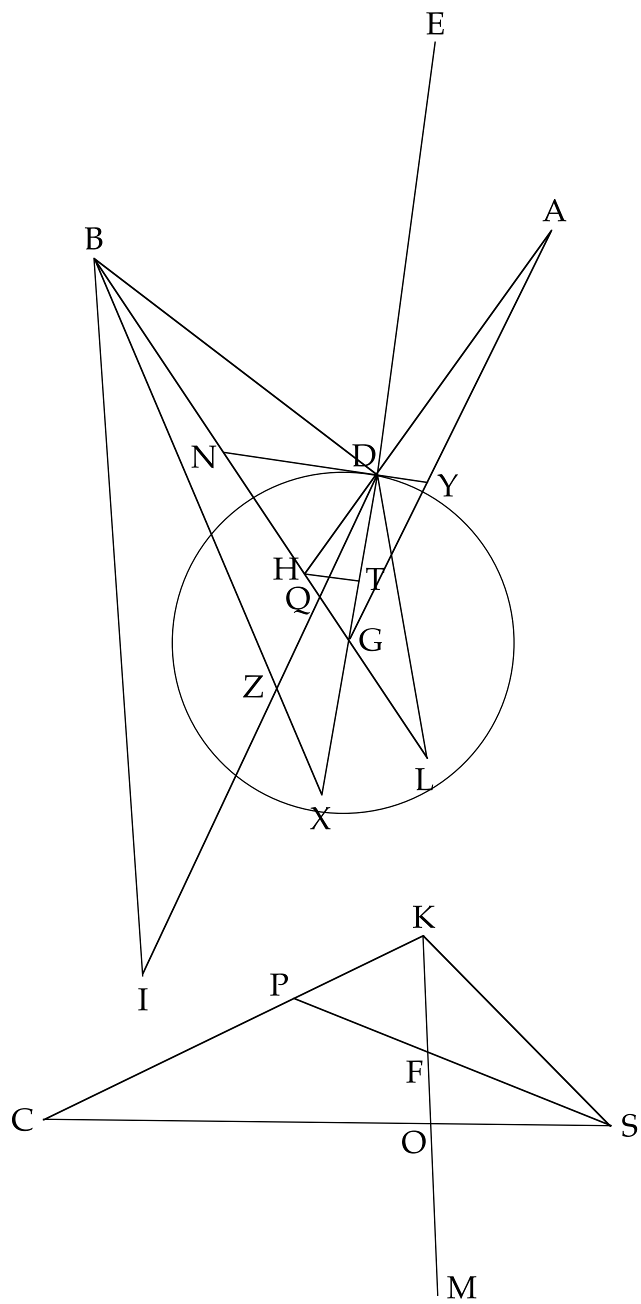


FIGURE 5.2.25

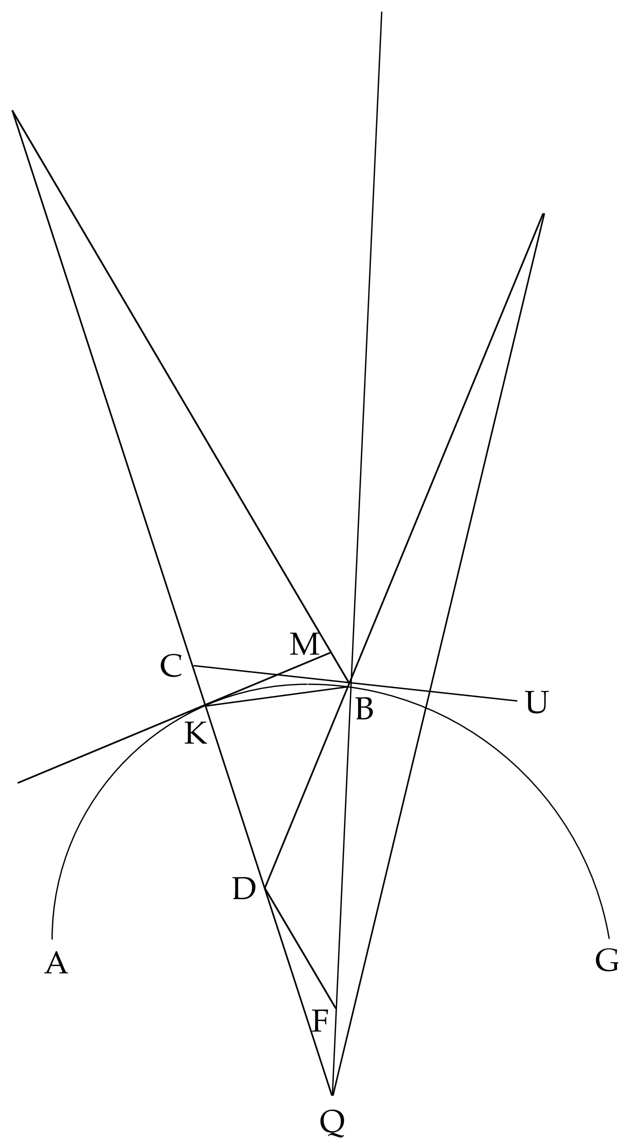


FIGURE 5.2.26

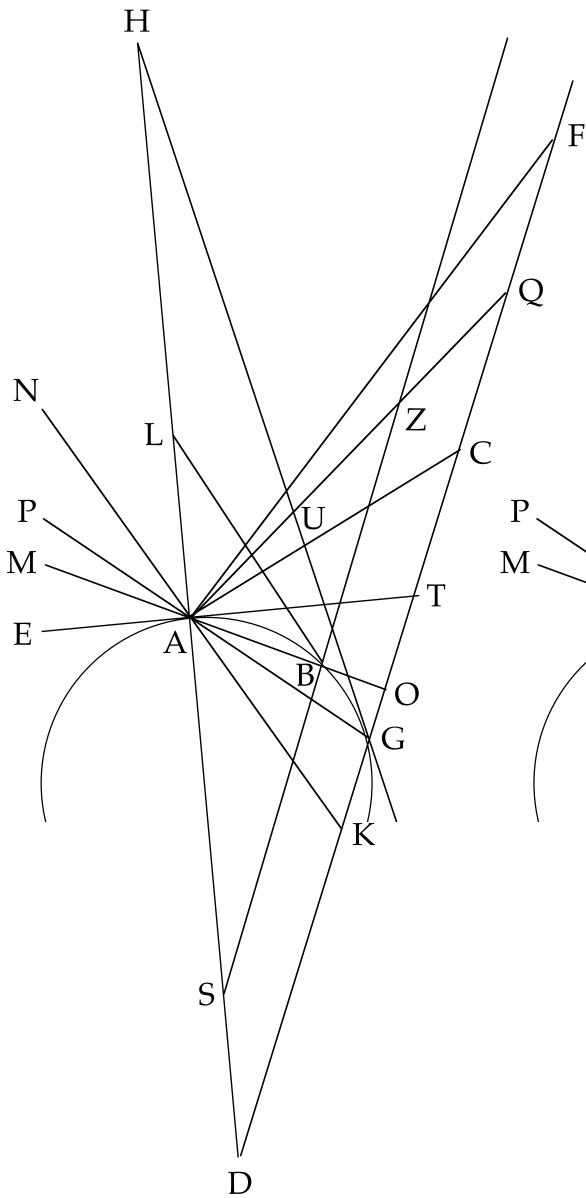


FIGURE 5.2.27

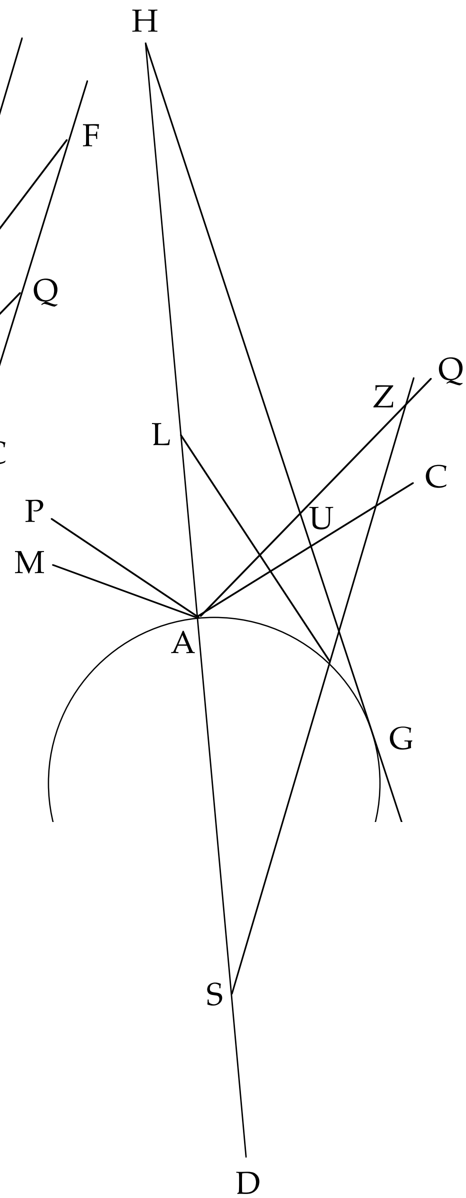


FIGURE 5.2.27a

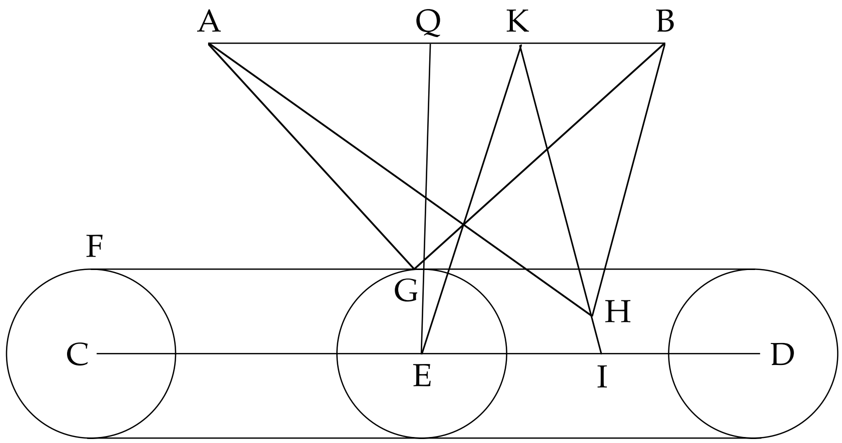


FIGURE 5.2.28

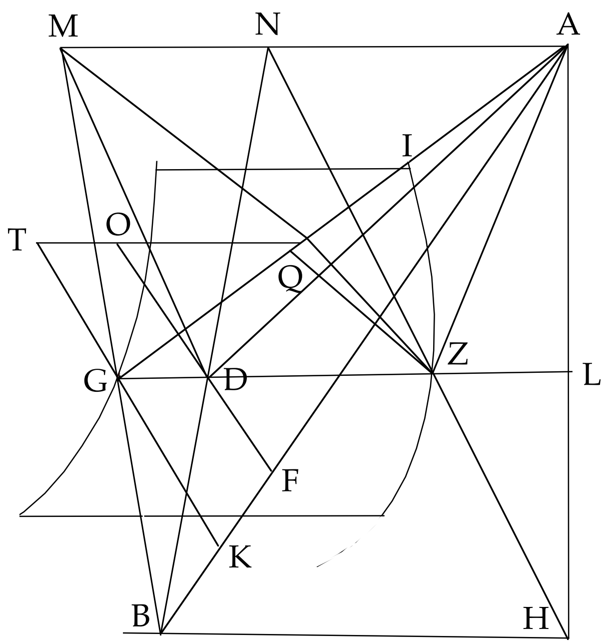


FIGURE 5.2.28a

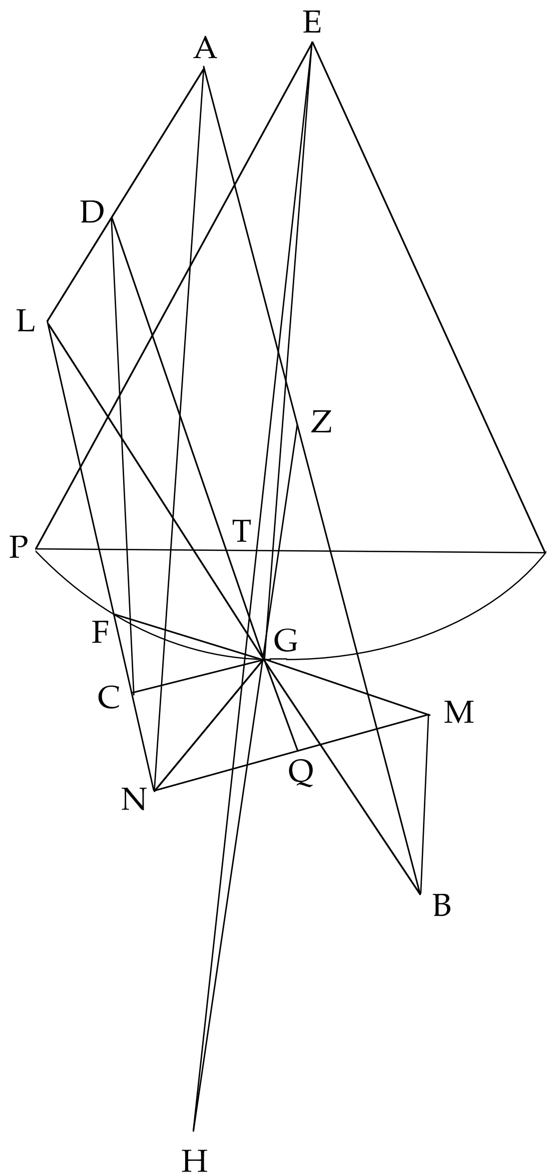


FIGURE 5.2.30

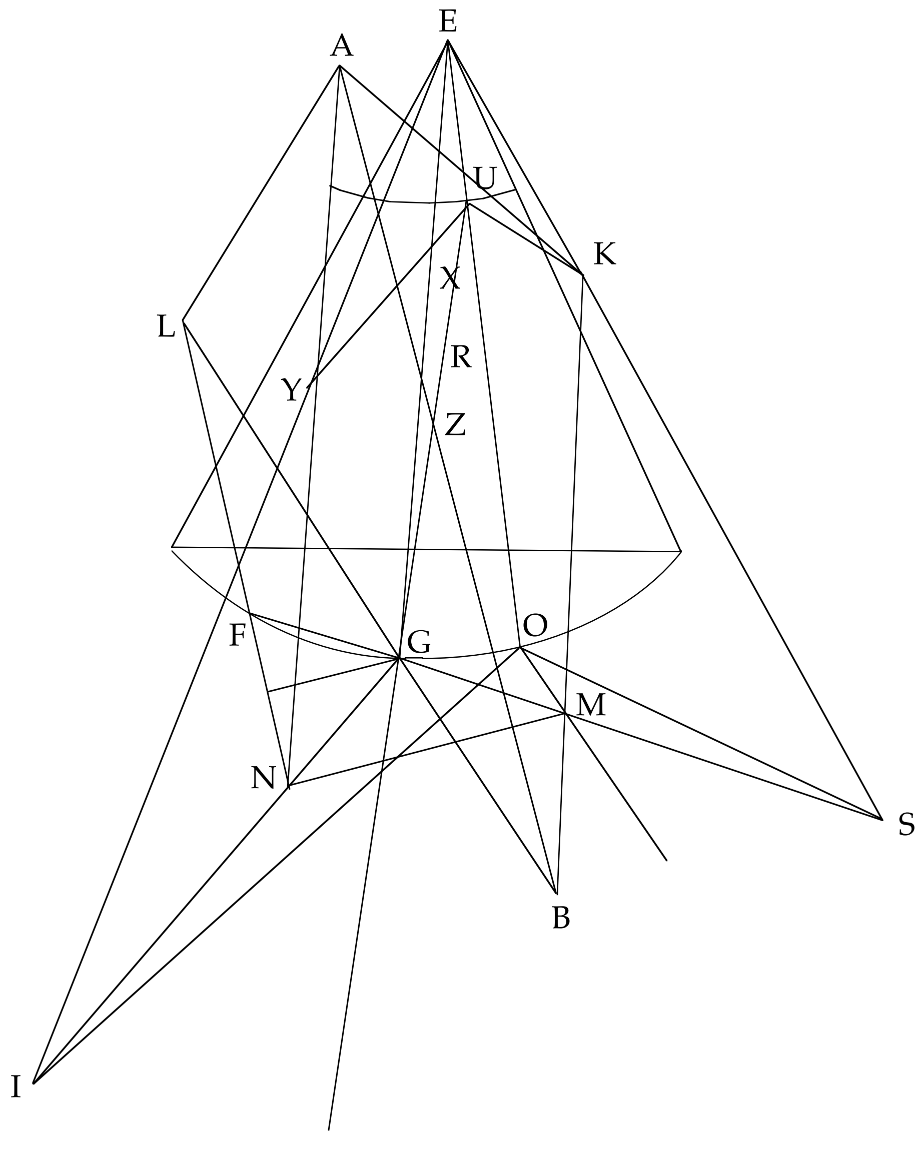


FIGURE 5.2.30a

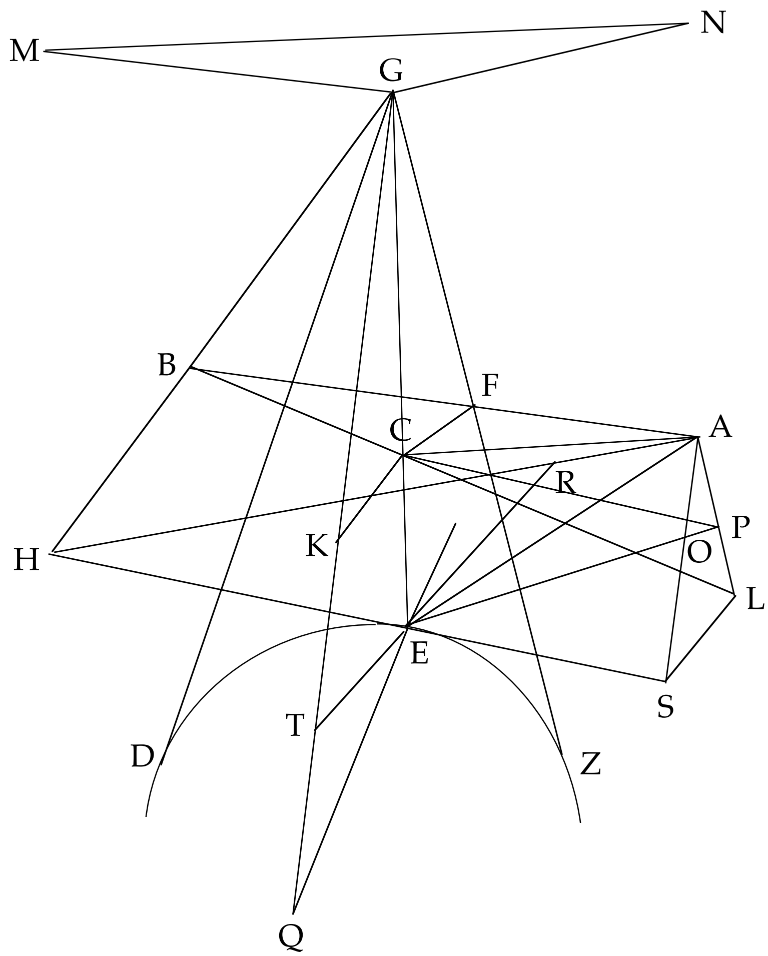


FIGURE 5.2.31

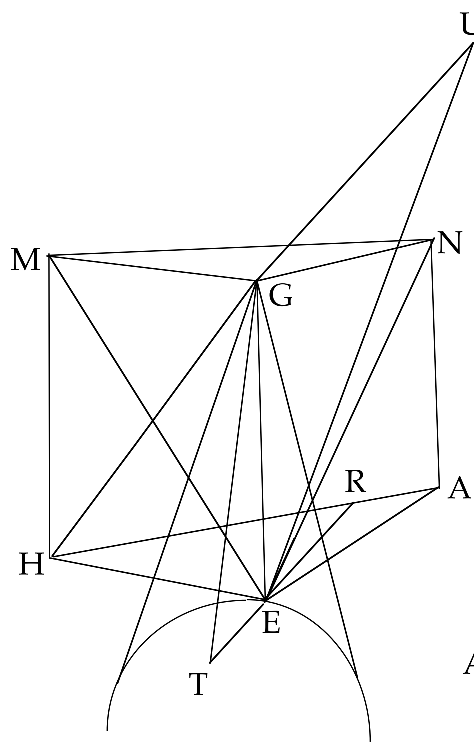


FIGURE 5.2.31a

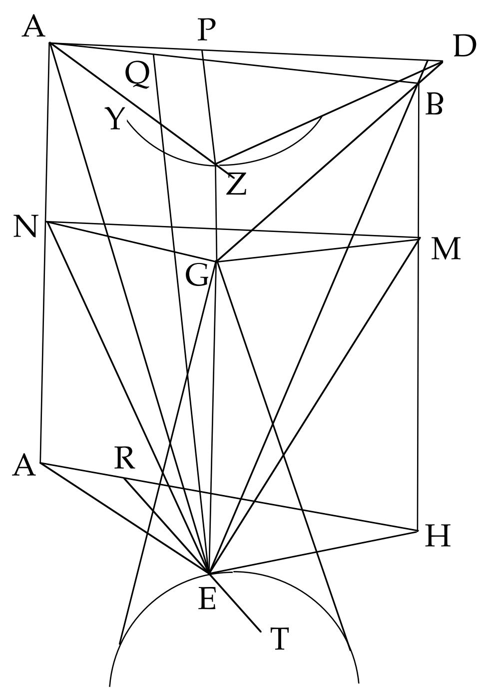


FIGURE 5.2.31b

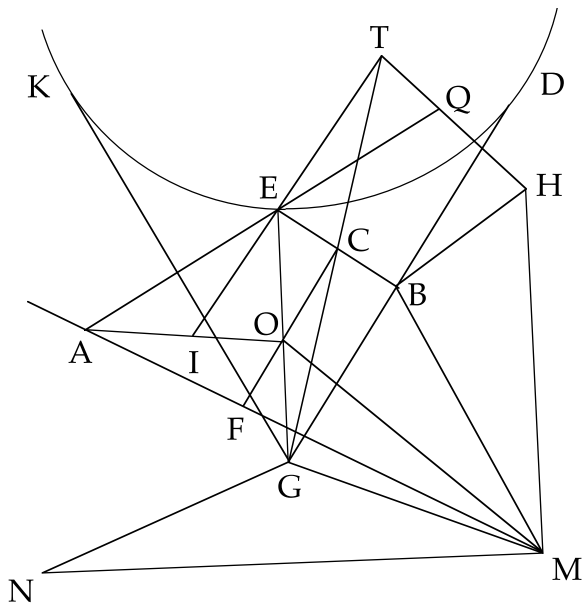


FIGURE 5.2.31c

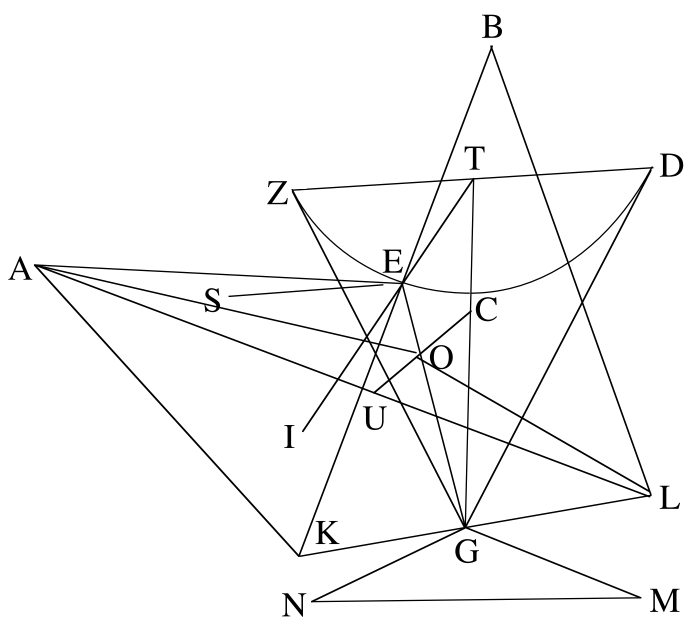


FIGURE 5.2.31e

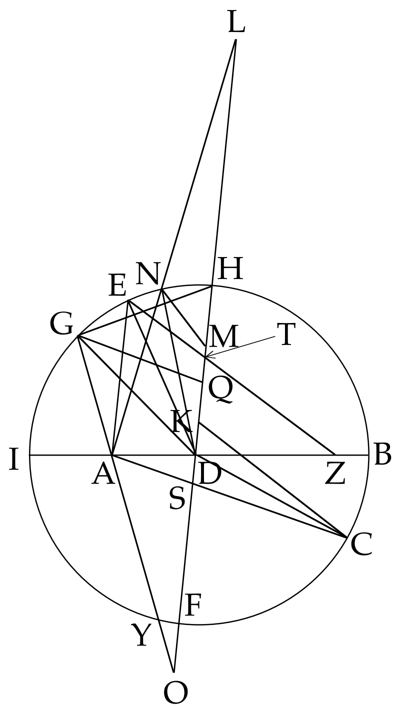


FIGURE 5.2.32

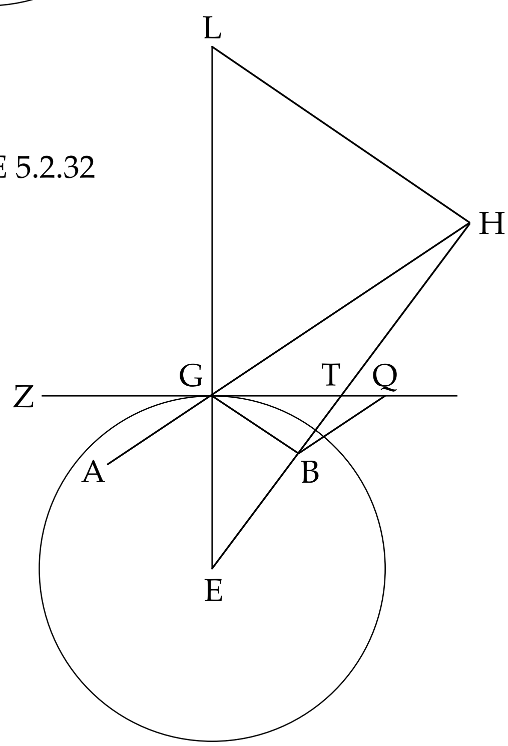


FIGURE 5.2.33

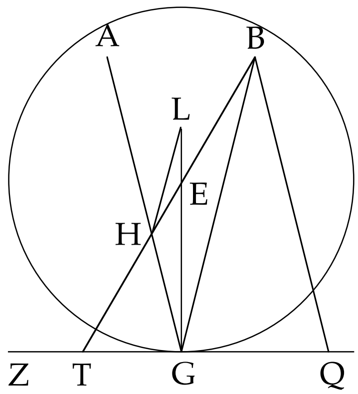


FIGURE 5.2.33a

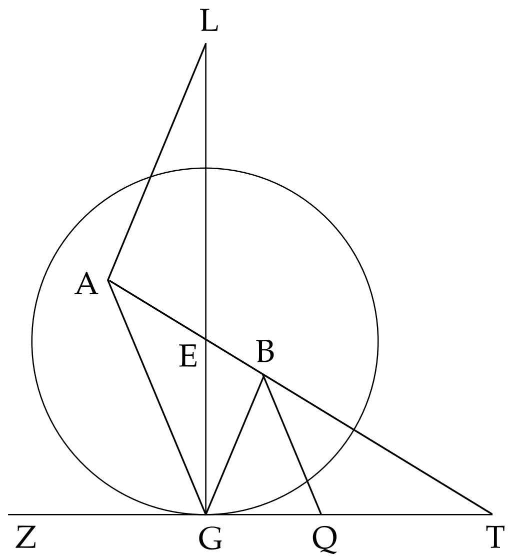


FIGURE 5.2.33b

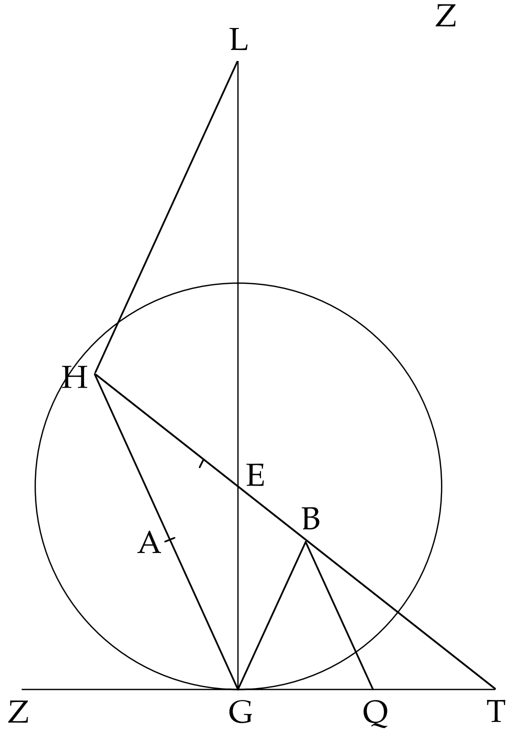


FIGURE 5.2.33c

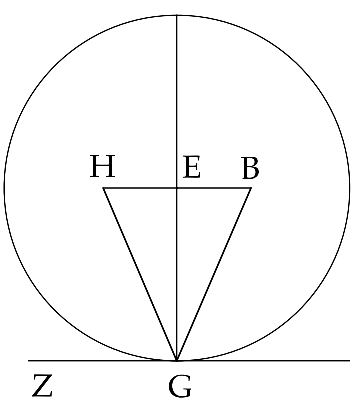


FIGURE 5.2.33d

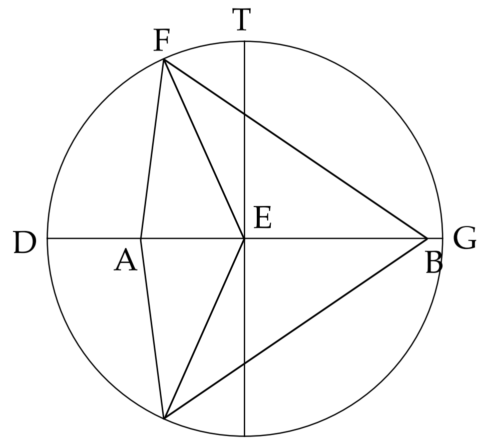


FIGURE 5.2.34

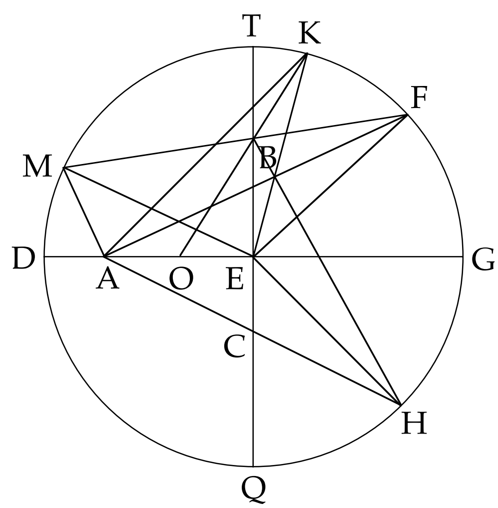


FIGURE 5.2.34a

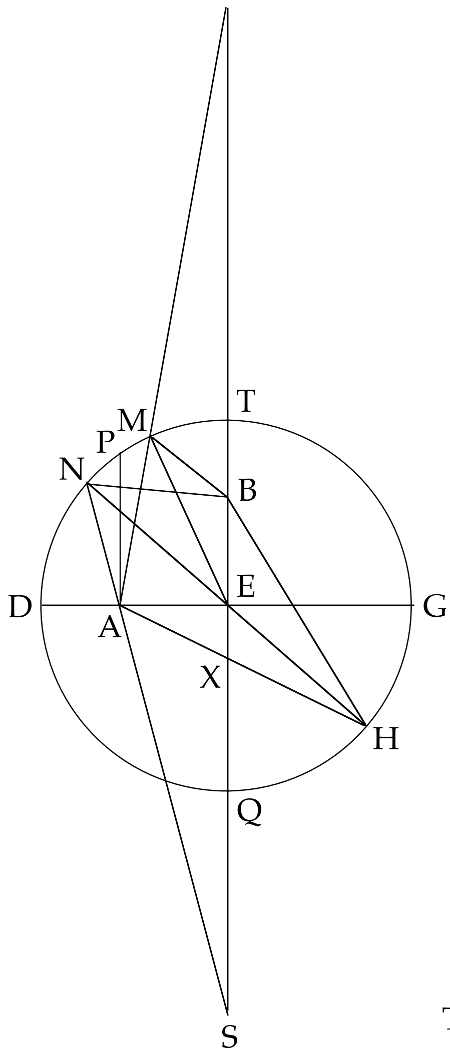


FIGURE 5.2.34b

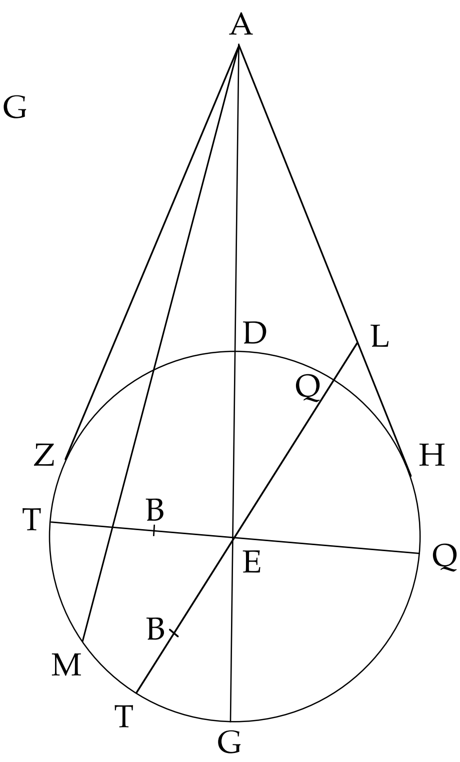


FIGURE 5.2.34c

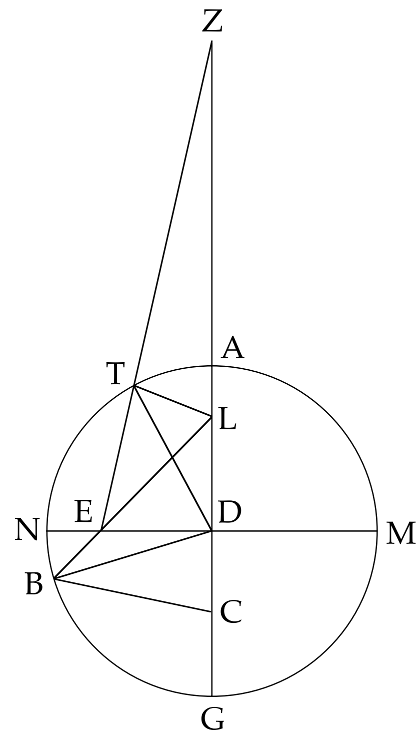


FIGURE 5.2.35

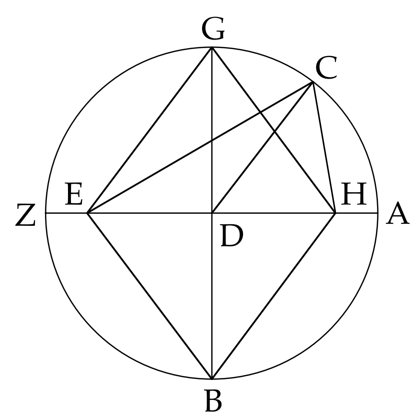


FIGURE 5.2.36

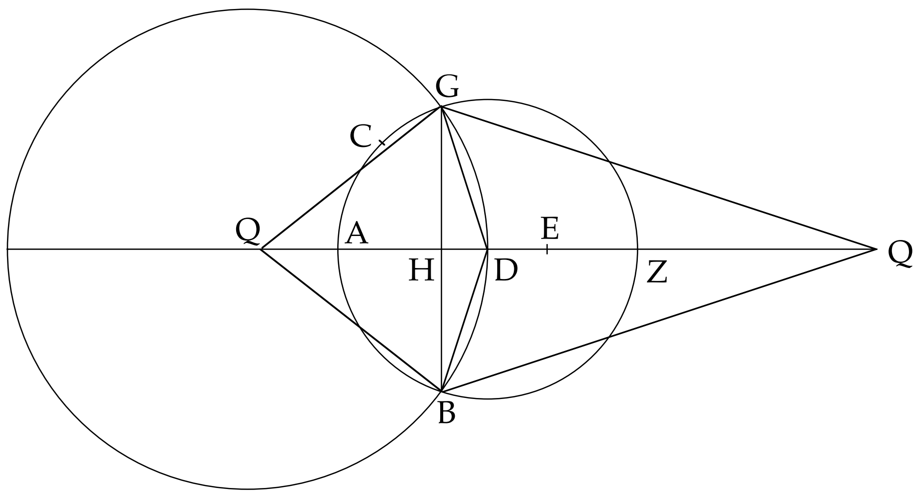


FIGURE 5.2.36a

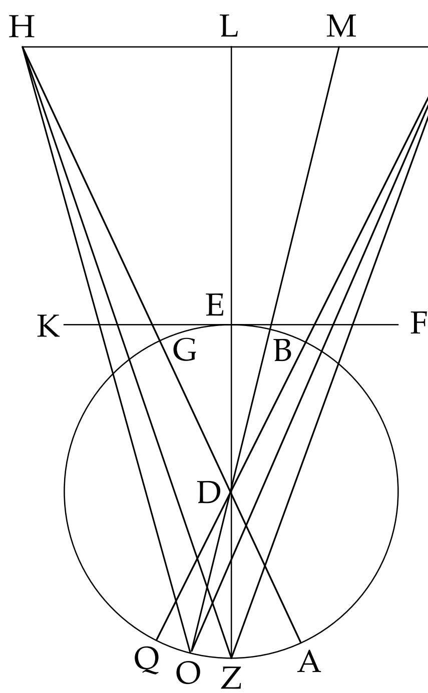


FIGURE 5.2.37

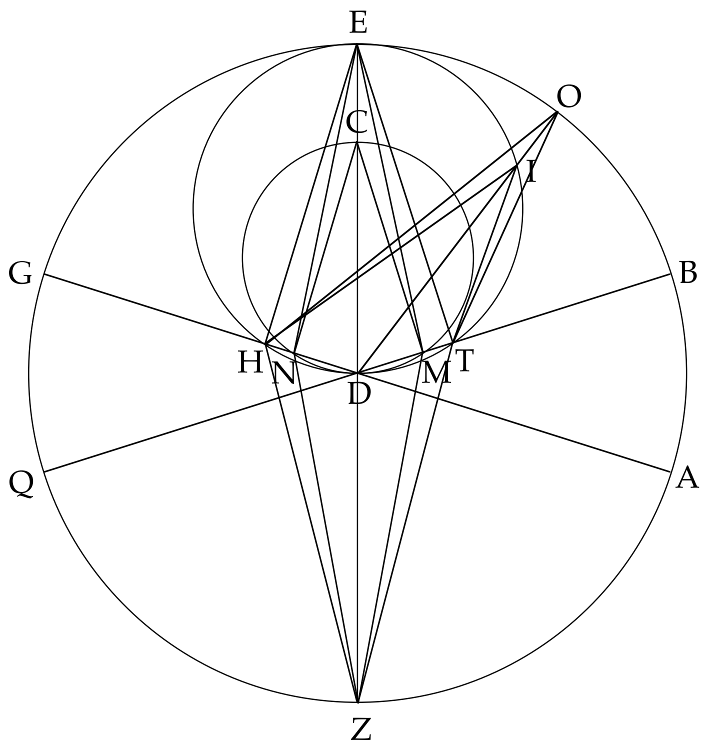


FIGURE 5.2.37b

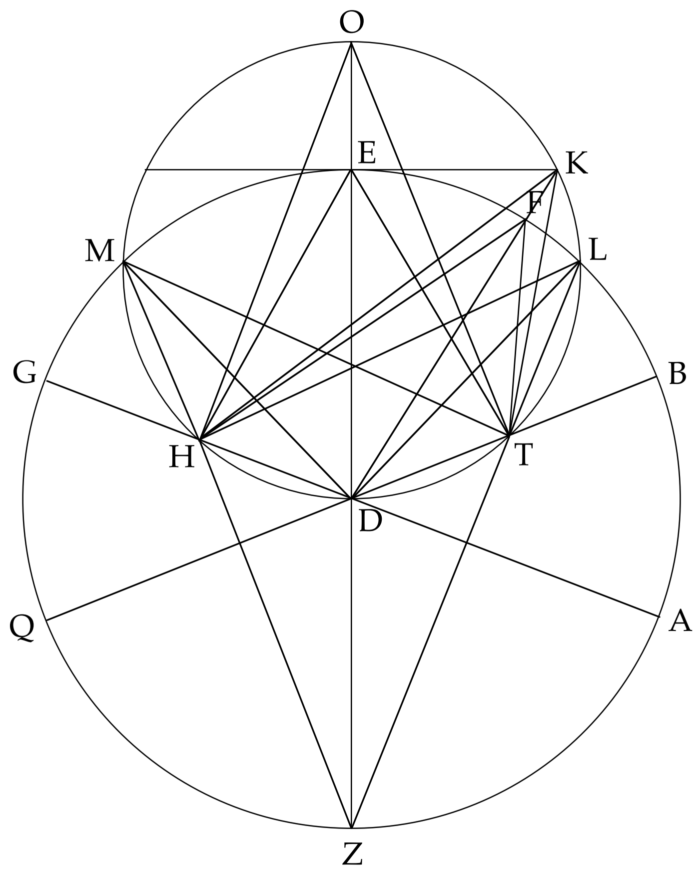


FIGURE 5.2.37c

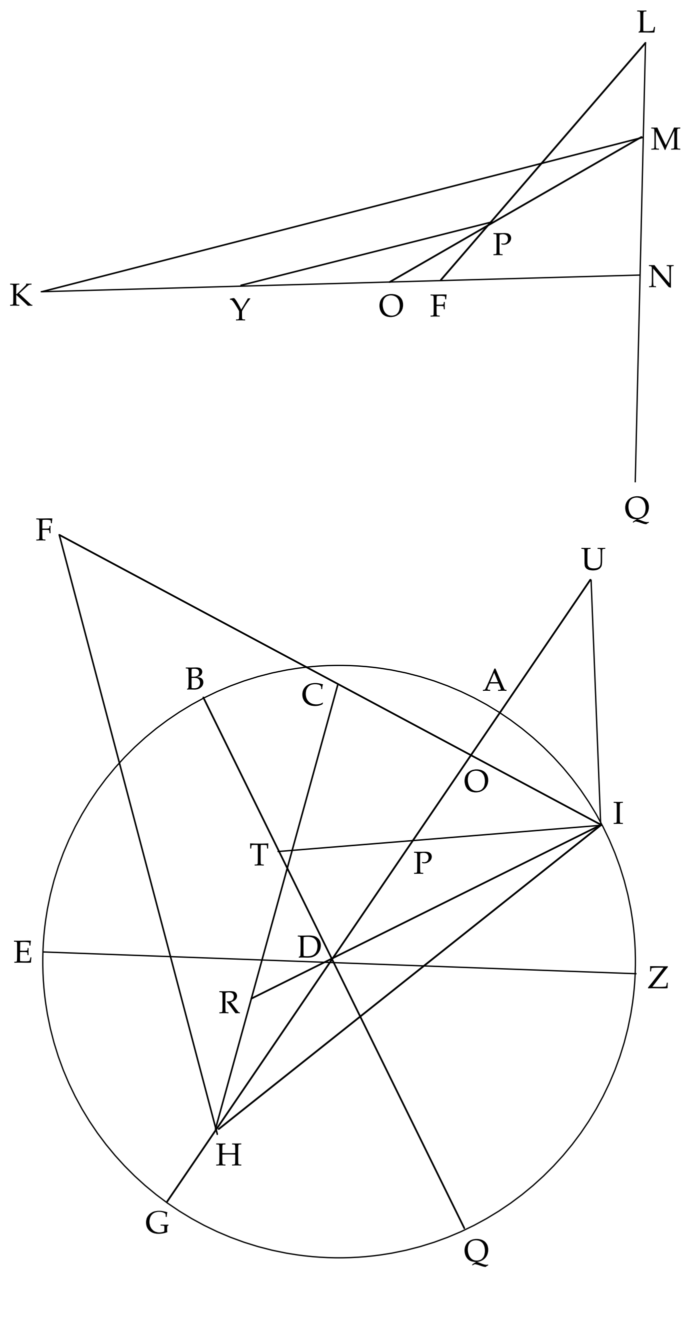


FIGURE 5.2.38

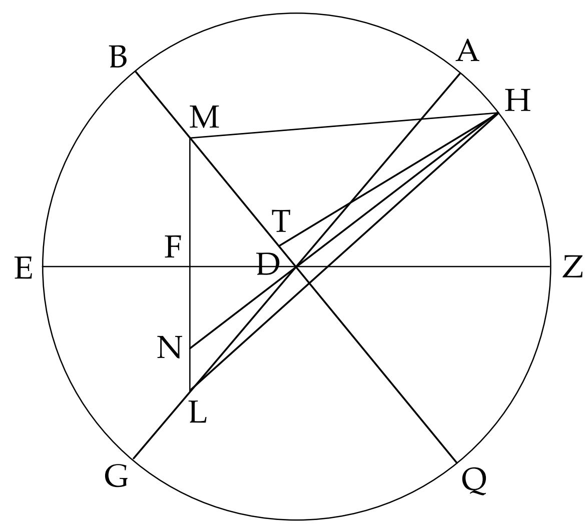


FIGURE 5.2.39

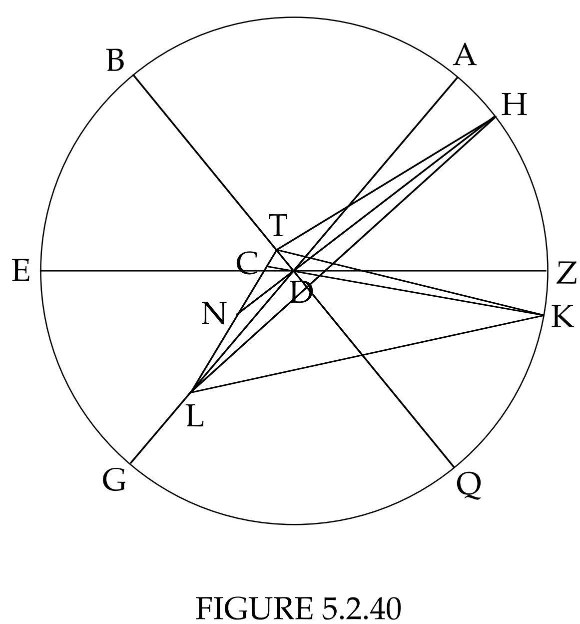


FIGURE 5.2.40

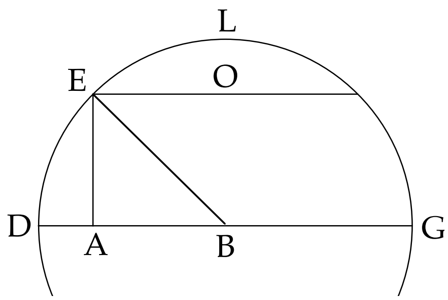


FIGURE 5.2.41

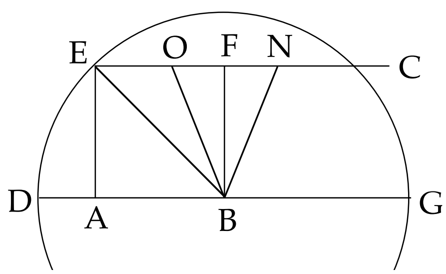


FIGURE 5.2.42

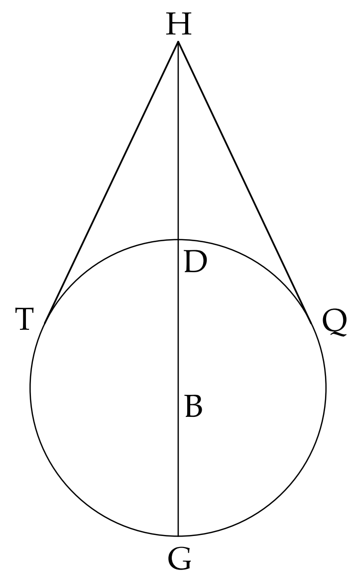


FIGURE 5.2.41a

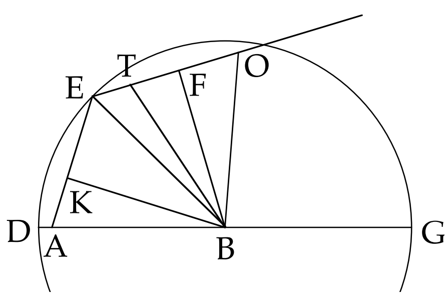


FIGURE 5.2.42a

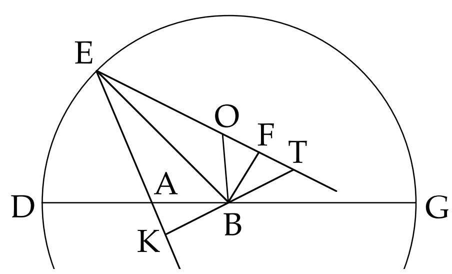


FIGURE 5.2.42b

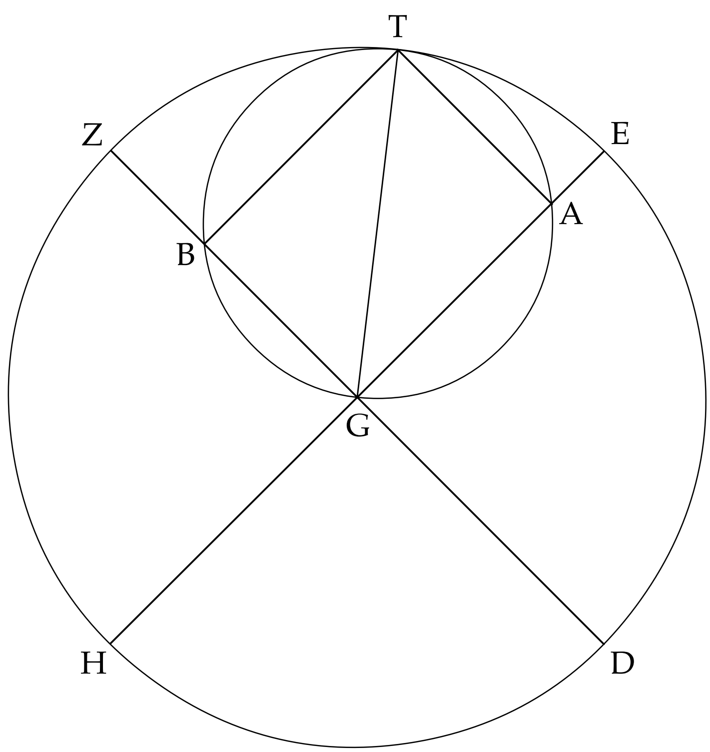


FIGURE 5.2.43

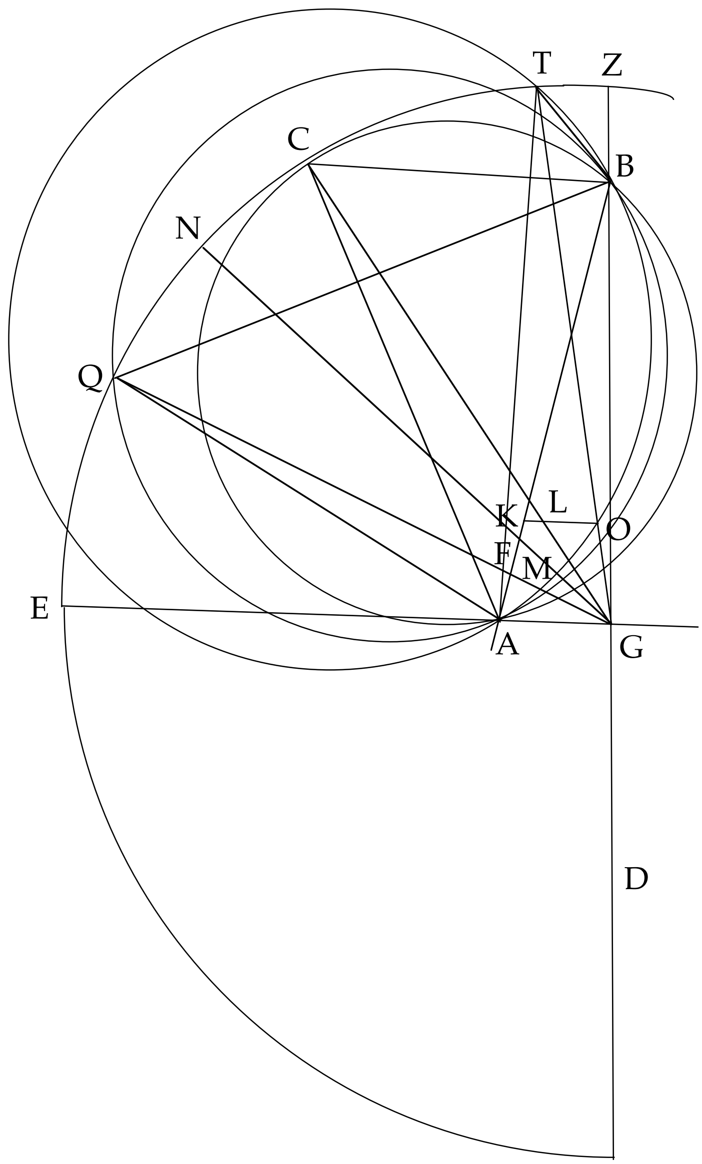


FIGURE 5.2.44

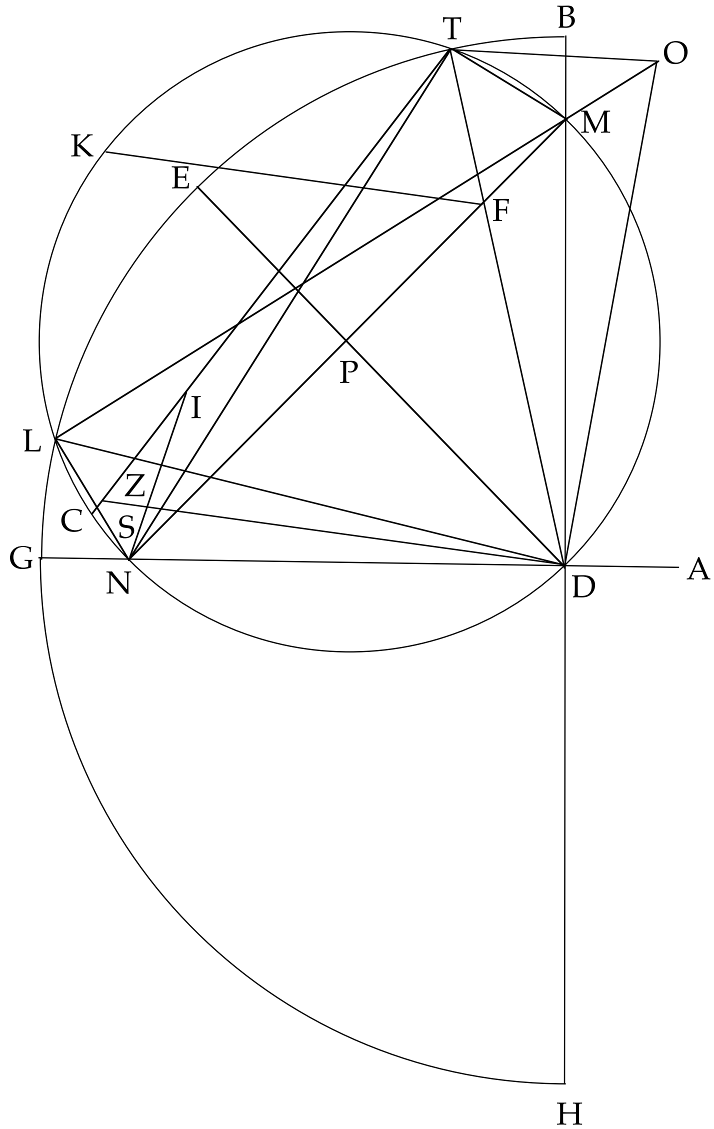


FIGURE 5.2.45

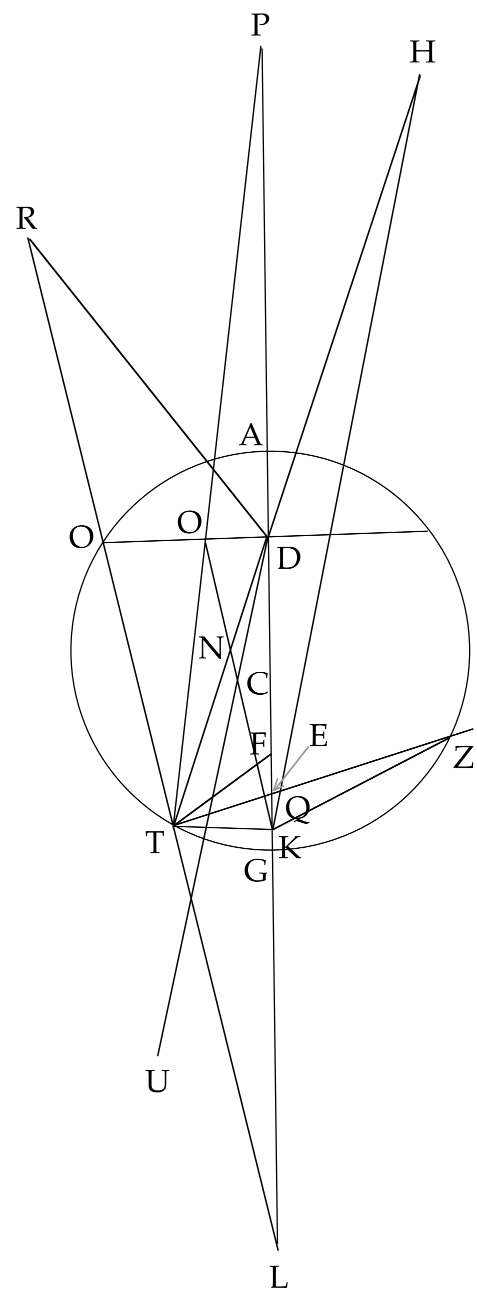


FIGURE 5.2.46

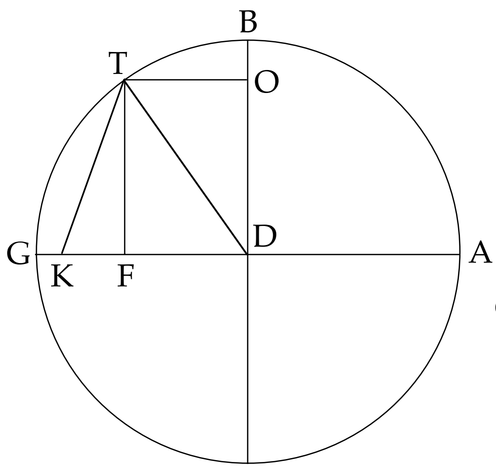


FIGURE 5.2.46a

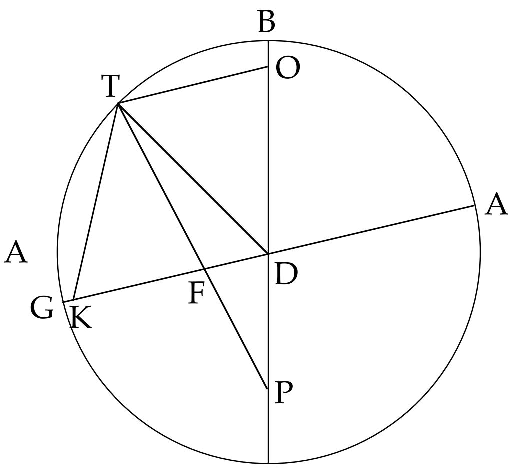


FIGURE 5.2.46b

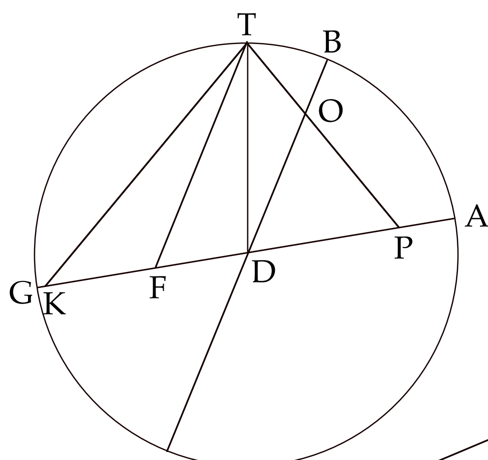


FIGURE 5.2.46c

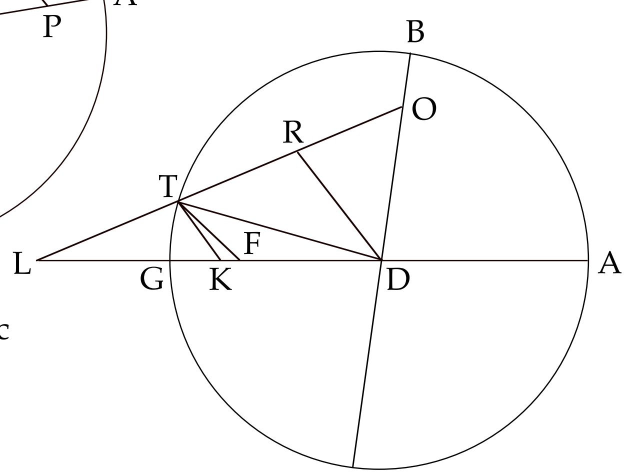


FIGURE 5.2.46d

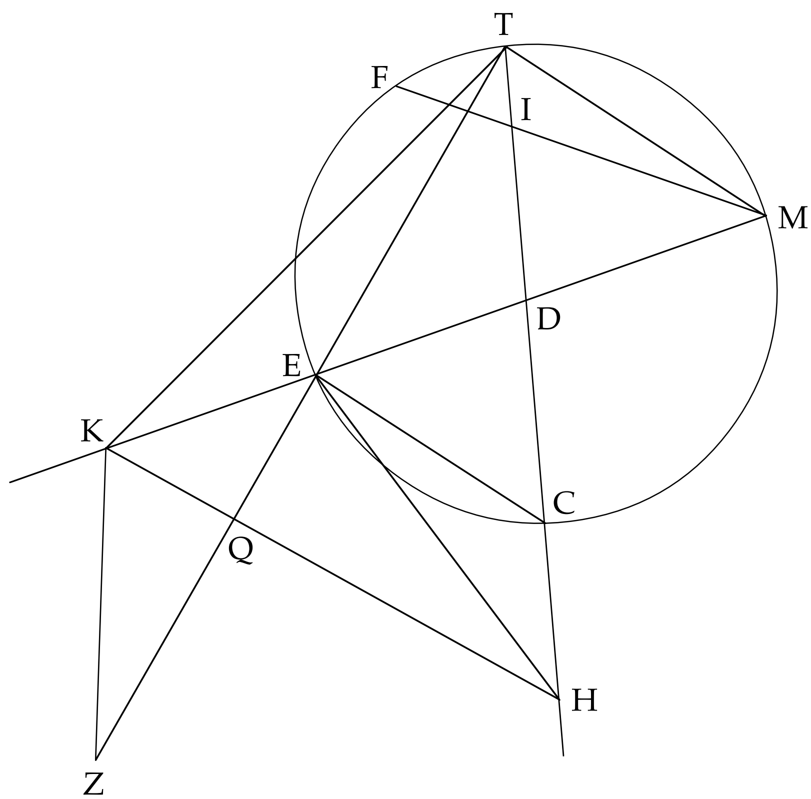


FIGURE 5.2.46e

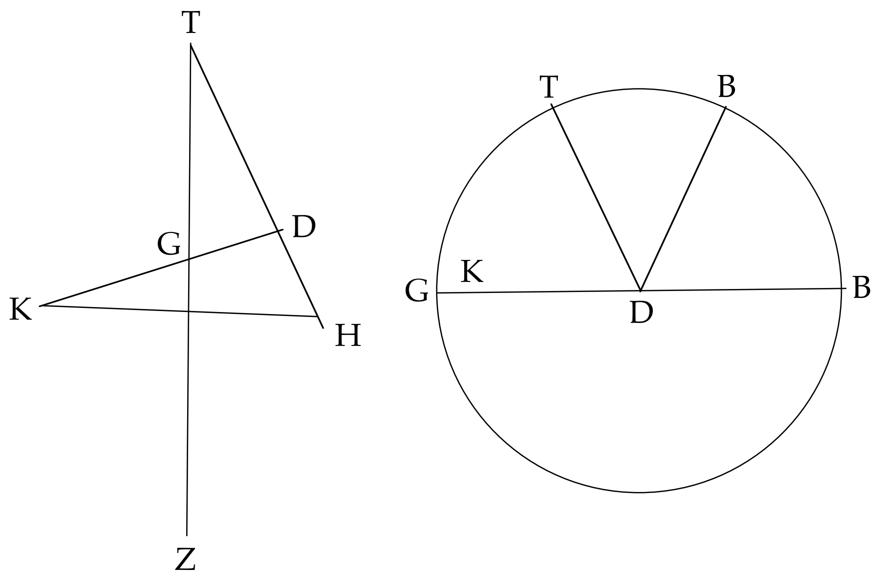


FIGURE 5.2.47

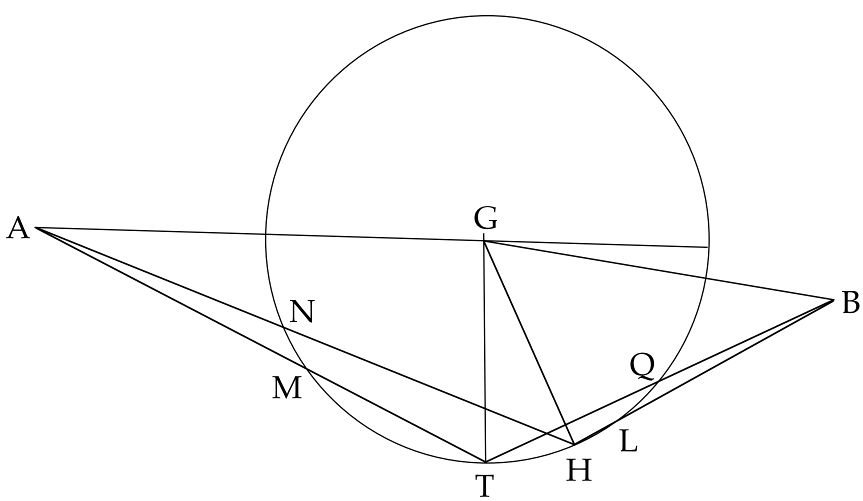


FIGURE 5.2.48

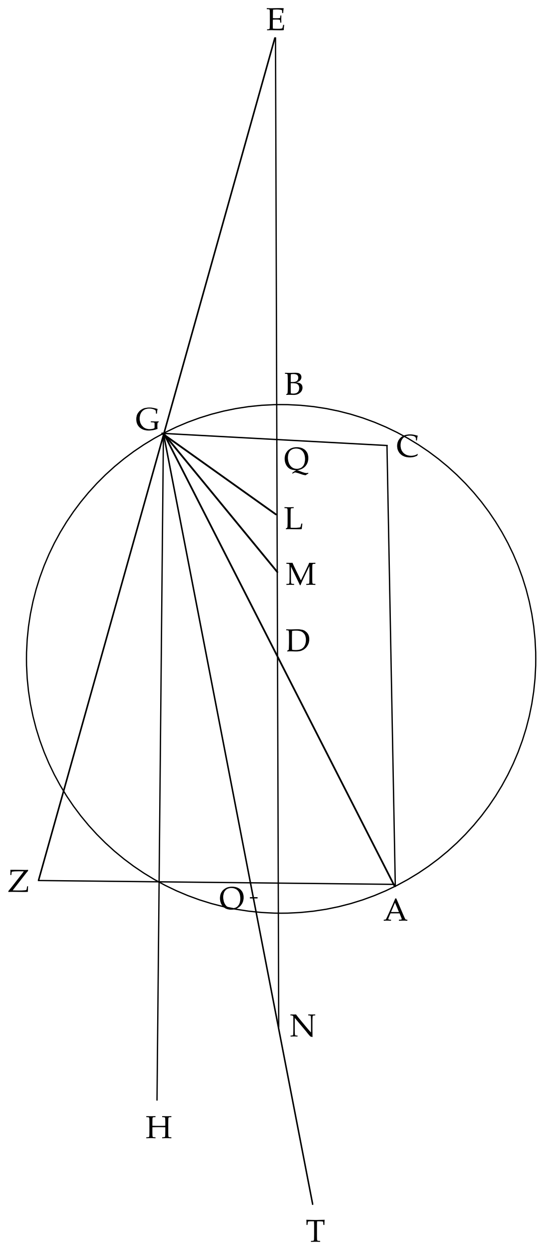


FIGURE 5.2.50

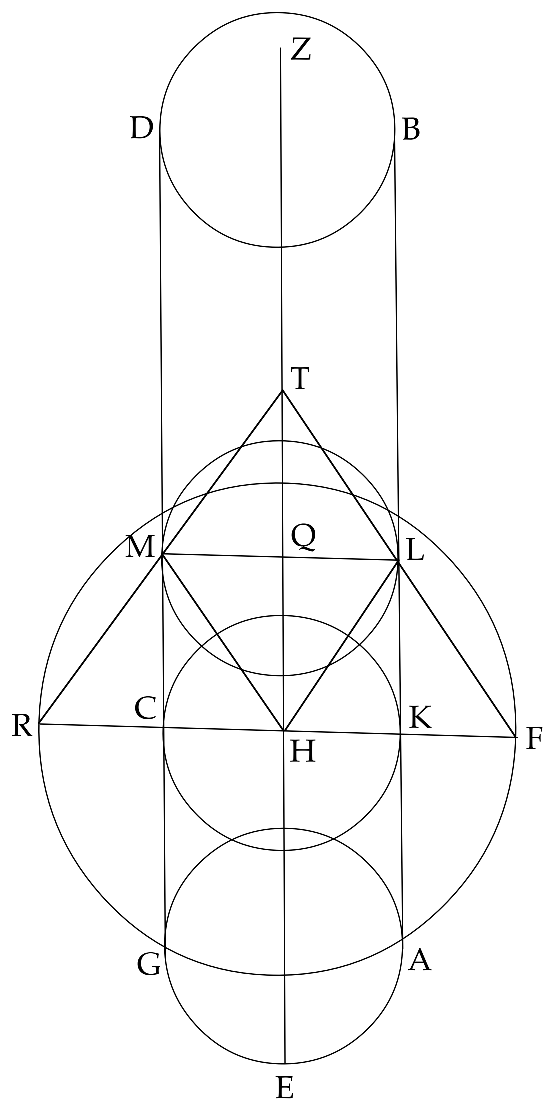


FIGURE 5.2.51

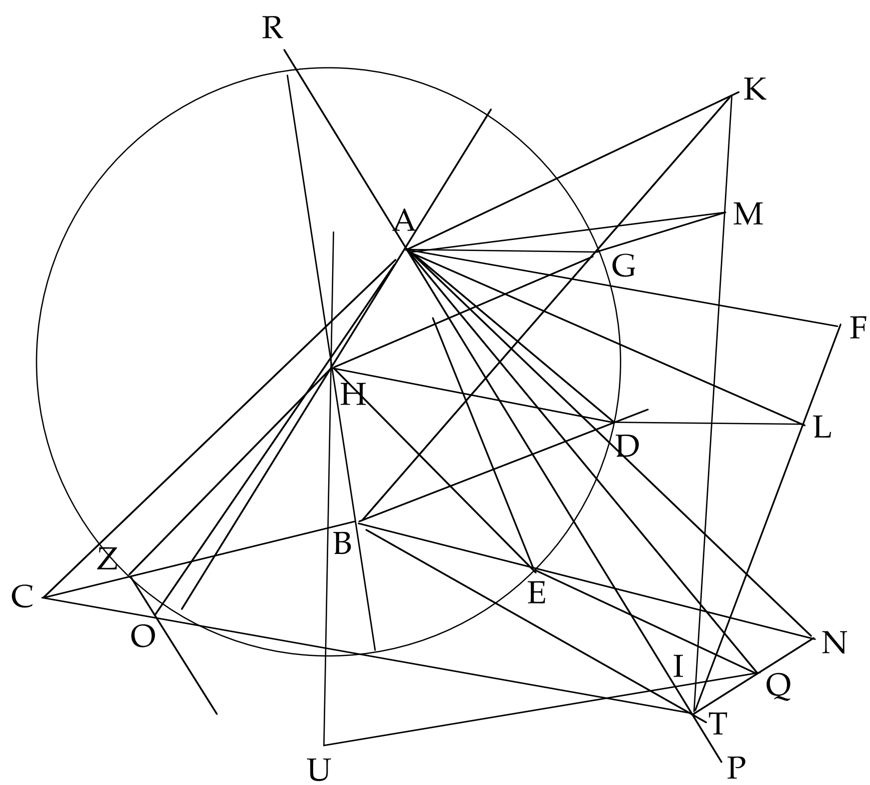


FIGURE 5.2.52

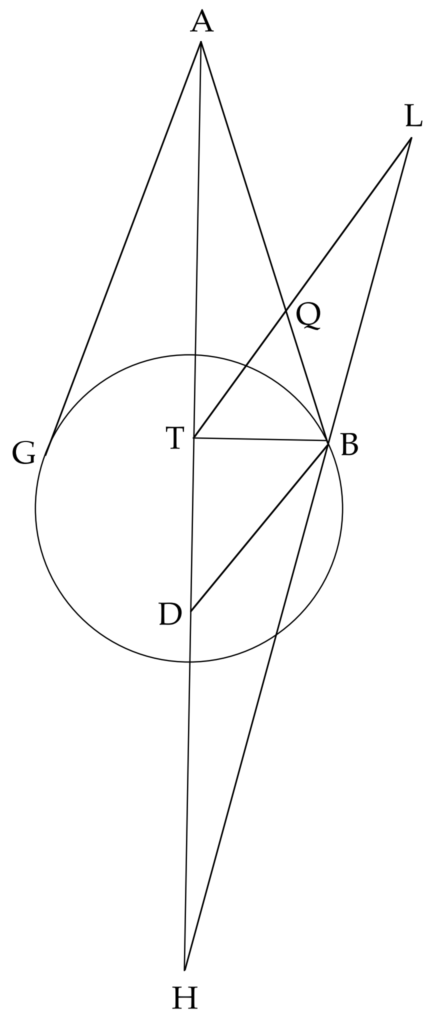


FIGURE 5.2.53

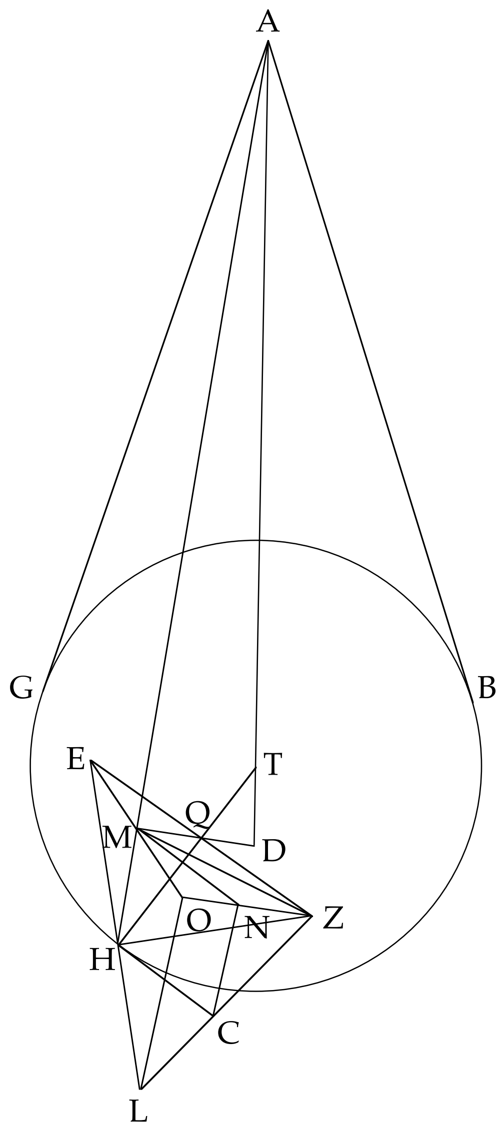


FIGURE 5.2.54

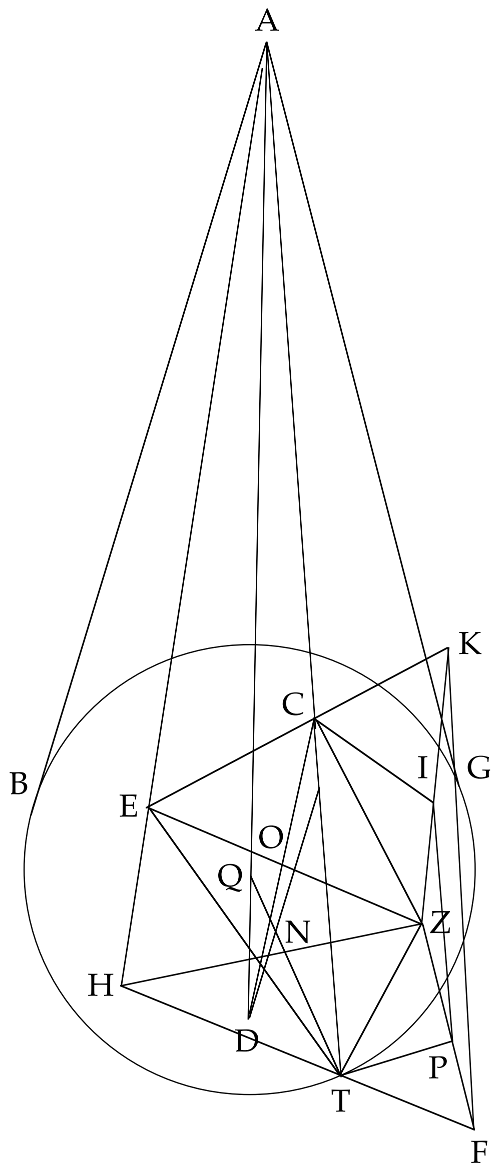


FIGURE 5.2.54a

LATIN-ENGLISH INDEX

LATIN-ENGLISH INDEX

ENGLISH-LATIN GLOSSARY

ENGLISH-LATIN GLOSSARY

a bit/a little bit modicus
above elatus
absence privatio
absolute inseparabilis
to absorb recipere
abutting contiguus
accidental/secondary accidentalis
to accomplish facere, peragere
account modus, ratio
account of reflection modus reflexionis
to account for determinare
accuracy veritas
actuality veritas
acute acutus
to add addere, adiungere
adjacent collateralis
to adjust mutare
air aer
alike similis
alternate colaternus
altitude altitudo
analysis modus
analysis of reflection modus reflexionis
angle angulus
angle of reflection angulus reflexionis
anterior anterior
apex caput
Apollonius of Perga Ablonius
apparatus instrumentum
apparent/appearing apparens
to appear apparere, videre

appearance apparentia
to apply adhibere, applicare, opponere
to apply before predicere
to apprehend adquirere
apprehension adquisitio
to approach accedere, descendere
appropriate proprius
arc arcus, pars, sectio
area pars
to argue before/previously predicere
to arise accidere
arrangement dispositio, ordinatio
to arrive accedere, accidere, procedere, venire
arrow sagitta
to ascertain facere
to assume ponere
to attach adhibere, cogere, consolidare, figurare, infigere
attachment applicatio
to attain accedere
to attribute assignare
axis axis, dyameter

to balk repellere
barley ordeum (*see also* **grain of barley**)
base basis, caput
base-block tabula
to be accidere, manere
to be apparent/evident/observed/seen/visible apparere, patere
to be as habere
to be bounded by interiacere
to be broader/larger/wider excedere
to be demonstrated/proven/shown patere
to be directly in line opponere
to be disparate diversificare
to be due to accidere
to be equidistant equidistare
to be established patere
to be far from elongare

to be formed procedere
to be hidden latere
to be incident descendere, exire, venire
to be invisible latere, occultare
to be left remanere
to be obligated debere
to be oblique declinare
to be open patere
to be parallel equidistare
to be rigid figere
to be tangent (to) contingere, tangere
to be unnoticeable occultare
beam radius
to bear in mind intelligere, notare
beautiful speciosus
beginning principium
below/lower inferior
between medius
beyond exitus, exterior
to bisect dividere, secare
bisection divisio
black niger, nigredo
block tabula
to block abscondere, obturare, occultare
blocking/facing disposition oppositio
body corpus, res
body of reflection corpus reflexionis
bolt repagulum
book liber
to bore extrahere
both duplex
bottom fundus, inferior
to bounce back revertere
bowed arcuatus
breadth latitudo
bright fortis
brightness maioritas
to bring together cogere

bronze eneus

to call nominare
to carry out facere
case situs
category partitio
cause causa
to cause efficere, facere, inducere
cavity concavatio, concavitas
celestis sky
center/centerpoint centrum, medius, punctum/punctus, punctus medius
center of sight centrum, centrum visus, visus
center of the eye centrum visus
change mutatio
to change aufere, mutare
chapter pars
character/characteristic proprietas, proprius, status
to choose sumere
circle circularis, circulus, spera
circular circularis
circumference circumferentia, latitudo
claim res
to claim affirmare
to claim earlier/previously predicere
clarity certitudo
clear lucidus, planus
to close claudere
to coincide applicare, cadere, continuare
coincidental continuus
color color, coloratio
to color colorare
coming together agregatio
common communis
common axis axis communis
common line linea communis
common section linea communis, sectio communis
compass circinus (*see also* **point of a compass**)
complement complementum

to complete complere
concave surface concavitas, concavum
to concentrate agregare
concentrated continuus
concentration agregatio
cone piramidalis, piramis
conic/conical piramidalis
conic section sectio piramidalis (*see also* **hyperbola**)
to conjoin coniungere
to connect/join interiacere
consideration consideratio
to construct facere
construction ypotesis
contact contactus
to contain continere, includere
continuation continuus
to continue gradi, producere
continuity continuitas
continuous continuus
continuum continuitas, continuus
convex exterior
to convey vehere
to convey back mittere
to correspond respicere, respondere
corresponding similis
counterpart oppositus
course processus
to cover cooperire
to create facere
to cut secare
to cut off aufere
cutting sectio
cylinder columpna, columpnalis
cylindric/cylindrical columpnaris
cylindric section sectio columpnaris (*see also* **ellipse**)

dark fuscus, nigredo, tenebrosus
to darken obumbrare

deception error
to decrease diminuere
deduction sillogismus
to define facere, includere
definite proprius
degree mensura
to demonstrate accidere, demonstrare, ostendere, probare
demonstration demonstratio, probatio, ratio
depth altitudo, profunditas
to define assignare, determinare
to describe assignare, signare
to describe previously predicere
to detect videre
determination assignatio, certitudo, probatio
to determine assignare, certificare, debere, determinare, habere, invenire, perpendere, videre
to devote assignare
diameter dyameter, latitudo, quantitas
to differ diversificare
difference differentia, diversitas, proportio
different/not the same diversus, inequalis/inequaliter
digit digitus
dim modicus
direct directus, rectus
to direct adhibere, dirigere, figere, intueri, tendere
direction pars
directly facing rectus
directly in front oppositus
directly/right in line continuus
to disappear abscondere
to discuss exponere
to discuss previously predicere
discussion sermo
disparate diversus
to dispose accidere, adhibere, disponere
disposition dispositio, situs
disproof improbatio
to disprove improbare

distance altitudo, distantia, elongatio, longitudo, remotio
distinct diversus
to distinguish determinare
to divide dividere, partire
division divisio
to do facere, operari, peragere, preparare
to draw ducere, educere, erigere, extrahere, facere, producere, protrahere
to draw away elongare, removere
drill instrumentum
to drop descendere, ducere, erigere, producere

each/every thing singulus
edge acus, caput, extremitas, finis, latus, processus
to elevate elevare
elevated elatus
ellipse sectio columpnaris (*see also* **cylindric section**)
to emanate exire, procedere
to enclose includere
to encompass includere
to encounter invenire
end/endpoint caput, extremitas, finis, pars, summitas, terminus
to enter descendere, intrare, subintrare
entering descensus
entry gressus
enumerate numerare
to envelop occupare
to equal valere
equality equalitas, paritas
equivalence ydemptitas
to erase aufere
erect/perfectly erect ortogonalis
to erect erigere
erected erectus
error error
to establish (by demonstration) demonstrare, determinare, habere
to establish earlier predicere
estimation estimatio
even planus

evenness equalitas
evident planus
example verbum
excess excrementum
to exclude excludere
to exhaust finire
to explain assignare
explanation assignatio
to expose adhibere, exponere
to extend accedere, considerare, descendere, ducere, extendere, occurrere, pervenire, procedere, producere, prolongere, protendere, protrahere, secare
to extend between interiacere
extension continuus
exterior extrinsecus
extrinsic accidentalis
eye oculus, visus

face facies
to face adhibere, declinare, opponere, respicere
facing oppositus
faint debilis, modicus
faintness debilitas
to fall between interiacere
to fall upon cadere, descendere, venire
familiar notus
far longus/longe, remotio
far enough modicus
to fashion facere
few rarus
fewness paucitas
figure figura
to fill extendere, obturare, occupare
find inventio
to find invenire
finger digitus
to finish complere
to fit adaptare, adhibere, applicare

to fit into intrare
fitting conveniens
to fix figere, infigere
fixed immotus
flat planitus, planus
to focus assignare
to follow accedere, accidere, cedere, mittere, movere, observare, operari, patere, postponere
force motus, vis
to forementíon predicere
form forma, modus
to form continere, efficere, facere, habere, includere, observare, tenere
form of a cone piramidalitas
forming a rectangle ductus…in… (*see also* **rectangle**)
four-fold quadruplex
function proprietas

to gain adquirere
gap divisio
general universalis
generally universaliter
geometer geometer
geometrical geometricis/geometricus
to give sumere
to go procedere
goblet ciphus
grain of barley granum ordei
to grasp adquirere, comprehendere
great magnus
great circle circulus, circulus magnus
green viriditas
ground terra

half medietas, medius
hand manus
to happen accidere
to have/possess habere
hardness duricies

head caput
heavy honerosus, ponderosus
height altitudo, elevatio, eminentia, longitudo
held in place immotus
to hide occultarre
to hold accidere
to hold up adhibere
hole foramen, vacuum
hollow (n.) concavitas**,** concavum, foramen
hollow (adj.) concavus
to hollow out fodere
horizon orizon
horizontality orizon
hyperbola sectio piramidalis (*see also* **conic section**)

identical similis
identity ydemptitas
to illuminate dirigere, illuminare, mittere
illuminated lucidus
illusion error
image forma, ymago
image-location locus reflexionis, locus ymaginis
imaginary/imagined intellectualis, intelligibilis
to imagine intelligere, sumere
immobile immotus
imperceptable/imperceptably imperceptibilis/imperceptibiliter
to impinge descendere
to impress (upon) figere, imprimere, inficere, infigere
impression fixio, impressio
to impute assignare
incidence accessus, descensus, motus
to incline declinare
inclination declinatio
to include continere, includere
increase augmentum
indefinite/indefinitely infinitum, numerus
individual singularis
induction sillogismus

infinite infinitus
infinity infinitum
ink/red ink incaustum
inner/inside/interior inferior, interior
insensible insensibilis
to insert adhibere, descendere, immittere, intrare, ponere
instance verbum
instant hora
instrument instrumentum
to insure perpendere
intellectual intellectualis
intense fortis, magnus
to intensify redere, redire
intensity fortitudo
to interfere impedire
interference interpositio
interior segment interior
intermediate medietas
interrelationship intricatio
interruption discretio
to intersect concurrere, dividere, secare, tangere, transire
intersection concursus, sectio
intersection-point punctus sectionis
intrinsic proprius
invariable/invariably generalis/generaliter
iron ferreus

to keep manere
kind modus, proprietas, qualitas, quantitas, species
to know cognoscere, habere, scire
known manifestus, notus

large magnus
lathe instrumentum
latter posterior, postremus
to lay applicare, deprimere, descendere, statuere
to leave relinquere, remanere
length altitudo, distantia, longitudo, quantitas

lengthwise altitudo, longitudo
letter scriptura
level planities
to level inducere
to lie adhibere, cadere, existere, ponere, remanere
to lie between intercidere, interiacere, interponere
to lie far from elongare, removere
to lie outside egredi, preterere, preterire
light lux
likewise similiter
limit meta, terminus
limit-point terminus
line linea, protractio, via
line of longitude latus
line of reflection linea reflexionis
line of sight intuitus, linea visualis, radius visualis, visualis
to locate statuere
location locum/locus, pars, situs
location of reflection locus reflexionis, situs reflexionis
long longus/longe
longitude longitudo
to look (at/into) adhibere, inspicere, videre
to lower cadere, deprimere, descendere
luminous illuminosus, lucidus, luminosus

magnitude magnitudo
to maintain conservare, durare, habere, observare, servare, tenere
maintained immutabilis
maintenance observatio
to make efficere, facere
to make beforehand/first precipere, preponere
to make out percipere
to make sure scire
man homo
manifest planus
manner modus
manual manus
mark mensura, nota, signum

to mark/mark off assignare, facere, notare, signare, sumere
mass moles
to mean volere
measure mensura
to measure/measure off metiri, signare, sumere
to meet concurrere
method modus
middle medius
midline linea media, medietas, medius
midpoint medietas, medius, punctus medius
to mingle admiscere, miscere
mirror speculum
to miss divertere
mode modus
mode of reflection modus reflexionis
moderate moderatus
moderate-size modicus
motion motus
motion in reflection motus reflexionis
to move descendere, movere
to move about volvere
to move away declinare
to move back retrocedere
to move forth procedere
movement processus
movement away recessus
movement toward accessus
to multiply multiplicare

narrow gracilis, strictus
to narrow constringere
natural naturalis
nature natura, naturalis
near propinquus
needle acus
neither neuter
to nest conserere
normal dyameter, ortogonalis, perpendicularis

not to permit prohibere
noticable notabilis

object corpus, obiectio, res
object-point punctum/punctus, punctus visus
obliquity declinatio
to observe videre
obtuse obtusus
obvious manifestus, planus
to occupy occupare
to occur accidere, diversificare, facere
opacity soliditas
opaque solidus
open discoopertus
to open aperire
opening abscisio, foramen
operation res
to oppose opponere, respicere
opposed/opposite diversus, oppositus
orientation situs
origin/origin-point ortus
to originate incipere, oriri, procedere
orthogonal ortogonalis, perpendicularis, rectus
orthogonally ortogonaliter, perpendiculariter
other alteratio
outer (surface)/outside exterior
to outweigh addere, pondere
to overlap admiscere, concurrere

panel regula
parallel equidistans
parchment pargamenum
part/partial pars, partialis/partialiter
particular proprietas, singulus
to pass/to pass along/by/into/through descendere, incedere, intrare, movere, pervenire, procedere, secare, transire
to pass on redere, redire
to pass unseen preterere, preterire

passage transitus
passing through penetratio
peculiar proprius
peg baculus
to penetrate intrare, penetrare, transire
to perceive adquirere, comprehendere, percipere, sentire
perceptible perceptibilis
perception/process of perceiving adquisitio, comprehensio, perceptio
perfect finis
perpendicular ortogonalis, perpendicularis, rectus
person homo
to pertain accidere
phenomenon res
physical corporeus
place pars, situs
to place adhibere, applicare, facere, habere, ponere, statuere
to place between includere
to place upright erigere
plane planus, superficies
plank regula
plaque regula, tabula
to please libere
point acumen, locum/locus, pars, punctum/punctus, res, signum, situs, terminus
to point adhibere
point of reflection locus reflexionis, punctus reflexionis
point on a section punctus sectionis
point seen/viewed punctus visus
pointed acutus
pointer baculus
point of a compass pes circini
pole polus
polish politas, politio, politiva, politura
polished politus
polished body politiva, politum
polished surface politum
pore porus
portion pars, portio, sectio

position pars, situs
to position disponere
position below inferioritas
to precede predicere
to predict predicere
preliminary antecedens
to prescribe predicere
to present adhibere
to preserve observare
to press up to opponere
to prevent negare
principle principium
procedure modus, operatio, opus
to proceed movere, procedere
process iteratio
to produce ducere, efficere, facere, producere, referre
to project ducere, elevare, procedere, producere
to project outward egredi
proof probatio
to propagate ingredi, movere, occurrere, procedere
propagation motus
properness veritas
proportion proportio
to propose proponere
proposition propositio
to prove demonstrare, probare
proximity contiguitas
to punch perforare

quadrilateral quadrangulus
quantity quantitas

radial line radius
to radiate egredi, movere, procedere, venire
radiation accessus, processus
radius dyameter, semidyameter
to raise elevare, sublevare
rarely raro

ratio proportio
rationally demonstrative
ray linea, radius
reach accessus
to reach accedere, accidere, cadere, continuare, descendere, movere, occurrere, pervenire, procedere, venire
reality veritas
to realize perpendere
reason causa
reasonable rationabilis
reasoning ratio
rebound reflexio, regressio
to rebound reflectere, regredi
rectangle ductus…in… (*see also* **forming a rectangle**)
rectilinear linea recta, rectilineus, rectitudo, rectus
red rubor
to reflect referre, reflectere
reflected reflexus
reflected angle angulus reflexionis
reflected ray linea reflexionis
reflection modus reflexionis, reflexio, reflexus
to relate to respicere
related group concursus
to remain manere, remanere
remainder residuus
to remove aufere, removere
to repeat iterare, replicare
to repel repellere
repetition replicatio
to replace collocare, ponere
repulsion repulsio
resistance fortitudo, prohibitio, repulsio
to restore reducere, regredi
result comprehensio
to return redere, redire, regredi
reversal revolutio
to reverse revolvere
revolution motus

to revolve movere
right directus, ortogonius, rectus
ring anulus
to rise ascendere
rod baculus
room domus
to rotate movere
rotation girus, motus
rough asperus
round circulus
to round circulari
roundness circulatio
ruler regula

same similis
sameness equalitas, ydemptitas
scientific understanding scientia
to scoop cavare
screen paries
to scrutinize adhibere
scrutiny intuitus
section linea, pars, portio, sectio
to see comprehendere, videre
to see that intelligere
seen apparens
segment pars, partialis, portio, sectio
semicircle semicirculus
to send back remittere
to sense sentire
sensible sensibilis, sensualis
sensible line/shaft linea sensualis
to separate removere, separare
to serve usitari
to set forth/out proponere
to set up adhibere, applicare, observare, statuere
setting-up erectio
scope via
shaft radius

shape figura, forma
sharp/sharp-edged or pointed acutus, rectus
sharp edge/point acuitas, acumen
to sharpen acuere
sharpness acuitas
to shift mutare
to shine accedere, cadere, declinare, descendere, mittere, procedere
to shine through intrare
shining descensus
show ostensio
to show ostendere, patefacere, probare
side latitudo, latus, pars
sight intuitus, oculus, visus
silver argenteus
similar similis
similarly similiter
situation situs
size magnitudo, mensura, quantitas
slant declinatio
to slant declinare
slight/slightly modicus
small exiguus, modicus, parvus
smallness parvitas
smooth lenis, politus
smoothness lenitas
softness mollities
solid solidus
source origo, ortus
spatial disposition situs
specific to proprietas
to specify determinare
sphere spera
spherical spericus
spot locum/locus, pars, punctum/punctus, situs
square quadratus
square (x^2) quadrangulus, quadratus
squared off ortogonalis
to squeeze out of shape immutare

to stand cadere, erigere, statuere
standing erectus
stationary immotus
stone lapis
stop obstaculus
straight directus, rectus
straight line linea recta
to strike cadere
strong fortis
to subdivide dividere, subiacere
to subtend intercidere, intercipere, respicere, tenere
to subtract aufere, removere
to sum up to valere
sun sol
sunlight sol
to suppose ponere
supposition ypotesis
surface altitudo, facies, pars, superficies

tablet regula, tabula
to take habere, sumere
to take out extrahere
to take place facere
template tabula
to tend (in a direction) tendere
theorematically demonstrative
theory opinio
thick spissus
thickness latitudo, spissitudo
thin gracilis, subtilis
thing res
to tilt downward declinare
time hora
tiny modicus
token modus
top caput
to touch cadere, contingere, descendere, secare, tangere
to transect secare

to transfer transfere
transparency raritas
transparent rarus
trap fixio
to travel procedere
triangle triangulus
triangular triangularis
to truncate secare
truth fides, veritas
tube columpna
to turn inclinare
to turn out evenire
twofold duplex

uncovered discoopertus
to understand intelligere, scire
unequal inequalis/inequaliter
uniformity similitudo, ydemptitas
uniformly equidistans
unity unitas
untouched vacuum
upright erectus, ortogonalis, rectitudo, rectus

variation diversitas
variety varietas
various diversus
to vary diversificare
veridity veritas
to verify verificare, videre
vertex acumen, caput, conus
vessel concavum, vas
vice-versa econversus
to view adhibere, comprehendere, inspicere, videre
viewed object/visible body corpus visum
viewer visus
viewpoint intuitus
visibility apparentia
visible apparens

visible object res, res visa
visible point punctus visus
vision visus
visual visualis
visual axis dyameter visualis, linea visualis, perpendicularis visualis
visual faculty visus
visual plane visualis superficies
void vacuum

wall paries
to want volere
water aqua
wax cera, cereus
way modus
weak debilis
to weaken debilitare, diminuere, subtrahere
weakening debilitas, debilitatio, minoritas
white albus
width latitudo, quantitas
to will volere
window foramen
to wish volere
to wobble vacillare
wooden ligneus
wooden rod lignum
writing scriptura

to yield pretendere

BIBLIOGRAPHY

Ackrill, J. L. S*ee* Aristotle.

Alberti, Leon Battista. *See* Spencer.

Akdogan, Cemil. *Albert's Refutation of the Extramission Theory of Vision and His Defence of the Intromission Theory.* Kuala Lumpur: International Institute of Islamic Thought and Civilization, 1998.

al-Kindi. *See* Björnbo and Vogel.

Alverny, M-Th. d' and F. Hudry, "Al-Kindi, *De Radiis,*" *Archives d'histoire doctrinale et littéraire du moyen âge* 41 (1974): 139-260.

Anawiti, G. C. and A. Z. Iskandar, "Hunayn ibn Ishaq," *Dictionary of Scientific Biography,* vol. 15, supplement 1, ed. Charles Coulston Gillispie, pp. 230-249. New York: Scribner's, 1978.

Aristotle. *Categories.* Trans. J. L. Ackrill, in *The Complete Works of Aristotle: The Revised Oxford Translation,* Jonathan Barnes, ed. Princeton: Princeton University Press, 1984.

Aristotle. *De anima.* Trans. J. A. Smith, in *The Complete Works of Aristotle: The Revised Oxford Translation,* Jonathan Barnes, ed. Princeton: Princeton University Press, 1984.

Aristotle. *De memoria et reminiscentia.* Trans. J. I. Beare, in *The Complete Works of Aristotle: The Revised Oxford Translation,* Jonathan Barnes, ed. Princeton: Princeton University Press, 1984.

Aristotle. *De sensu et sensato.* Trans. J. I. Beare, in *The Complete Works of Aristotle: The Revised Oxford Translation,* Jonathan Barnes, ed. Princeton: Princeton University Press, 1984.

Aristotle. *Meteorology.* Trans. E. W. Webster, in *The Complete Works of Aristotle: The Revised Oxford Translation,* Jonathan Barnes, ed. Princeton: Princeton University Press, 1984.

Aristotle. *Physics.* Trans. R. P. Hardie and R. K. Gaye, in *The Complete Works of Aristotle: The Revised Oxford Translation,* Jonathan Barnes, ed. Princeton: Princeton University Press, 1984.

Aristotle. *Posterior Analytics.* Trans. Jonathan Barnes, in *The Complete Works of Aristotle: The Revised Oxford Translation,* Jonathan Barnes, ed. Princeton: Princeton University Press, 1984.

Avicenna. *See* Van Riet.

Baarmann, J., "Abhandlung über das Licht von Ibn al-Haitam," *Zeitschrift der Deutschen Morgenländischen Gesellschaft* 36 (1882): 195-237.

Bacon, Roger. *See* Lindberg.

Badawi, 'Abdurrahman. *Histoire de la Philosophie en Islam*. 2 vols. Paris: Vrin, 1972.

Barnes, Jonathan, ed. *The Complete Works of Aristotle: The Revised Oxford Translation.* Princeton: Princeton University Press, 1984. *See also* Aristotle, *Posterior Analytics.*

Barrow, Isaac. *Isaac Barrow's Optical Lectures, 1667*. Ed. and trans. A. G. Bennett, D. F. Edgar, and H. C. Fay. London: The Worshipful Company of Spectacle Makers, 1987.

Bartholomaeus Anglicus. *See* Long.

Beare, J. I. *See* Aristotle.

Bennett, A. G. *See* Barrow.

Biagioli, Mario. *Galileo Courtier*. Chicago: University of Chicago Press, 1993.

Bierens de Haan, D. *See* Huygens.

Björnbo, Axel and Sebastian Vogel, ed. and trans. *Alkindi, Tideus und Pseudo-Euklid: Drei optische Werke.* Leipzig: Teubner, 1912.

Bosscha, J. *See* Huygens.

Boyer, Carl B. *The Rainbow: From Myth to Mathematics.* New York: Yoseloff, 1959.

Brownson, C. D., "Euclid's *Optics* and its Compatibility with Linear Perspective," *Archive for the History of Exact Sciences* 24: 165-194

Clagett, Marshall, "A Medieval Latin Translation of a Short Arabic Tract on the Hyperbola," *Osiris* 11 (1954): 359-366.

Clagett, Marshall. *The Science of Mechanics in the Middle Ages.* Madison: University of Wisconsin Press, 1959.

Clagett, Marshall. *Archimedes in the Middle Ages*. 5 vols. Madison: University of Wisconsin Press, 1964, vol. 1. Philadelphia: American Philosophical Society, 1965-1984, vols. 2-5.

Copernicus. *De revolutionibus orbium caelestium. See* Duncan.

Dales, Richard C., "Robert Grosseteste's Scientific Works," *Isis* 52 (1961): 394-402.

Dales, Richard C. *The Problem of the Rational Soul in the Thirteenth Century.* Leiden: E. J. Brill, 1995.

De Lacy, Phillip, ed. and trans. *Galen on the Doctrines of Hippocrates and Plato.* Corpus Medicorum Graecorum 4.4.1.2. Berlin: Akademie Verlag, 1980-1984.

Denery, Dallas. *Seeing and Being Seen in the Later Medieval World: Optics, Theology and Religious Life*. Cambridge: Cambridge University Press, 2005

Donahue, William H., trans. *Johannes Kepler, Optics:* Paralipomena to Witelo and Optical Part of Astronomy. Santa Fe, NM: Green Lion Press, 2000.

Duncan, A. M., trans. *Copernicus, On the Revolution of the Heavenly Spheres.* New York: Barnes and Noble, 1976.

Eastwood, Bruce S., "Grosseteste's 'Quantitative' Law of Refraction: A Chapter in the History of Non-Experimental Science," *Journal of the History of Ideas* 28 (1967): 403-414.

Eastwood, Bruce S. *The Elements of Vision: The Micro-Cosmology of Galenic Visual Theory According to Hunayn Ibn Ishaq.* Transactions of the American Philosophical Society 72.5. Philadelphia: American Philosophical Society, 1982.

Eastwood, Bruce S., "Alhazen, Leonardo, and Late-Medieval Speculation on the Inversion of Images in the Eye," *Annals of Science* 43 (1986): 413-446.

Edgar, D. F. *See* Barrow.

Euclid *Optics. See* Heiberg, Kheirandish.

al-Farisi, Kamal al-Din. *Tanqih al-Manazir.* See Wiedemann.

Favaro, Antonio, ed. *Le opere di Galileo Galilei,* vol. 3.2. Florence: Barbèra, 1907.

Fay, H. C. *See* Barrow.

Fried, Michael, and Unguru, Sabetai. *Apollonius of Perga: Text, Context, Subtext*. Leiden: Brill, 2001.

Galen. *De usu partium. See* May

Galen. *De placitis Hippocratis et Platonis. See* De Lacy

Galileo. *Siderius Nuncius. See* Van Helden.

Galileo. *Theorica speculi concavi sphaerici. See* Favaro.

Gaukroger, Stephen. *Descartes: an Intellectual Biography.* Oxford: Clarendon Press, 1995.

Gilson, Étienne, "Les sources Gréco-Arabes de l'Augustinisme Avicennisant," *Archives d'histoire doctrinale et littéraire du moyen âge* 4 (1929): 5-149.

Gilson, Étienne. *Reason and Revelation in the Middle Ages.* New York: Scribner's, 1938.

Gilson, Simon. *Medieval Optics and Theories of Light in the Works of Dante*. Lewiston, Queenston, Lampeter: Edwin Mellen, 2000.

Girke, Dorothea, ed. *Eilhard Wiedemann: Gesammelte Schriften zur arabisch-islamischen Wissenschaftsgeschichte,* vol. 1. Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften an der Johann Wolfgang Goethe-Universität, 1984.

Goldstein, Bernard R., trans. *The Arabic Version of Ptolemy's* Planetary Hypotheses. Transactions of the American Philosophical Society 57.4. Philadelphia. American Philosophical Society, 1967.

Granger, Frank, trans. *Vitruvius:* De architectura, *Books I-IV*. Cambridge, MA: Harvard University Press, 1931.

Grant, Edward, "Jordanus Nemorarius," *Dictionary of Scientific Biography,* vol. 7, ed. Charles Coulston Gillispie, pp. 189-210. New York: Scribner's, 1973.

Grant, Edward. *Much Ado About Nothing: Theories of Space and Vacuum from the Middle Ages to the Scientific Revolution.* Cambridge: Cambridge University Press, 1981.

Grant, Edward. *In Defense of the Earth's Centrality and Immobility: Scholastic Reaction to Copernicanism in the Seventeenth Century.* Transactions of the

American Philosophical Society 74.4. Philadelphia: American Philosophical Society, 1984.

Grant, Edward. *The Foundations of Modern Science in the Middle Ages.* Cambridge: Cambridge University Press, 1996.

Hall, A. R. and M. B. *See* Oldenbourg

Hardie, R. P. and R. K. Gaye. *See* Aristotle, *Physics*.

Harvey, E. Ruth. *The Inward Wits: Psychological Theory in the Middle Ages and the Renaissance.* Warburg Institute Surveys, 6. London: University of London Press, 1975.

Heath, T. L. *Diophantus of Alexandria*, 2nd ed., 1910; reprint, New York: Dover, 1964.

Heath, T. L., trans. *The Thirteen Books of Euclid's* Elements, 3 vols, 2nd ed., 1925; reprint ed., New York: Dover, 1956.

Heiberg, I. L., ed. *Euclidis opera omnia*, 7 vols. Leipzig: Teubner, 1883-1888.

Heiberg, I. L., ed. *Apollonii Pergaei Quae Graece Existant Cum Commentariis Antiquis*. 2 Volumes. Leipzig: Teubner, 1891-1893.

Heiberg, I. L., ed. *Sereni Antinoensis opuscula*. Leipzig: Teubner, 1896.

Heiberg, I. L. and Eilhard Weidemann, "Ibn al-Haitams Schrift über parabolische Hohlspiegel," *Bibliotheca Mathematica,* ser. 3, 10 (1909-1910): 201-237.

Hero of Alexandria. *Catoptrica.* Ed. and trans. W. Schmidt, *Heronis Alexandrini Opera Quae Supersunt Omnia*, vol. 2. Leipzig: Teubner, 1900.

Hogendijk, Jan P. *Ibn al-Haytham's* Completion of the Conics. New York: Springer, 1984.

Huxley, G. L. *Anthemius of Tralles: A Study in Later Greek Geometry.* Cambridge, MA: Harvard University Press, 1959.

Huygens, Christiaan. *Oeuvres complètes.* Ed. and trans. D. Bierens de Haan, J. Bosscha, D.J. Korteweg and J.A. Vollgraff. 22 Volumes. LaHaye: Nijhoff, 1988-1950.

.

Ibn al-Haytham. *De configuratione mundi.* *See* Langermann.

Ibn al-Haytham. *Kitab al-Manazir.* *See* Sabra.

Ibn Ishaq, Hunayn. *The Book of the Ten Treatises on the Eye. See* Meyerhof.

Ilardi, Vincent, "Renaissance Florence: The Optical Capital of the World," *The Journal of European Economic History* 22 (1993): 507-541.

Ilardi, Vincent. *Renaissance Vision from Spectacles to Telescopes*. Memoirs of the American Philosophical Society. Forthcoming.,

Jolivet, Jean, "L'Intellect selon al-Farabi: quelques remarques." *Philosophie médiévale arabe et latine,* pp. 211-219. Paris: Vrin, 1995.

Kemp, Martin. *The Science of Art: Optical Themes in Western Art from Brunelleschi to Seurat.* New Haven: Yale University Press, 1990.

Kepler, Johannes. *Kepler's Conversation with Galileo's Sidereal Messenger.* Trans. Edward Rosen. New York: Johnson Reprint, 1965.

Kepler, Johannes. *Ad Vitellionem paralipomena. See* Donahue.

Kheirandish, Elaheh. *The Arabic Version of Euclid's* Optics. New York: Springer Verlag, 1999.

Kohl, Karl, "Über den Aufbau der Welt nach Ibn al-Haitam," *Sitzungsberichte der Physikalisch-medizinischen Sozietät in Erlangen,* 54-55 (1922-1923): 140-179.

Kohl, Karl, "Über das Licht des Mondes, eine Untersuchung von Ibn al-Haitham," *Sitzungsberichte der Physikalisch-medizinischen Sozietät in Erlangen,* 56-57 (1924-1925): 305-398.

Korteweg, D. J. *See* Huygens.

Kretzmann, Norman, Anthony Kenney, and Jan Pinborg, eds. *The Cambridge History of Later Medieval Philosophy.* Cambridge: Cambridge University Press, 1982.

Kuhn, Thomas. *The Structure of Scientific Revolutions*. Chicago: University of Chicago Press, 1962.

Langermann, Y. Tzvi. *Ibn al-Haytham's* On the Configuration of the World. New York: Garland, 1990.

Leff, Gordon. *The Dissolution of the Medieval Outlook: An Essay on Intellectual and Spiritual Change in the Fourteenth Century.* New York: New York University Press, 1976.

Leonardo da Vinci. *See* Richter.

Lindberg, David C., "The Cause of Refraction in Medieval Optics," *British Journal for the History of Science* 4 (1969): 23-38.

Lindberg, David C., ed. and trans. *John Pecham and the Science of Optics.* Madison: University of Wisconsin Press, 1970.

Lindberg, David C., " Introduction" to the 1972 reprint of *Opticae thesaurus.* See Risner.

Lindberg, David C. *A Catalog of Medieval and Renaissance Optical Manuscripts.* Toronto: Pontifical Institute of Mediaeval Studies, 1975.

Lindberg, David C. *Theories of Vision from Al-Kindi to Kepler.* Chicago: University of Chicago Press, 1976.

Lindberg, David C., "The Transmission of Greek and Arabic Learning," *Science in the Middle Ages,* David C. Lindberg, ed., pp. 52-90. Chicago: Chicago University Press, 1978.

Lindberg, David C., "Medieval Latin Theories of the Speed of Light," *Roemer et la Vitesse de la Lumière,* René Taton, ed. Paris: Vrin, 1978.

Lindberg, David C., ed. and trans. *Roger Bacon's Philosophy of Nature: A Critical Edition, with English Translation, Introduction, and Notes, of* De multiplicatione specierum *and* De speculis comburentibus. Oxford: Clarendon Press, 1983.

Lindberg, David C., "Optics in Sixteenth-Century Italy," *Novità celesti e crisi del sapere,* supplement to *Annali dell'Istituto e Museo di Storia della Scienza* fasc. 2 (1983): 131-148.

Lindberg, David C., "The Genesis of Kepler's Theory of Light: Light Metaphysics from Plotinus to Kepler," *Osiris* 2 (1986): 5-42.

Lindberg, David C., ed. and trans. *Roger Bacon and the Origins of* Perspectiva *in the Middle Ages.* Oxford: Clarendon Press, 1996.

Lohne, J. A., "Alhazens Spiegelproblem," *Nordisk Matematisk Tidskrift* 18 (1970): 5-35.

Long, R. James, ed. *Bartholomaeus Anglicus, De proprietatibus rerum, Books 3-4: On the Properties of Soul and Body.* Toronto: Pontifical Institute of Mediaeval Studies, 1979.

Maier, Anneliese. *Die Vorläufer Galileis im 14. Jahrhundert: Studien zur Naturphilosophie der Spätscholastik,* 2nd ed. Rome: Edizioni di storia e letteratura, 1966. *See also* Sargent

Manetti, Vincent. *Vita di Filippo di Ser Brunelleschi.* Ed. and trans. Howard Saalmann and Catherine Enggass. University Park, PA: Penn State University Press, 1970.

Maurer, Armand. *Medieval Philosophy.* New York: Random House, 1962.

Maurolyco, Francesco. *Photismi de lumine et umbra ad perspectivam et radiorum incidentiam facientes.* Naples, 1611.

May, Margaret Talmadge, trans. *Galen on the Usefulness of the Body.* Ithaca, NY: Cornell University Press, 1968.

McMurrich, J. Playfair. *Leonardo da Vinci, the Anatomist (1452-1519).* Baltimore: Williams & Wilkins, 1930.

Meyerhof, Max, ed. and trans. *The Book of the Ten Treatises on the Eye Ascribed to Hunain ibn Is-haq.* Cairo, 1928.

Milhaud, Gaston. *Descartes savant.* Paris: Librairie Félix Alcan, 1921.

Millás Vallicrosa, José Maria. *Las traducciones orientales en los manuscritos de la Biblioteca Catedral de Toledo.* Madrid, 1942.

Mogenet, Joseph, ed. *Autolycus de Pitane. Histoire du texte suivie de l'édition critique de Traitées de la sphère en mouvement and de levers et couches*. Receuil de travaux d'histoire et de philologie de l'Université de Louvain 3, 37. Louvain, 1950.

Moody, E. A., "Empiricism and Metaphysics in Medieval Philosophy," *Philosophical Review* 67 (1958): 145-163.

Morrow, Glenn R., trans. *Proclus: A Commentary on the First Book of Euclid's* Elements. Princeton: Princeton University Press, 1970.

Nebbia, Giorgio, ''Ibn al-Haytham nel millesimo anniversario della nascita," *Physis* 9 (1967): 165-214

Newhauser, Richard, "Der 'Tractatus moralis de oculo' des Petrus von Limoges und seine *Exempla," Exempel und Exempel-sammlungen,* Walter Haug and Burghard Wachinger, eds., pp. 95-136. Tübingen: Max Niemeyer Verlag, 1991.

Nuñes, Pedro. *De crepusculis liber unus.... Item Allacen ... De causis crepusculorum liber unus, a Gerardo Cremonensi iam olim latinitate donatus.* Lisbon, 1542.

Oldenbourg, Henry. *The Correspondence of Henry Oldenbourg*. 9 Vols. Ed. A. R. and M. B. Hall. Madison: University of Wisconsin Press: 1965-1973.

Omar, Saleh. *Ibn al-Haytham's Optics: A Study of the Origins of Experimental Science*. Minneapolis: Bibliotheca Islamica, 1977.

Panofsky, Erwin. *Meaning in the Visual Arts.* Garden City, NY: Doubleday Anchor, 1955.

Papakhristu, O. See Rehren.

Pecham, John. *Perspectiva communis. See* Lindberg.

Phillips, Heather, "John Wyclif and the Optics of the Eucharist," *From Ockham to Wyclif,* Anne Hudson and Michael Wilks, eds., pp. 245-258. Oxford: Basil Blackwell, 1987.

Porta, Giambattista della. *De refractione opticae partes libri novem*. Naples, 1593.

Proclus. *See* Morrow.

Pseudo-Euclid. *See* Björnbo and Vogel.

Ptolemy *Almagest. See* Toomer.

Ptolemy. *Optics. See* Smith.

Ptolemy. *Planetary Hypotheses. See* Goldstein.

Rashed, Roshdi, "Le 'Discours de la Lumière' d'Ibn al-Haytham (Alhazen)," *Revue d'histoire des sciences et de leurs applications* 21 (1968): 197-224.

Rashed, Roshdi, "La modèle de la sphère transparente et l'explication de I'arc-en-ciel," *Revue d'Histoire des Sciences et de leurs Applications* 23 (1970): 109-140.

Rashed, Roshdi, "Futhitos et al-Kindi sur 'l' illusion lunaire'," in ΣΟΦΙΗΣ ΜΑΙΗΤΟΡΕΣ*"Chercheurs de sagesse": Hommage à Jean Pépin.* Paris: Institut d'Études Augustiniennes, 1972.

Rashed, Roshdi, "A Pioneer in Anaclastics: Ibn Sahl on Burning Mirrors and Lenses," *Isis* 81 (1990): 464-491.

Rashed, Roshdi. *Géometrie et dioptrique au Xe siècle: Ibn Sahl, al-Quhi, and Ibn al-Haytham.* Paris: Les Belles Lettres, 1992.

Rashed, Roshdi. *Les mathématiques infinitésimales du IXe au XIe siécle,* vol. 2. London: Al-Furqan, 1993.

Rashed, Roshdi and Jean Jolivet, eds. *Oeuvres philosophiques et scientifiques d'al-Kindi,* vol. 2. Leiden: Brill, 1998.

Reeves, Eileen Adair. *Painting the Heavens: Art and Science in the Age of Galileo.* Princeton: Princeton University Press, 1997.

Rehren, Thilo and Papakhristu, O., "Cutting-Edge Technology--The Ferghana Process of medieval crucible steel smelting," *Metalla* 7 (2000): 55-69.

Richter, Jean-Paul, ed. *The Literary Works of Leonardo da Vinci.* 2 vols. London/ New York: Oxford University Press, 1939.

Riedl, Clare C. *Robert Grosseteste on Light.* Milwaukee: 1942.

Risner, Friedrich, ed. *Opticae thesaurus. Alhazeni arabis libri septem, nunc primum editi. Eiusdem liber De crepusculis et Nubium ascensionibus. Item Vitellonis thuringopoloni libri X.* Basel, 1572, reprint, New York: Johnson Reprint, 1972.

Risner, Friedrich. *Opticae libri quattuor ex voto Petri Rami novissimo*. . . . Kassel, 1606.

Rosen, Edward. *See* Kepler.

Rouse, Mary A. and Richard H. Rouse. *Authentic Witnesses: Approaches to medieval texts and manuscripts.* Notre Dame, IN: University of Notre Dame Press, 1991.

Saalmann, Howard and Catherine Enggass. *See* Manetti.

Sabra, A. I., "Ibn al Haytham's Criticisms of Ptolemy's *Optics*," *Journal of the History of Philosophy* 4 (1966): 145-149.

Sabra, A. I., "The Authorship of the *Liber de crepusculis,* an 11th-Century Work on Atmospheric Refraction," *Isis* 58 (1967): 77-85.

Sabra, A. I. *Theories of Light from Descartes to Newton*. London: Oldbourne,1967. Reprint, Cambridge: Cambridge University Press, 1981.

Sabra, A. I., "Ibn al-Haytham," *Dictionary of Scientific Biography,* vol. 6, ed. Charles Coulston Gillispie, pp. 189-210. New York: Scribner's, 1972.

Sabra, A. I., ed., "Treatise on the Marks Seen on the Surface of the Moon," *Journal for the History of Arabic Science* 1 (1977): 5-19.

Sabra, A. I., "Form in Ibn al-Haytham's Theory of Vision," *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften* 5 (1980): 115-140.

Sabra, A. I., "Ibn al-Haytham's Lemmas for Solving 'Alhazen's Problem,'" *Archive for History of Exact Sciences* 26 (1982): 299-324.

Sabra, A. I., ed. *Ibn al-Haytham, Al-Manazir I-II-III= Kitab-al-Manazir. Books I-II-III.--On Direct Vision.* Kuwait, 1983.

Sabra, A. I., "Psychology versus mathematics: Ptolemy and Alhazen on the moon illusion." *Mathematics and its applications to science and natural philosophy in the Middle Ages,* Edward Grant and John Murdoch, eds., pp. 217-247. Cambridge: Cambridge University Press, 1987.

Sabra, A. I., trans. *The Optics of Ibn al-Haytham: Books I-III on Direct Vision.* 2 vols. London: Warburg Institute, 1989.

Sabra, A. I., "On Seeing the Stars II: Ibn al-Haytham's 'Answers' to the 'Doubts' Raised by Ibn Ma'dan," *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften* 10 (1995/96): 1-59.

Sabra, A. I., "One Ibn al-Haytharn or Two? An Exercise in Reading the Bio-Bibliographical Sources," *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften* 12 (1998): 1-40.

Sargent, Steven D., ed. and trans. *On the Threshold of Exact Science: Selected Writings of Anneliese Maier on Late Medieval Natural Philosophy.* Philadelphia: University of Pennsylvania Press, 1982.

Sarton, George. *An Introduction to the History of Science*. 2 Volumes. Baltimore: Williams & Wilkins, 1927.

Sarton, George. *A History of Science*. 2 Volumes. New York: Norton, 1970.

Schechner, Sara, "Between Knowing and Doing: Mirrors and their Imperfections in the Renaissance," *Early Science and Medicine* 10 (2005): 137-162.

Schmidt, W. *See* Hero of Alexandria

Shastid, Thomas, "History of Ophthalmology." *The American Encyclopedia and Dictionary of Ophthalmology,* vol. 11, Casey A. Wood, ed., pp. 8524-8904. Chicago: Cleveland Press, 1917.

Simon, Gérard, "Derrière le miroir," *Le Temps de la Réflexion* 2 (Paris: Gallimard, 1981): 298-332.

Simon, Gérard, "*L'Optique* d'Ibn al-Haytham et la tradition Ptoléméenne," *Arabic Sciences and Philosophy* 2 (1982): 203-235.

Simon, Gérard. *Le regard, l'être et l'apparence dans l'Optique de l'Antiquité.* Paris: Seuil, 1988.

Simon, Gérard. *Sciences et savoirs aux XVIe et XVIIe siècles.* Paris: Presses Universitaires du Septentrion, 1996.

Simon, Gérard. *L'Archéologie de la vision: L'optique, le corps, le peinture*. Paris: Seuil, 2003.

Smith, A. Mark, "Getting the Big Picture in Perspectivist Optics," *Isis* 72 (1981): 568-589.

Smith, A. Mark, ed. and trans. *Witelonis* Perspectivae *liber quintus.* Studia Copernicana 23. Wroclaw: Ossolineum Press, 1983.

Smith, A. Mark. *Ptolemy's Theory of Visual Perception.* Transactions of the American Philosophical Society 86.2. Philadelphia: American Philosophi cal Society, 1986.

Smith, A. Mark. *Descartes's Theory of Light and Refraction: A Discourse on Method.* Transactions of the American Philosophical Society 77.3. Philadelphia: American Philosophical Society, 1987.

Smith, A. Mark, "Alhazen's Debt to Ptolemy's *Optics*," *Nature, Experiment, and the Sciences,* T. H. Levere and W. R. Shea, eds., pp. 147-164. Dordrecht: Kluwer, 1990.

Smith, A. Mark, "Knowing Things Inside Out: The Scientific Revolution from a Medieval Perspective," *The American Historical Review* 95 (1990): 726-744.

Smith, A. Mark, "Picturing the Mind: The Representation of Thought in the Middle Ages and Renaissance," *Philosophical Topics* 20 (1992): 149-170.

Smith, A. Mark, "The Latin Version of Ibn Mu'adh's Treatise 'On Twilight and the Rising of Clouds,'" *Arabic Sciences and Philosophy* 2 (1992): 83-132.

Smith, A. Mark, "Extremal Principles in Ancient and Medieval Optics," *Physis* 31 (1994): 113-140.

Smith, A. Mark, "Ptolemy, Alhazen, and Kepler and the Problem of Optical Images," *Arabic Sciences and Philosophy* 8 (1998): 8-44.

Smith, A. Mark, "The Physiological and Psychological Grounds of Ptolemy's Visual Theory: Some Methodological Consideration," *Journal of the History of the Behavioral Sciences* 34 (1998): 231-246.

Smith, A. Mark. *Ptolemy and the Foundations of Ancient Mathematical Optics.* Transactions of the American Philosophical Society 89.3. Philadelphia: American Philosophical Society, 1999.

Smith, A. Mark, "The Latin Source of the Fourteenth Century Italian Translation of Alhacen's *De aspectibus* (Vat. Lat. 4595)," *Arabic Sciences and Philosophy* 11 (2001): 27-43.

Smith, A. Mark, ed. and trans. *Alhacen's Theory of Visual Perception*. Transactions of the American Philosophical Society 91.5 and 6. Philadelphia: American Philosophical Society, 2001.

Smith, A. Mark, "What is the History of Medieval Optics Really About?" *Proceedings of the American Philosophical Society* 148 (2004): 180-194.

Smith, A. Mark, "Reflections on the Hockney-Falco Thesis: Optical Theory and Artistic Practice in the Fifteenth and Sixteenth Centuries," *Early Science and Medicine* 10 (2005): 163-185.

Smith, A. Mark and Bernard R. Goldstein, "The Medieval Hebrew and Italian Versions of Ibn Mu'adh's 'On Twilight and the Rising of Clouds'," *Nuncius* 8 (1993): 613-643.

Smith, J. A. *See* Aristotle.

Spencer, John R., trans. *Leon Battista Alberti On Painting*. New Haven: Yale University Press, 1966.

Stock, Brian. *The Implications of Literacy: Written Language and Models of Interpretation in the Eleventh and Twelfth Centuries.* Princeton: Princeton University Press, 1983.

Stratton, George M., trans. *Theophrastus and the Greek Physiological Psychology Before Aristotle.* London: Allen and Unwin, 1917.

Tachau, Katherine. *Vision and Certitude in the Age of Ockham: Optics, Epistemology and the Foundations of Semantics, 1250-1345.* Leiden: E. J. Brill, 1988.

Takahashi, Ken'ichi. *The Medieval Latin Traditions of Euclid's* Catoptrica. Fukuoku: Kyushu University Press, 1992.

Theophrastus. *On the Senses. See* Stratton.

Thiesen, Wilfred. "The Mediaeval Tradition of Euclid's *Optics,*" PhD dissertation: University of Wisconsin, 1974.

Tideus. *See* Björnbo and Vogel.

Toomer, G. J., trans. *Ptolemy's Almagest.* Berlin, Heidelberg, New York: Springer, 1984.

Travaglia, Pinella. *Magic, Causality and Intentionality: The Doctrine of Rays in Al-Kindi.* Micrologus Library, vol. 3. Florence: Edizioni del Galuzzo, 1999.

Unguru, Sabetai, "A Very Early Acquaintance with Apollonius of Perga's Treatise on Conic Sections in the Latin West," Centaurus 20 (1976): 112-128.

Unguru, Sabetai, ed. and trans. *Witelonis* Perspectivae *liber Primus.* Studia Copernicana 28. Wroclaw: Ossolineum Press, 1977.

Unguru, Sabetai, ed. and trans. *Witelonis* Perspectivae *liber secundus et liber tertius.* Studia Copernicana 15. Wroclaw: Ossolineum Press, 1991.

Van Helden, Albert, trans. *The Sidereal Messenger.* Chicago: University of Chicago Press, 1989.

Van Riet, Simone, ed. *Avicenna Latinus: Liber de anima seu sextus de naturalibus I-II-III.* Louvain: E. Peeters, 1972.

Vitruvius. *De architectura. See* Granger.

Vollgraff, J. A. *See* Huygens.

Voss, Don L., "Ibn al-Haytham's *Doubts Concerning Ptolemy:* a Translation and Commentary," PhD dissertation: University of Chicago, 1985.

Wallace, William A. *Causality and Scientific Explanation,* vol. 1. Ann Arbor: University of Michigan Press, 1972.

Webster, E. W. *See* Aristotle.

Wedin, Michael V. *Mind and Imagination in Aristotle.* New Haven: Yale University Press, 1988.

Wiedemann, Eilhard, "Über das Licht der Sterne nach Ibn Al Haitham," *Wochenschrift für Astronomie, Meteorologie und Geographie,* ns, 33 (1890): 129-133.

Wiedemann, Eilhard, "Ibn al-Haitam, ein arabischer Gelehrter," *Festschrift J. Rosenthal.* Leipzig, 1906, pp. 147-178.

Wiedemann, Eilhard, "Über ein Schrift von Ibn al Haitam 'Über die Beschaffenheit der Schatten'," *Sitzungsberichte der Physikalisch-medizinischen Sozietät in Erlangen* 39 (1907): 226-248.

Wiedemann, Eilhard, "Ibn al-Haitams Schrift über die sphärischen Hohlspiegel," *Bibliotheca Mathematica* ser. 3, 10 (1909-1910): 293-307.

Wiedemann, Eilhard, "Über die *Camera obscura* bei Ibn al Haitam," *Sitzungsberichte der Physikalisch-medizinischen Sozietät in Erlangen* 46 (1914): 155-169.

Wiedemann, Eilhard, "Theorie des Regensbogens von Ibn al Haitam," *Sitzungsberichte der Physikalisch-medizinischen Sozietät in Erlangen* 46 (1914): 39-56.

Wiedemann, Eilhard. *Aufsätze zur arabischen Wissenschafts-Geschichte,* vol. 2. Hildesheim: Georg Olms, 1970.

Winter, H. J. J. and W. 'Arafat, "Ibn al-Haitham on the Paraboloidal Focussing Mirror," *Journal of the Royal Asiatic Society of Bengal* ser. 3, 15 (1949): 25-40.

Winter, H. J. J. and W. 'Arafat, "A Discourse on the Concave Spherical Mirror of Ibn al-Haitham," *Journal of the Royal Asiatic Society of Bengal* ser. 3, 16 (1950): 1-16.

Wippel, John F., "The Condemnations of 1270 and 1277 at Paris," *Journal of Medieval and Renaissance Studies* 7 (1977): 169-201.

Witelo. *See* Smith, Unguru.

Wolf-Devine, Celia. *Descartes on Seeing: Epistemology and Visual Perception.* Carbondale and Edwardsville, IL: Southern Illinois University, 1993.

Wolfson, H. A., "The Internal Senses in Latin, Arabic, and Hebrew Philosophical Texts," *Harvard Theological Review* 25 (1935): 69-133.

GENERAL INDEX